Common Fixed-Point Theorem in Cone Metric Spaces for Rational Contractions

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***Abstract*—In this paper we prove the common fixed-point theorem in cone metric space for rational expression in normal cone setting. Our results generalized the main result of Jaggi [7] and Dass, Gupta [3].we studied the concept of contractions mappings to obtain common fixed point in cone metric spaces for rational contractions.**

***Index Terms*—Fixed Point, Cone Metric Spaces, Common Fixed Point, Rational Contractions.**

I. INTRODUCTION

The Banach contraction principle with rational expressions have been expanded and some fixed and common fixed-point theorems have been obtained in [4], [5]. Huang and Zhang [6] initiated cone metric spaces, which is a generalization of metric spaces, by substituting the real numbers with ordered Banach spaces. Later, various authors have proved some common fixed-point theorems with normal and non-normal cones in these spaces [4],[5],[8],[10]. Recently Muhammad arshad et al[2] have introduced almost Jaggi and Gupta contraction in Partially ordered metric spaces to prove the fixed-point theorem.

1. Basic facts and definitions.

Definition 2.1[6]: Let Ealways be a real Banach space and P a subset of $E. P$ is called a cone if and only if:

(i) P is closed, nonempty and $P \neq \{0\}$;

(ii) $a, b \in R,\ a, b \geq 0,\ x, y \in P \Rightarrow ax + by \in P$;

(iii) $x \in P$and $-x \in P \Rightarrow x = 0$.

For Given a cone $P \subset E$, one can define a partial ordering $\leq$ with respect to $P$by $x \leq y$ if and only if $y - x \in P$. The notation $x < y$ indicates that $x \leq y$ but $x \neq y$, while $x << y$ will show$y - x \in intP$, where $intP$ denotes the interior of P. From now on, it is assumed that int $P \neq \emptyset$.

The cone P is callednormal if there is a number $K > 0$ such that for all $x, y \in E$,

$0 \leq x \leq y$ implies $\|x\| \leq K\|y\|$.

The cone P is calledregular if every increasing sequence which is bounded from above is convergent. That is, if $\{x_n\}$ is sequence such that

$$x_1 \leq x_2 \leq \cdots \leq x_n \leq \cdots \leq y,$$

for some $y \in E$, then there is $x \in E$ such that,

$\|x_n - x\| \to 0, (n \to \infty)$.

Equivalently the cone P is regular if and only if every decreasing sequence which is bounded from below is convergent. It is well known that a regular cone is a normal cone.

Definition 2.2[6]: Let X be a nonempty set. Suppose the mapping $d : X \times X \to E$ satisfies

(d1) $0 < d(x, y)$ for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(d2) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(d3) $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in X$.

Then d is called a cone metric on X, and (X, d) is called a cone metric space.

Example2.1[6]:Let $E = R^2, P = \{(x, y) \in E \mid x, y \geq 0\} \subset R^2, X = R$ and $d : X \times X \to E$ suchthat$d(x, y) = (|x - y|, \alpha|x - y|)$,where $\alpha \geq 0$ is a constant. Then (X, d) is a cone metric space.

Definition 2.3[6]: A point x of set X is said to be Common Fixed Point of mapping S, T if, $Sx = Tx = x$.

Definition 2.4[6]: Let (X, d) be a cone metric space, $\{x_n\}$ a sequence in X, $\{x_n\}$ is a Cauchy sequence if there is some $k \in \mathbb{N}$ such that, for all $n, m \geq k$,

$$d(x_n, x_m) \ll c;$$

Definition 2.5[6]: Let (X, d) be a cone metric space, $\{x_n\}$ a sequence in X, $\{x_n\}$ is a convergent sequence if there is some $k \in \mathbb{N}$ such that, for all $n \geq k$,

$$d(x_n, x) \ll c;$$

Then x is called limit of the sequence $\{x_n\}$.

Note that: - (i) Every convergent sequence in a cone metric space X is a Cauchy sequence.

(ii) A cone metric space X is said to be complete if every Cauchy sequence in $X$is convergent in X.

Definition 2.6[6]: Let $X = (X, d)$ be a metric space. A mapping $T: X \to X$ is called a contraction on X if there is a positive real number $\alpha < 1$ s .t for all $x, y \in X$,

$$d\,(Tx, Ty) \leq \alpha\, d\,(x\,, y)\,, (\alpha < 1).$$

Geometrically this means that any points x and y have images that are together than those points x and y; more precisely, the ratio $d\,(Tx, Ty)\,/\,d\,(x\,, y)$ does not exceed a constant α which is strictly less than 1.

2. Main Results

Definition 3.1: Let(X, d) be a cone metric space. A self-mapping T on $X$is called an almost jaggi contraction it satisfies the following condition:

$$d\,(Tx, Ty) \leq \alpha\, d(y, Ty) + \beta \max\left\{\frac{d(y, Tx)}{d(x, y)}, d\,(x, y)\right\} + Lmin\,\{d(x, Ty), d(y, Tx)\}$$

For all $x, y \in X$ where $L \geq 0$ and $\alpha, \beta \in [0,1)$ with $\alpha + \beta < 1$.

Theorem 3.1:Let(X, d) be a complete cone metric space and P a normal cone with normal constant M. Let $T: X \to X$ be an almost jaggi contraction for all $x, y \in X$ where $L \geq 0$ and $\alpha, \beta \in [0,1)$ with $\alpha + \beta < 1$.Then T has a unique fixed point in X.

Proof: Choose $x_0 \in X$. Set $x_1 = Tx_0\,, x_n = Tx_{n-1}$

$$d(x_n\,, x_{n+1}) = d(Tx_{n-1}\,, Tx_n)$$

$$\leq \alpha d(x_n\,, Tx_n) + \beta\, max\left\{\frac{d(x_n\,, T\,x_{n-1})}{d(x_{n-1},\; x_n)}, d(x_{n-1},\; x_n)\right\} + Lmin\,\{d(x_{n-1}\,, T\,x_n), d(x_n\,, T\,x_{n-1})\}$$

$$\leq \alpha d(x_n\,, x_{n+1}) + \beta \max\left\{\frac{d(x_n\,, x_n)}{d(x_{n-1},\; x_n)}, d(x_{n-1},\; x_n)\right\} + Lmin\,\{d(x_{n-1}\,, x_{n+1}), d(x_n\,, x_n)\}$$

$$d(x_n\,, x_{n+1}) \leq \alpha d(x_n\,, x_{n+1}) + \beta d(x_{n-1},\; x_n)$$

$$d(x_n\,, x_{n+1}) - \alpha d(x_n\,, x_{n+1}) \leq \beta d(x_{n-1},\; x_n)$$

$$(1-\alpha) d(x_n\,, x_{n+1}) \leq \beta d(x_{n-1},\; x_n)$$

$$d(x_n\,, x_{n+1}) \leq \frac{\beta}{(1-\alpha)} d(x_{n-1},\; x_n)$$

$$K = \frac{\beta}{(1-\alpha)}\,, \alpha + \beta < 1, 0 < k < 1$$

And by induction

$$d(x_n\,, x_{n+1}) \leq k d(x_{n-1},\; x_n)$$

$$.$$

$$\leq k^n d(x_0,\; x_1)$$

$$d(x_n\,, x_m) \leq d(x_n,\; x_{n+1}) + d(x_{n+1},\; x_{n+2}) + \cdots + d(x_{n+m-1}\,, x_m)$$

$$\leq (k^n + k^{n+1} + k^{n+2} + \cdots + k^{n+m-1}) d(x_0,\; x_1)$$

$$\leq \frac{k^n}{1-k} d(x_0, \ x_1)$$

We get $\|d(x_n, x_{m)}\| \leq M \frac{k^n}{1-k} \|d(x_0, \ x_1)\|$ which implies that $d(x_0, \ x_1) \to 0 \; as \; n \to \infty$. hence $x_n$is a Cauchy sequence, so by completeness of X this sequence must be convergent in X.

$$d(u, Tu) \leq d(u, x_{n+1}) + d(x_{n+1}, Tu)$$
$$d(u, Tu) \leq d(u, x_{n+1}) + d(Tx_n, Tu)$$

$$d(u, Tu) \leq d(u, x_{n+1}) + \alpha d(u, Tu) + \beta \max \left\{ \frac{d(u\ , T\ x_n)}{d(x_n, \ u)}, d(x_n, u) \right\} + \ Lmin \{d(x_n \ \ , Tu), d(u\ , Tx_n)\}$$
$$d(u, Tu) \leq d(u, x_{n+1}) + \alpha d(u, u) + \beta \max \left\{ \frac{d(u\ , x_{n+1})}{d(x_n, \ u)}, d(x_n, u) \right\} + \ Lmin \{d(x_n \ \ , u), d(u\ , x_{n+1})\}$$
$$d(u, Tu) \leq d(u, x_{n+1}) + \beta \max \left\{ \frac{d(u\ , x_{n+1})}{d(x_n, \ u)}, d(x_n, u) \right\} + \ Lmin \{d(x_n \ \ , u), d(u\ , x_{n+1})\}$$

So, using the condition of normality of cone

$$\|d(u, Tu)\| \leq \|d(u, x_{n+1})\| + \beta max \left\{ \frac{\|d(u\ , x_{n+1})\|}{\|d(x_n, \ u)\|}, \|d(x_n, u)\| \right\} + \ Lmin \{\|d(x_n \ \ , u)\|, \|d(u\ , x_{n+1})\|\}$$

As $n \to 0$ we have $\|d(u, Tu)\| \leq 0$.
Hence, we get $u = Tu, u$ is a fixed point of T.

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