Set-Valued Mapping on Digital Metric-Space

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Abstract: **In this paper, we consider mappings F and G in a complete metric space into the class $B(X)$ of non-empty and bounded subsets of X and Set valued mapping on Digital metric space has been proved.**

Keywords: **Complete Metric Space, Non-empty and Bounded Subsets, Singleton Set and Digital Metric Space.**

INTRODUCTION

Digital Metric Space is a concept related to the definition of a metric space in Digital geometry.

Digital image processing area of engineering research. It has tremendous potential of utilizing available computing power with help of optimized algorithms and produce wonderful results of much importance.

In 1906, Frechet, a French mathematician, first addressed the idea of an abstract space with metric space. His work helped in establishing the understanding the key concepts in non-geometric spaces such as continuity and convergence.

Digital metric space is a generalization metric space that is used in fixed-point theory and digital topology. In a Digital metric space, points are considered to be images that are subsets of integers.

Now we discuss the basics of the Digital metric space from the topological point of view. Adjacency and neighbourhood are discussed next to give fundamental background on the topic. The fundamental object in digital topology is a Lattice, which is used to represent a digital image in n-dimensions. The lattice has lattice points (with integer co-ordinates) which are called pixels(2d) or Voxels (3d). A Digital Plane $\mathbb{Z}^2$ is a set of all the points in $\mathbb{R}^2$ and 3-D Digital space $\mathbb{Z}^3$ is a set of all the points in $\mathbb{R}^3$ having integer co-ordinates.

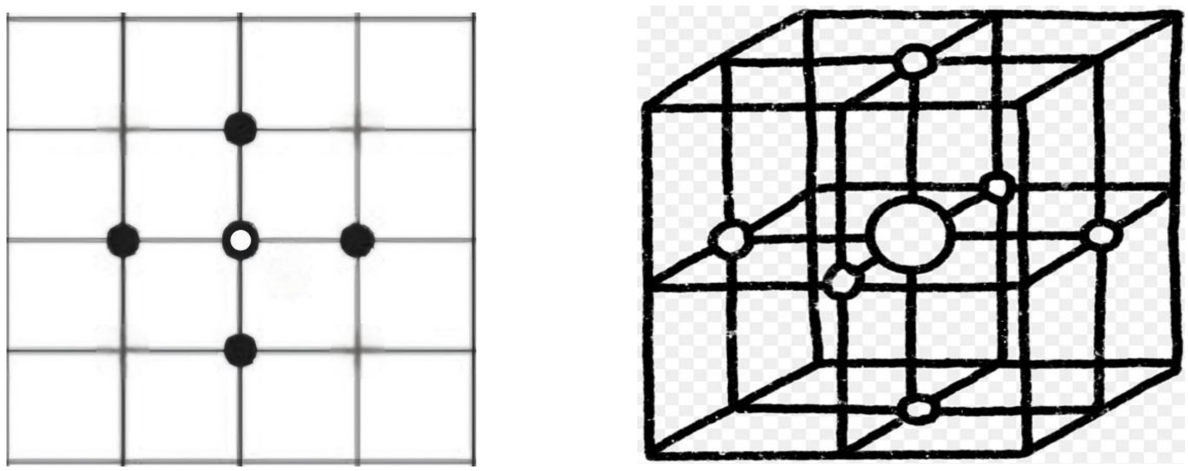


Figure-1 2D and 3-D digital planes

Let (X, d) be a complete metric space. Let $B(X)$ be the family of all non-empty bounded subsets of X. Let A, B in $B(X)$. We define the function $\delta(A, B)$ by

$$\delta(A, B) = \text{Sup}\,\{d(a, b) \,:\, a \in A, b \in B\}$$

If A is singleton set

$$\delta(A, B) = \delta(a, B)$$

If B is also a singleton set then

$$\delta(A, B) = d(a, b)$$

We see that $\delta(A, B) = \delta(B, A) \geq 0$

And

$$\delta(A,B) \le \delta(A,C) + \delta(C,B) \quad , \forall \; A,B,C \in B(X)$$

If A is non-empty set, subset of X we define

$$F(A) = \cup F(a), \quad a \in A$$

If $F: X \to B(X)$ then fixed-point of F is defined as $z \in X$ Such that $z \in F(z)$.

B. Fisher [1] Proved the following theorem.

Theorem: Let F and G be mappings of a complete metric space (X,d) into $B(X)$ Satisfying the inequality

$$\delta(Fx, Gy) \le c.\max\{\delta(x,Fx), \delta(y,Gy), d(x,y)\}$$

$\forall x,y \in X$ Where $0 \le c < 1$ then F and G have a common unique fixed point.

Inspired by above theorem we prove following theorem. (in main result)

2. PRELIMINARIES

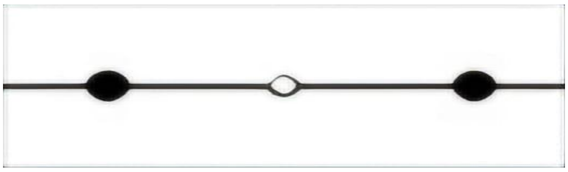
Definition 2.1 Let (X,d) be a Complete metric space. Then a sequence $\{x_n\}$ in X is called a Cauchy sequence $\Leftrightarrow$ $\forall \epsilon > 0 \; \exists \; n_0 \in N$ such that for each $n,m \ge n_0$ we have $d(x_n, x_m) < \epsilon$.

Definition 2.2 Let (X,d) be a Complete metric space. Then a sequence $\{x_n\}$ in X is called convergent sequence $\Leftrightarrow$ $\exists \; x \in X$ such that $\forall \; \exists \; n_0 \in N$ such that $\forall \; n \ge n_0$ we have $d(x_n, x) < \epsilon$, in this case we write $\lim_{n\to\infty} x_n = x$.

Definition 2.3 Let (X,d) be a Complete metric space is complete if every Cauchy sequence convergent.

Definition 2.4 Let l, n be positive integers with $1 \le l \le n$. Consider two distinct point $p = \{p_1, p_2, \ldots\ldots, p_n\}, q = \{q_1, q_2, \ldots\ldots, q_n\} \in \mathbb{Z}^n$. The points p and q are $k_l - adjacent$ if there are at most l indices i such that $|p_i - q_i| = 1$ and for all other indices j, $|p_j - q_j| \ne 1, p_j = q_j$.

2.1.1 $(2 - adjacency)$ Two points on digital plane are said to be $2 - adjecent$ if $|p - q| = 1$.



$Figure\ 2.2 - adjacency$

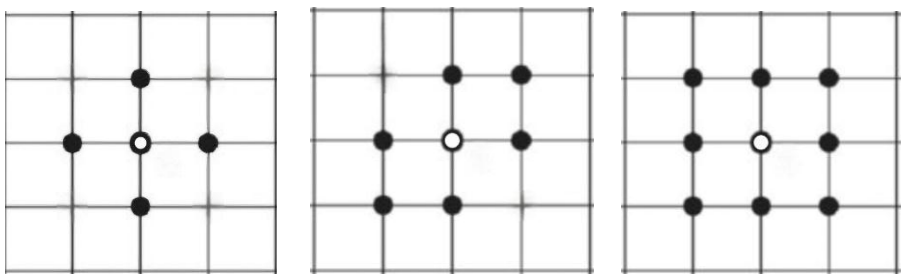
2.1.2 $(4 - neighbours)$ The $4 - neighbours$ of a point p_{ij} are its four horizontal and vertical neighbours $(i \pm 1, j)$and $(i, j \pm 1)$. $4 - neighbours$ of a point p_{ij} and denoted by $N_4(p_{ij})$.

2.1.3 $(6 - neighbours)$ The $6 - neighbours$ of a point p_{ij} are its four horizontal and vertical neighbours $(i \pm 1, j)$ and $(i, j \pm 1)$ along with 2-neighbours $(i+1, j+1)$ and $(i-1, j-1)$ or $(i-1, j+1)$ and $(i+1, j-1)$ i.e. from either of the diagonals. $6 - neighbours$ of a point p_{ij} and denoted by $N_6(p_{ij})$.

2.1.4 $(8 - neighbours)$ The $8 - neighbours$ of a point p_{ij} consist of its $4 - neighbours$ together with its four diagonal neighbours $(i+1, j \pm 1)$ and $(i-1, j \pm 1)$ and are denoted by $N_8(p_{ij})$. The diagonal neighbours of a point p_{ij} are denoted by $N_D(p_{ij})$. The $4 - neighbours$ $N_4(p_{ij})$ and 4 diagonal neighbours, $N_D(p_{ij})$ are together called as $8 - neighbours$ of the point p_{ij} are denoted by $N_8(p_{ij})$.

2.1.5 $(4 - adjacency)$ Two points p and q on a digital plane $\mathbb{Z}^2$ are said to be $4 - adjacent$ if $q \in N_4(p_{ij})$.

2.1.6 $(6 - adjacency)$ Two points p and q on a digital plane $\mathbb{Z}^2$ are said to be $6 - adjacent$ if $q \in N_6(p_{ij})$.



$Figure\ 3.4 - adjacency(left), 6 - adjacency(center), 8 - adjacency(right)$

2.1.7 ($8-adjacency$) Two points p and q on a digital plane $\mathbb{Z}^2$ are said to be $8-adjacent$ if $q \in N_8(p_{ij})$.

2.1.8 ($6-adjacency\ in\ \mathbb{Z}^3$) Two points p and q are $6-adjacent$ in 3-D Digital space($\mathbb{Z}^3$) if point q is located at co-ordinates $(i\pm1,j,k),(i,j\pm1,k)or\ (i,j,k\pm1)$ from the point p_{ijk}.

2.1.9 ($18-adjacency\ in\ \mathbb{Z}^3$) Two points p and q are $18-adjacent$ in 3-D Digital space($\mathbb{Z}^3$) if point q is located at co-ordinates $(i\pm1,j\pm1,k),(i\pm1,j\mp1,k),(i\pm1,j,k\pm1),(i\pm1,j,k\mp1),(i,j\pm1,k\pm1)or\ (i,j\pm1,k\mp1)$ from the point p_{ijk}.

2.1.10 ($26-adjacency\ in\ \mathbb{Z}^3$) Two points p and q are $26-adjacent$ in 3-D Digital space($\mathbb{Z}^3$) if point q is located at co-ordinates $(i\pm1,j\pm1,k\pm1),(i\pm1,j\pm1,k\mp1),(i\pm1,j\mp1,k\pm1)or\ (i\mp1,j\pm1,k\pm1)$ from the point p_{ijk}.

Definition 2.5 A digital image is a pair (X,k), $\Phi\neq X\subset\mathbb{Z}^n$ for some positive integer n and k is an adjacency relation on X. Technically, a digital image (X, k) is an undirected graph whose vertex set is the set of members of X and whose edge set is the set of unordered pairs $\{x_0,x_1\}\subset X$ such that $x_0\neq x_1$ and x_0 are x_1 are $k-adjacent$.

Definition 2.6 Let $(X,k)\subset\mathbb{Z}^{n_0}$ and $(Y,k_1)\subset\mathbb{Z}^{n_1}$ be digital images and $f:X\to X$ be a function.

- If for every $k_0-connected$ subset $U\ of\ X, f(U)$ is a $k_1-connected\ subset$ of Y then f is said to be $(k_0,k_1)-continuous$.
- f is $(k_0,k_1)-continuous$ if for every $k_0-adjacent$ points $\{x_0,x_1\} of\ X$, either $f(x_0)=f(x_1)\ or\ f(x_0) and\ f(x_1)$ are $k_1-adjacent$ in Y.
- If f is $(k_0,k_1)-continuous$, bijective and f^{-1} is $(k_0,k_1)-continuous$, then f is called $(k_0,k_1)-isomorphism$ and denoted by $X_{(k_0,k_1)}\cong Y$.

Now we present basic terminology required for further discussion. Let (X, d, k) be a digital metric space with $\mathrm{k-adjacency}$ and Euclidean metric d for $\mathbb{Z}^n$.

Definition 2.7 Let (X,k) be a digital image set. Let d be a function from $(X,k)\times(X,k)\to\mathbb{Z}^n$ satisfying all the properties of metric space. The triplet (X,d,k) is called a digital space.

Definition 2.8 A sequence $\{x_n\}$ of points of a digital metric space (X,d,k) is a Cauchy sequence if and only if there is $\alpha\in\mathbb{N}$ such that $d(x_n,x_m)<1,\forall\ n,m>\alpha$.

Theorem 2.9: For a digital space (X,d,k), if a sequence $\{x_n\}\subset X\subset\mathbb{Z}^n$ is a Cauchy Sequence then there is $\alpha\in\mathbb{N}$ such that we have $x_n=x_m,\forall\ n,m>\alpha$.

Proposition 2.10 A sequence $\{x_n\}$ of points of a digital metric space (X,d,k) converges to a limit $l\ \in X\ if\ for\ all\ \in>0$, there is $\alpha\in\mathbb{N}$ such that $d(x_n,l)<\epsilon,\ \forall\ n>\alpha$.

Proposition 2.11 A sequence $\{x_n\}$ of points of a digital metric space (X,d,k) converges to a limit $l\ \in X$ if there is $\alpha\in\mathbb{N}$ such that $x_n=l,\forall\ n>\alpha$.

Theorem 2.12 A digital metric space (X,d,k) is complete.

Definition 2.13 Let (X,d,k) be any digital metric space. A self-map f on a digital metric space is said to be a digital contraction if there exists a $\lambda\in[0,1)$ such that for all $x,y\ \in X$,

$$d\big(f(x),f(y)\big)\leq\ \lambda\, d(x,y)$$

Definition 2.14 Every digital contraction map $f:(X,d,k)\to(X,d,k)$ is digitally continuous.

Definition 2.15 In a digital metric space (X,d,k), consider two points x_i,x_j in a sequence $\{x_n\}\subset X$ such that they are $k-adjacent$. Then they have the Euclidean distance $d(x_i,x_j)$ which is greater than or equal to 1 and at most $\sqrt{t}$ depending on the position of the two points.

3. RESULT

Theorem 3.1: Assume that $(\mathrm{X,k})$ is a digital image where $X\ \subset\ \mathbb{Z}^n$ and k is an adjacency relation in X. Also assume that (X,d,k) be a digital metric space. Let F,G are mappings from $X\ to\ B(X)$,Satisfying the inequality

$$\delta(F(x,y),G(u,v)\leq\alpha.\frac{\delta\big((x,y).F(x,y)\big)\big[1+\delta\big((u,v),G(u,v)\big)\big]}{1+d\big((x,y),(u,v)\big)}+\ \beta.d\big((x,y),(u,v)\big)+\gamma.\delta((u,v),G(u,v))$$

Where $x, y, u, v \in X$ and $(\alpha + \beta + \gamma) < 1, \alpha > 0, \beta > 0, \gamma > 0$ Then F, G have a unique fixed-point.

Proof: Assume that $x_0, y_0 \in X$ are some arbitrary elements in X. We have sequences $\{x_n\}$ and $\{y_n\}$ in X. So that

$G(x_0, y_0) = X_1$ Choose a point $x_1, y_1 \in X_1$ Then $x_2, y_2 \in X_2 = F(x_1, y_1)$

$G(x_1, y_1) = X_3$ Choose a point $x_3, y_3 \in X_3$ Then $x_4, y_4 \in X_4 = F(x_3, y_3)$

$G(x_4, y_4) = X_5$ Choose a point $x_5, y_5 \in X_5$ Then $x_6, y_6 \in X_6 = F(x_5, y_5)$

Say in general

$G(x_{2n-2}, y_{2n-2}) = X_{2n-1}$ Choose a point $x_{2n-1}, y_{2n-1} \in X_{2n-1}$ Then $x_{2n}, y_{2n} \in X_{2n} = F(x_{2n-1}, y_{2n-1})$

Then

$$\delta(X_{2n}, X_{2n-1}) = \delta(F(x_{2n-1}, y_{2n-1}), G(x_{2n-2}, y_{2n-2}))$$

$$\leq \alpha. \frac{\delta\{(x_{2n-1}, y_{2n-1}), F(x_{2n-1}, y_{2n-1})\}[1+\delta\{(x_{2n-2}, y_{2n-2}), G(x_{2n-2}, y_{2n-2})\}]}{1+d[(x_{2n-1}, y_{2n-1}), (x_{2n-2}, y_{2n-2})]} + \beta. d\{(x_{2n-1}, y_{2n-1}), (x_{2n-2}, y_{2n-2})\} +$$
$$\gamma. \delta\{(x_{2n-2}, y_{2n-2}), G(x_{2n-2}, y_{2n-2})\}$$

$$\leq \alpha. \frac{\delta\{(x_{2n-1}, y_{2n-1}), X_{2n}\}[1+\delta\{(x_{2n-2}, y_{2n-2}), X_{2n-1}\}]}{1+d[(x_{2n-1}, y_{2n-1}), (x_{2n-2}, y_{2n-2})]} + \beta. d\{(x_{2n-1}, y_{2n-1}), (x_{2n-2}, y_{2n-2})\} +$$
$$\gamma. \delta\{(x_{2n-2}, y_{2n-2}), G(x_{2n-2}, y_{2n-2})\}$$

$$\delta(X_{2n}, X_{2n-1}) \leq \alpha. \frac{\delta\{(x_{2n-1}, y_{2n-1}), X_{2n}\}[1+\delta\{X_{2n-2}, X_{2n-1}\}]}{1+\delta[(X_{2n-1}, X_{2n-2})]} + \beta. \delta\{(X_{2n-1}, X_{2n-2})\} + \gamma. \delta\{(X_{2n-2}, X_{2n-1})\}$$

$$\delta(X_{2n}, X_{2n-1})(1+\alpha) \leq (\beta+\gamma). \delta(X_{2n-1}, X_{2n-2})$$

$$\delta(X_{2n}, X_{2n-1}) \leq \frac{(\beta+\gamma)}{(1+\alpha)}. \delta(X_{2n-1}, X_{2n-2})$$

$$\delta(X_{2n}, X_{2n-1}) \leq h. \delta(X_{2n-1}, X_{2n-2})$$

$$\frac{(\beta+\gamma)}{(1+\alpha)} = h \; where \; |h| \leq 1$$

Or

$$\delta(X_{2n-1}, X_{2n}) \leq h. \delta(X_{2n-1}, X_{2n-2})$$

Similarly

$$\delta(X_{2n-2}, X_{2n-1}) \leq h^2. \delta(X_{2n-2}, X_{2n-3})$$

And finally

$$\delta(X_{2n-2}, X_{2n-1}) \leq h^{2n-1}. \delta(X_0, G(X_0))$$

$$\delta(X_n, X_{n+r}) \leq \delta(X_n, X_{n+1}) + \cdots \ldots \ldots \ldots \ldots \ldots . + \delta(X_{n+r-1}, X_{n+r})$$

$$\delta(X_{2n-2}, X_{2n-1}) \leq [h^n + \cdots \ldots \ldots \ldots . + h^{n+r-1}]. \delta(X_0, G(X_0))$$

$$\delta(X_{2n-2}, X_{2n-1}) \leq \frac{h^n}{1-h}. \delta(X_0, G(X_0))$$

As $h \leq 1$ then given $\epsilon > 0, \exists\, n_0 \in N$ Such that $d(x_m, x_n) \leq \delta(X_m, X_n) < \epsilon, \; \forall \; m, n \geq n_0$.

It follows that $\{x_n\}$ and $\{y_n\}$ is a Cauchy Sequence in digital metric space (X, d, k) and as (X, d, k) is complete So $\{x_n\}$ and $\{y_n\}$ Converges to some $z \in X$.

Now $\delta(z, X_n) \leq \delta\,(z, x_m) + \delta(x_m, X_n)$

$\delta(z, X_n) \leq \delta\,(z, x_m) + \delta(X_m, X_n)$

$\leq \delta\,(z, x_m) + \epsilon\ , \quad \forall\, m, n\ \geq n_0$

Let $m \to \infty$ we get $\delta\,(z, X_n) \leq \epsilon,\ \forall\, n \geq n_0$

Thus $\delta(Fz, X_{2n+1}) = \delta\,(Fz, G(x_{2n}, y_{2n}))$

$$\leq\ \alpha.\frac{\delta(z, Fz)[1 + \delta\{(x_{2n}, y_{2n}), G(x_{2n}, y_{2n})\}]}{1 + d(x_{2n}, y_{2n})} + \beta.d(z, \big((x_{2n}, y_{2n})\big) +\ \gamma.\delta\{(x_{2n}, y_{2n}), G(x_{2n}, y_{2n})\}$$

$$\leq\ \alpha.\frac{\delta(z, Fz)[1 + \delta\{(x_{2n}, y_{2n}), G(x_{2n}, y_{2n})\}]}{1 + \delta(x_{2n}, y_{2n})} + \beta.d(z, \big((x_{2n}, y_{2n})\big) +\ \gamma.\delta\{(x_{2n}, y_{2n}), G(x_{2n}, y_{2n})\}$$

Taking $n \to \infty$, $\delta(Fz, z) \leq \delta(z, Fz)$

Or $\delta(Fz, z) = 0$

$Fz = z$

Similarly, $Gz = z$

Or z is the common fixed-point of $F\ and\ G$.

Uniqueness of Fixed-point

Let $z\ and\ w$ be two fixed-points of $F\ and\ G$.

Then $\delta(z, w) = \delta(Fz, Fw)$

$\leq\ \alpha.\frac{\delta(z, Fz)[1 + \delta(w, Gw)]}{1 + d(z, w)} + \beta.d(z, w) + \gamma.\delta(w.Gw)$

$\delta(z, w) \leq 0 + \beta.d(z, w) + 0$

$\delta(z, w) \leq \beta.d(z, w)$

$\delta(z, w) \leq \beta.\delta(z, w)$

$\delta(z, w)(1 - \beta) \leq\ 0$

$\because (1 - \beta)\ \neq 0$

So $\delta(z, w) = 0$

Or $z = w$

4. CONCLUSION

In this Paper, we have proved a theorem in the context of Set Valued mapping on Digital Metric Space.

ACKNOWLEDGEMENT

The author grate fully acknowledges the help of Dr. S.S. Pagey, Ret. Professor of Mathematics. Institute for Excellence in higher education Bhopal in preparing this paper.

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