Ramanujan's Theta-Function Identities Involving Lambert Series

Nandini C S[1], Sowmya M[2]
[1]*Lecturer in science, Government Polytechnic, Karkala- 574104 Udupi District*
[2]*Lecturer in Science, Government CPC Polytechnic, Mysore*

Abstract: M.D. Hirschhorn [10] has shown that Jacobi's two – square theorem is an immediate consequence of a famous identity of Jacobi. Motivated by this, we show how several identities of Ramanujan involving Lambert series, product and Quotient of theta functions follows easily from Jacobi's triple product identity.

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1.INTRODUCTIONS

In Chapter 16 of his second notebook [2], [3], Ramanujan develops the theory of theta functions and his theta function is defined by

$$f(a,b) = \sum_{n=-\infty}^{\infty} a^{n(n+1)/2} b^{n(n-1)/2}, \quad |ab| < 1.$$

Following Ramanujan, we define

$$\Phi(q) := f(q,q) = \frac{(-q;-q)_\infty}{(q;-q)_\infty},$$

$$\Psi(q) := f(q,q^3) = \frac{(q^2;q^2)_\infty}{(q;q^2)_\infty},$$

and

$$f(-q) := f(-q,-q^2) = (q;q)_\infty$$

The product representations of these theta functions can be derived by using the Jacobi triple product identity:

$$f(a,b) = (-a;ab)_\infty(-b;ab)_\infty(ab;ab)_\infty, \quad |ab| < 1.$$

Ramanujan recorded [11, pp.353-355], several identities involving Lambert series and products of quotients of theta functions. Some of them are

$$1+4\sum_{n=1}^{\infty}\left(\frac{q^{4n-3}}{1-q^{4n-3}} - \frac{q^{4n-1}}{1-q^{4n-1}}\right) = \Phi^2(q), \tag{1.1}$$

$$1-6\sum_{n=1}^{\infty}(-1)^n\left(\frac{q^{3n-2}}{1+(-1)^n q^{3n-2}} + \frac{q^{3n-1}}{1-(-1)^n q^{3n-1}}\right) = \frac{\Phi^3(q)}{\Phi(q^3)}, \tag{1.2}$$

$$1+4\sum_{n=1}^{\infty}\left(\frac{q^{12n-8}}{1-q^{12n-8}} - \frac{q^{12n-4}}{1-q^{12n-4}}\right)+2\sum_{n=1}^{\infty}\left(\frac{q^{3n-1}}{1-q^{3n-1}} - \frac{q^{3n-2}}{1-q^{3n-2}}\right) = \Phi(q)\,\Phi(q^3), \tag{1.3}$$

$$\sum_{n=1}^{\infty}\left(\frac{q^{6n-5}}{1-q^{12n-10}} - \frac{q^{6n-1}}{1-q^{12n-2}}\right) = q\Psi(q^2)\Psi(q^6), \tag{1.4}$$

$$1-6\sum_{n=1}^{\infty}\left(\frac{q^{3n-1}}{1-q^{3n-1}} - \frac{q^{3n-2}}{1-q^{3n-2}}\right) = a(q) \tag{1.5}$$

Where

$$a(q)=\sum_{j,k=-\infty}^{\infty} q^{j^2+jk+k^2},$$

$$\sum_{n=1}^{\infty}\left(\frac{q^{4n-3}}{1-q^{8n-6}} - \frac{q^{4n-1}}{1-q^{8n-2}}\right) = q\Psi^2(q^4), \tag{1.6}$$

$$1+3\sum_{n=1}^{\infty}\left(\frac{q^{6n-5}}{1-q^{6n-5}} - \frac{q^{6n-1}}{1-q^{6n-1}}\right) = \frac{\Psi^3(q)}{\Psi(q^3)}, \tag{1.7}$$

$$\sum_{n=1}^{\infty}\left(\frac{n}{3}\right)\frac{q^n}{1-q^{2n}} = q\frac{\Psi^3(q^3)}{\Psi(q)} \tag{1.8}$$

Where $\left(\frac{n}{3}\right)$ denotes the Legendre symbol,

$$1+2\sum_{n=1}^{\infty}\left(\frac{q^{3n-2}}{1-q^{3n-2}}-\frac{q^{3n-1}}{1-q^{3n-1}}\right)+4\sum_{n=1}^{\infty}\left(\frac{q^{6n-4}}{1-q^{6n-4}}-\frac{q^{6n-2}}{1-q^{6n-2}}\right) = \frac{\Phi^3(-q^3)}{\Phi(-q)} \qquad (1.9)$$

Most of these identities are equivalent to important classical results in Number theory. Infact (1.1) is equivalent to Jacobi's two square theorem [9] which states that the number of representations of n as a sum of two square is four times the difference between the number of divisors of n congruent to 1 modulo 4 and the number of divisors of n congruent to 3 modulo 4. C. Adiga [1] has obtained formulas for the numbers of representations of an integer N≥1 as a sum of two or four triangular numbers. Recently Shaun Cooper[7], Shaun Cooper and H. Y Lam[8] have established formulas for the number of representations of an integer N≥1 as a sum of two, four, six and eight squares and triangular numbers and also sums of an even number of squares and an even number of triangular numbers.

The purpose of this paper is to give a simple unified approach to proving (1.1)-(1.9)

This has been motivated by the recent works of M.D. Hirschhorn [10]

2. Some Preliminary Results

In the following theorem, we collect the various results needed for proving our main results.

Theorem 2.1. In 0<q<1, then the following theta-function identities hold.

1. If $|q|$ <1, then

$$\prod_{n\geq1}(1+aq^{2n-1})(1+a^{-1}q^{2n-1})(1-q^{2n}) = \sum_{n=-\infty}^{\infty} a^n q^{n^2}, \qquad (2.1)$$

2. $\prod_{n\geq1}(1-q^n)^3 = \sum_{n=0}^{\infty}(-1)^n(2n+1)\,q^{n(n+1)/2}$, (2.2)
3. $\Phi^2(q) - \Phi^2(-q) = 8q\Psi^2(q^4)$, (2.3)
4. $\Phi^2(q)f(-q)=\sum_{n=-\infty}^{\infty}(6n+1)q^{(3n^2+n)/2}$, (2.4)
5. $\Psi(q^2)\,f^2(-q) = \sum_{n=-\infty}^{\infty}(3n+1)q^{3n^2+2n}$, (2.5)
6. $4q\,\Psi(q^2)\,\Psi(q^6) = \Phi(q)\,\Phi(q^3) - \Phi(-q)\,\Phi(-q^3)$, (2.6)
7. $\frac{\Phi^3(q)}{\Phi(q^3)} + 2\,\frac{\Phi^3(-q^2)}{\Phi(-q^6)} = 3\,\Phi(q)\,\Phi(q^3)$, (2.7)
8. $a(q) = \Phi(q)\,\Phi(q^3)+4q\,\Psi(q^2)\,\Psi(q^6)$, (2.8)
9. $a(q)= \frac{\Phi^3(-q^3)}{\Phi(-q)} + 4q\,\frac{\Psi^3(q^3)}{\Psi(q)}$, (2.9)

and

10. $a(q) = \frac{\Psi^3(q)}{\Psi(q^3)} + 3q\,\frac{\Psi^3(q^3)}{\Psi(q)}$, (2.10)

Proof: For a proof of (2.1) see [9, Theorem 352, p.282]; for a proof of (2.2) see [9, Theorem 357, p.285]; for a proof of (2.3), see [3, p.40]; for a proof of (2.4), see [3,p.114]; for a proof of (2.5), see [3, p.115]; for proofs of (2.6) and (2.7), see[3,p.232]; for a proof of (2.8), see [4,p.93]; for a proof of (2.9), see [4,p.110]; for a proof of (2.10), see [4,p.111].

3. Main Theorems

We first prove two theorems, which are useful for the derivation of theta-function identities (1.1)-(1.9).

Theorem 3.1. For 0<q<1,

$$x + p\sum_{n=1}^{\infty}\left(\frac{q^{kn-((k-m)/2)}}{1+q^{kn-((k-m)/2)}}-\frac{q^{kn-((k+m)/2)}}{1+q^{kn-((k+m)/2)}}\right)$$

$$= \frac{\sum_{n=-\infty}^{\infty}(pn+x)q^{(kn^2+mn)/2}}{f(q^{(k-m)/2)},q^{(k+m)/2)})}, \qquad (3.1)$$

Proof: Replacing q by q^k and a by $a^p q^m$ in (2.1), we find that

$$\prod_{n\geq1}(1+a^p q^{2kn-k+m})(1+a^{-p}q^{2kn-k-m})(1-q^{2kn}) = \sum_{n=-\infty}^{\infty} a^{pn}q^{kn^2+mn}. \qquad (3.2)$$

Changing q to $q^{1/2}$ in (3.2) and then multiplying throughout by a^x, we obtain

$$a^x \prod_{n\geq1}[1+a^p q^{kn-((k-m)/2}][1+a^{-p}q^{kn-((k+m)/2}][1-q^{kn}]$$

$$=\sum_{n=-\infty}^{\infty} a^{pn+x} q^{(kn^2+mn)/2}. \quad (3.3)$$

Differentiating this with respect to a, and then setting a=1, we obtain (3.1).

Theorem 3.2. For 0 < q < 1,

$$x - p\sum_{n=1}^{\infty}\left(\frac{q^{kn-(k-m)/2}}{1-q^{kn-((k-m)/2)}} - \frac{q^{kn-(k+m)/2}}{1-q^{kn-((k+m)/2)}}\right)$$

$$= \frac{\sum_{n=-\infty}^{\infty}(-1^n)(pn+x)q^{(kn^2+mn)/2}}{f(-q^{(k-m)/2},-q^{(k+m)/2})}, \quad (3.4)$$

Proof: Replacing q by q^k and a by $-a^p q^m$ in (2.1), we deduce that

$$\prod_{n\geq 1}(1 - a^p q^{2kn-k+m})(1-a^{-p} q^{2kn-k-m})(1-q^{2kn})$$

$$= \sum_{n=-\infty}^{\infty}(-1)^n a^{pn} q^{kn^2+mn}. \quad (3.5)$$

Replacing q by $q^{1/2}$ in (3.5) and then multiplying throughout by a^x, we obtain

$$a^x \prod_{n\geq 1}[1 - a^p q^{kn-((k-m)/2)}][1-a^{-p} q^{kn-((k+m)/2)}][1-q^{kn}]$$

$$= \sum_{n=-\infty}^{\infty}(-1)^n a^{pn+x} q^{(kn^2+mn)/2}. \quad (3.6)$$

Differentiating this with respect to a, and then setting a=1, we obtain (3.4).

Proof of (1.1). [10] Putting x =1, p = k = 4 and m = 2 in (3.1), we obtain

$$1+4\sum_{n=1}^{\infty}\left(\frac{q^{4n-1}}{1+q^{4n-1}} - \frac{q^{4n-3}}{1+q^{4n-3}}\right) = \frac{\sum_{n=-\infty}^{\infty}(4n+1)q^{2n^2+n}}{f(q,q^3)}. \quad (3.7)$$

Now using (2.2) in (3.7) and changing q to –q, we obtain the required result.

Proof of (1.2). [3, Entry 4(iv), p.227] Putting x = 1, p = 6, k = 3 and m = 1 in (3.1) and the changing q to –q, we find that

$$1-6\sum_{n=1}^{\infty}(-1)^n\left(\frac{q^{3n-2}}{1+(-1)^n q^{3n-2}} + \frac{q^{3n-1}}{1-(-1)^n q^{3n-1}}\right) = \frac{\sum_{n=-\infty}^{\infty}(-1)^n(6n+1)q^{(3n^2+n)/2}}{f(-q,q^2)}$$

Using (2.4) in the above identity, we obtain the required result.

Proof (1.3).[5] Changing q to $-q^2$ in (1.2), we obtain

$$1-6\sum_{n=1}^{\infty}\left(\frac{q^{6n-4}}{1+q^{6n-4}} - \frac{q^{6n-2}}{1+q^{6n-2}}\right) = \frac{\Phi^3(-q^2)}{\Phi(-q^6)} \quad (3.8)$$

Adding (1.2) to twice of (3.8), we find that

$$3- 6\sum_{n=1}^{\infty}(-1)^n\left(\frac{q^{3n-2}}{1+(-1)^n q^{3n-2}} + \frac{q^{3n-1}}{(1-(-1)^n q^{3n-1})}\right)$$

$$-12\sum_{n=1}^{\infty}\left(\frac{q^{6n-4}}{1+q^{6n-4}} - \frac{q^{6n-2}}{1+q^{6n-2}}\right) = \frac{\Phi^3(q)}{\Phi(q^3)} + 2\frac{\Phi^3(-q^2)}{\Phi(-q^6)}.$$

Using (2.7) in the above identity and on simplification, we obtain the required result.

Proof of (1.4). [3, Entry 3(i), p.223] Changing q to –q in (1.3), we obtain

$$1+4\sum_{n=1}^{\infty}\left(\frac{q^{12n-8}}{1-q^{12n-8}}-\frac{q^{12n-4}}{1-q^{12n-4}}\right) - 2\sum_{n=1}^{\infty}(-1)^n\left(\frac{q^{3n-1}}{1+(-1)^n q^{3n-1}}+\frac{q^{3n-2}}{(1-(-1)^n q^{3n-2})}\right)$$

$$= \Phi(-q)\Phi(-q^3)$$

Using (1.3) and the above identity, we find that

$$2\sum_{n=1}^{\infty}\left(\frac{q^{3n-1}}{1-q^{3n-1}}-\frac{q^{3n-2}}{1-q^{3n-2}}\right) + 2\sum_{n=1}^{\infty}(-1)^n\left(\frac{q^{3n-1}}{1+(-1)^n q^{3n-1}}+\frac{q^{3n-2}}{1-(-1)^n q^{3n-2}}\right)$$

$$= \Phi(q)\Phi(q^3) - \Phi(-q)\Phi(-q^3). \tag{3.9}$$

Using (2.6)and (3.9), we obtain the required result.

Proof of (1.5). [4, p.93] Adding (1.3) and (3.9) and then using (2.6), we obtain

$$1+4\sum_{n=1}^{\infty}\left(\frac{q^{12n-8}}{1-q^{12n-8}}-\frac{q^{12n-4}}{1-q^{12n-4}}\right) +4\sum_{n=1}^{\infty}\left(\frac{q^{3n-1}}{1-q^{3n-1}}-\frac{q^{3n-2}}{(1-q^{3n-2})}\right)$$

$$+ 2\sum_{n=1}^{\infty}(-1)^n\left(\frac{q^{3n-1}}{1+(-1)^n q^{3n-1}}+\frac{q^{3n-2}}{(1-(-1)^n q^{3n-2})}\right) == \Phi(q)\Phi(q^3) + 4q\Psi(q^2)\,\Psi(q^6).$$

Now, using (2.8) and after some simplification, we get the required result.

Proof of (1.6). [12, p.356] Using (1.1) and (2.3), we obtain the required result.

Proof of (1.7). [3, Entry 4(iii), p.226] Putting x = 1, p = 3, k = 6 and m = 4 in (3.4) we find that

$$1- \quad 3\sum_{n=1}^{\infty}\left(\frac{q^{6n-1}}{1-q^{6n-1}}-\frac{q^{6n-5}}{1-q^{6n-5}}\right) = \frac{\sum_{n=-\infty}^{\infty}(-1)^n(3n+1)q^{(3n^2+2n)}}{f(-q,-q^5)}.$$

Using (2.5) in the above identity, we obtain the required result.

Proof of (1.8). [12, p.357] Using (1.5) and (1.7) in (2.10), we obtain the required result.

Proof of (1.9). [12, p.357] Using (1.5) and (1.8) in (2.9), we obtain the required result.

Different proofs of (1.6), (1.8) and (1.9) have also been given by B. C. Berndt [5].

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