Chromatic Index of Vertex Induced Subgraph with Edge Coloring

G. RAJA RAO[1], PRO. G. SHOBHALATHA[2]
[1]*Research scholar, Department of mathematics, S.K. University, Anantapur-515003, A.P., India.*
[2] *Professor, Department of mathematics, S.K. University, Anantapur, 515003, A.P., India*

***Abstract*—A proper edge coloring of a graph is an edge coloring such that no two adjacent edges are assigned the same color. The minimum number of colors needed to color edges of G is called the chromatic index $\chi'(G)$ of G. we discussed the chromatic index of the graph, chromatic index of the induced sub graph and the domination number, the equality between these three values by using the Ce – V table which is newly introduced term for finding the dominating set.**

***Index Terms*—Interval graph, dominating set, Domination number of the graph G, Edge coloring, Chromatic index, Chromatic index of the vertex induced sub graph.**

I. INTRODUCTION

In graph theory, a proper edge coloring of a graph is an assignment of "colors" to the edges of the graph so that no two incident edges have the same color. For example, the figure to the right shows an edge coloring of a graph by the color's red, blue, and green. Edge colorings are one of several different types of graph coloring. The edge-coloring problem asks whether it is possible to color the edges of a given graph using at most *k* different colors, for a given value of *k*, or with the fewest possible colors. The minimum required number of colors for the edges of a given graph is called the chromatic index of the graph.

A graph can be used to represent almost any physical situation involving discrete objects and a relationship among them. A graph structure is very suitable for representing relationships between objects in the abstract, and a large number of combinatorial problems can be modeled. Many Mathematicians have contributed to the growth of this theory and EULER (1707-1782) became the father of graph theory when he settled a famous unsolved problem of his days called the Konigsberg Bridge Problem. A graph G = (V, E) consists of two sets: a non-empty finite set V and a finite set E. The elements of V are called vertices (or points or nodes) and the elements of E are called edges (or lines). Each edge is identified with a pair of vertices. The set V (G) is called the vertex set of G's, and the set E(G) is called the edge set of E's(G). If e = {u, v} ∈ E(G) then we say that e joins u and v. The vertices u and v are called the ends of the edge uv.

The edge-coloring problem is to color all edges of a given graph with the minimum number of colors so that no two adjacent edges are assigned the same color. The edge-coloring problem was appeared in 1880 in relation with the four-color problem. The problem is that every map could be colored with four colors so that any neighboring countries have different colors. It took more than 100 years to prove the problem affirmatively in 1976 with the help of computers. The first paper dealing with the edge-coloring problem was written by Tait in 1880. In this paper Tait proved that if the four-color conjecture is true, then the edges of every 3-connected planar graph can be properly colored using only three colors. Several years later, in 1891 Petersen pointed out that there are 3-connected, cubic graphs which are not 3 colorable. The minimum number of colors needed to color edges of G is called the chromatic index $\chi'(G)$ of G. Obviously $\chi'(G) \geq \Delta(G)$, since all edges incident to the same vertex must be assigned different colors. In 1916, K¨nig has proved his famous theorem which states that every bipartite graph can be edge-colored with exactly $\Delta(G)$ colors, that is $\chi'(G) = \Delta(G)$. In 1949, Shannon proved that every graph can be edge colored with at most $3\Delta(G)/2$ colors, that is $\chi'(G) \leq 3\Delta(G)/2$. In 1964, Vizing proved that $\chi'(G) \leq \Delta(G) + 1$ for every simple graph i.e., the number of colors needed to edge color a simple graph is either its maximum degree Δ or Δ+1.

In this paper, we discussed the chromatic index of the graph, chromatic index of the induced sub graph and the domination number, the equality between these three values by using the Ce – V table which is the coloring of the edges mapped to the respective vertices.

II. PRELIMINARIES

An edge coloring of a graph is a function from its edge set to the set of natural numbers. An edge coloring of a graph G is a function f: E(G) → C, where C is a set of distinct colors. For any positive integer k, a k-edge coloring is an edge coloring that uses exactly k different colors. A proper edge coloring of a graph is an edge coloring such that no two adjacent edges are assigned the same color. Thus, a proper edge coloring f of G is a function f: E(G) → C such that f(e) ≠ f(e') whenever edges e and e' are adjacent in G. An edge coloring containing the smallest possible number of colors for a given graph is known as a minimum edge coloring. The edge chromatic number / chromatic index of a graph G, denoted $\chi^{/}(G)$, minimum number of different colors required for a proper edge coloring of G. For a graph G (V, E), a dominating set S is a subset of the vertices such that every vertex in V - S is adjacent to some vertex in S. Let $G=(V,E)$ be a graph. A set $D \subseteq V(G)$ is a dominating set of *G's* if every vertex in V/D is adjacent to some vertex in D. A dominating set D of the graph *G (V, E)* is a non - split dominating set if the induced sub graph < *V – D* > is connected. A subset $D \subseteq V(G)$ is called dominating set if N[D] = V(G). The minimum cardinality of such a set D is called the domination number $\gamma(G)$ of the graph G.

Iv. Explanation Of the C_e – V Table

We consider the vertices and the corresponding edge colors in vertical and horizontal rows respectively. We denote the edge colors with C_e and we mapped the colors with the vertices. For every vertex we consider the set of adjacent vertices with respective mapped colors. For the first vertex we take highest numbered adjacent vertex into the dominating set, remaining vertices which are containing the dominated vertex in the set of adjacent vertices need not to give the dominating vertex in the dominating set and also the dominating vertex itself also doesn't give the dominated vertex. If any vertex does not contain the dominated vertex in the set of adjacent vertices, then also, we take the highest numbered adjacent vertex from the set and consider it into the dominating set. These C_e – V table gives the dominating set and the domination number of the dominating set gives the equality with the chromatic index of the graph and induced graph.

III. MAIN THEOREMS

1. Theorem:

The interval graph *G* with the set of vertices $\{v_1, v_2, \ldots, v_n\}$ and with the edge coloring corresponding to an interval family I. If v_i is any vertex in the dominating set, which intersects the vertex v_k to its right and the vertex v_k not intersects the vertex v_n to its right then the chromatic index of the vertex induced sub graph gives the relation $\chi^{/}(G)=\gamma(G)+\chi^{/}(V/D)$ where the dominating set obtained from C_e – V table.

Proof:

The interval graph *G* with the set of vertices $\{v_1, v_2, \ldots, v_n\}$ corresponding to an interval family I. In the case v_i is the vertex in the dominating set not intersects the vertex v_k to its right then the domination number increases leads to contradiction and in the case v_k the vertex intersects the vertex v_n to its right then there was no change in the chromatic index of the graph G, but the chromatic index of the vertex induced sub graph increases leads to the inequality of $\chi^{/}(G)=\gamma(G)+\chi^{/}(V/D)$.

Illustration:

The interval graph *G* with the edge colors is as follows,

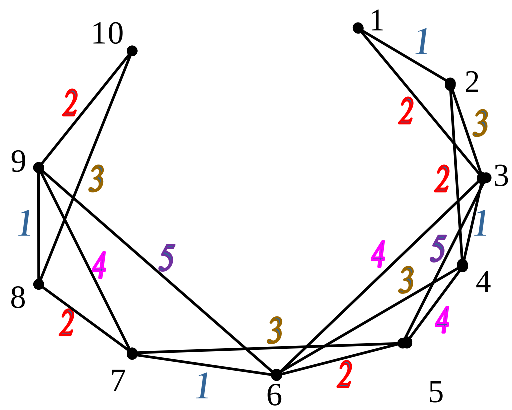

Fig.1: Interval graph *G* with the edge colors

In this interval graph *G* with the edge colors, we can see that the dominating set is {3, 9}.

The edge colors as follows from the above interval graph *G* is as follows, that no two edges sharing the same color.

$C(v_1,v_2) = 1$, $C(v_1,v_3) = 2$,
$C(v_2,v_3) = 3$, $C(v_2,v_4) = 2$,
$C(v_3,v_4) = 1$, $C(v_3,v_5) = 5$,
$C(v_3,v_6) = 4$, $C(v_4,v_5) = 4$,
$C(v_4,v_6) = 3$, $C(v_5,v_6) = 2$,
$C(v_5,v_7) = 3$, $C(v_6,v_7) = 1$,
$C(v_6,v_9) = 5$, $C(v_7,v_8) = 2$,
$C(v_7,v_9) = 4$, $C(v_8,v_9) = 1$,
$C(v_8,v_{10}) = 3$, $C(v_9,v_{10}) = 2$.

To find the dominating set of the following interval graph with edge colors we will introduce the new table namely vertices mapping with the edge colors shortly denoted with C_e – V table as follows,

V/C_e	1	2	3	4	5
v_1	v_2	v_3	-	-	-
v_2	v_1	v_4	v_3	-	-
v_3	v_4	v_1	v_2	v_6	v_5
v_4	v_3	v_2	v_6	v_5	-
v_5	-	v_6	v_7	v_4	v_3
v_6	v_7	v_5	v_4	v_3	v_9
v_7	v_6	v_8	v_5	v_9	-
v_8	v_9	v_7	v_{10}	-	-
v_9	v_8	v_{10}	-	v_7	v_6
v_{10}	-	v_9	v_8	-	-

Now we will find the dominating set by using the above C_e – V table as follows,

Step 1:

For the vertex v_1, the set of adjacent vertices with the respective edge colors is {v_2, v_3}, we consider the vertex v_3 which was the highest numbered adjacent vertex of v_1 into the dominating set,

i.e., Dominating set = {v_3}

Step 2:

For the vertex v_2, the set of adjacent vertices with the respective edge colors is {v_1, v_4, v_3}, the vertex v_3 already exists in the dominating set as well as in the above set, so we didn't take any vertex from these vertices into the dominating set.

Step 3:

The vertex v_3 itself exists in the dominating set, so can't consider any set of adjacent vertices for the dominating set.

Step 4:

For the vertices v_4, v_5, v_6, the vertex v_3 exists in the set of adjacent vertices with the respective edge colors.

Step 5:

For the vertex v_7, the set of adjacent vertices with the respective edge colors is {v_5, v_6, v_8, v_9}, in this set v_3 doesn't exists and v_9 is the highest numbered adjacent vertex for v_7 and giving the different edge color.

Then the dominating set is Dominating set = {v_3, v_9} and the domination number $\gamma(G) = 2$ and also the chromatic index of the graph G is $\chi'(G) = 5$

Now the vertex induced sub graph V\D is as follows,

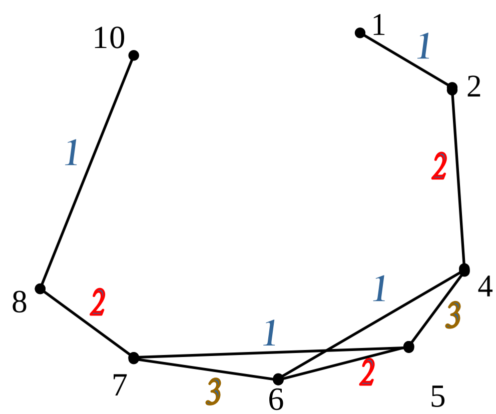

Fig.2: Interval graph induced sub graph V/D with the edge colors

The chromatic index of the induced sub graph V/D is $\chi'(V/D) = 3$

Then clearly, we can establish the relationship for the domination number of the graph G and for the chromatic index of the vertex induced sub graph G as

$$\chi^{/}(G) = \gamma(G) + \chi^{/}(V/D)$$

Hence the theorem is proved.

2. Theorem:

Let G be an edge coloring interval graph with interval family corresponding to I. If V_k is the last vertex in the dominating set and intersects the vertex v_i to its right then the chromatic number of the interval graph G increases for the equality $\chi^{/}(G) = \gamma(G) + \chi^{/}(V/D)$ with the dominating set obtained from the $C_e - V$ table.

Proof:

Let $\{v_1, v_2, \ldots, v_n\}$ be the set of vertices corresponding to an interval family I. The last vertex v_k not intersects the vertex v_i then the degree of the vertex will be decreases and the number of colors needed for the edges in the interval graph will also decreases as the remaining all vertices of the interval graph G intersects four or less than four vertices. So the last vertex must intersect the vertex v_i to its right for the equality $\chi^{/}(G) = \gamma(G) + \chi^{/}(V/D)$ and the dominating set obtained from the C_e-V table as follows,

Illustration:

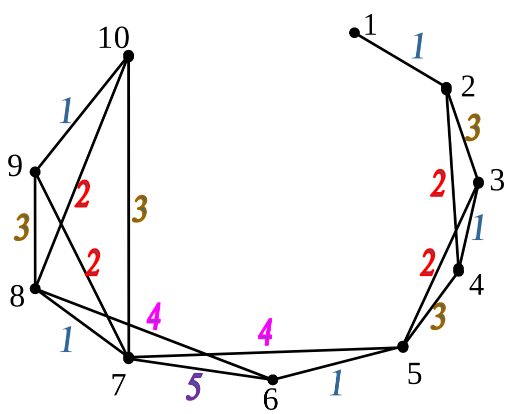

Fig.3: Interval graph *G* with the edge colors

In this interval graph *G* with the edge colors, we can see that the dominating set is {2, 7}.

The edge colors as follows from the above interval graph *G*, that no two edges sharing the same color.

$C(v_1,v_2) = 1$, $C(v_2,v_3) = 3$,
$C(v_2,v_4) = 2$, $C(v_3,v_4) = 1$,
$C(v_3,v_5) = 2$, $C(v_4,v_5) = 3$,
$C(v_5,v_6) = 1$, $C(v_5,v_7) = 4$,
$C(v_6,v_7) = 5$, $C(v_6,v_8) = 4$,
$C(v_7,v_8) = 1$, $C(v_7,v_9) = 2$,
$C(v_7,v_{10}) = 3$, $C(v_8,v_9) = 3$,
$C(v_8,v_{10}) = 2$, $C(v_9,v_{10}) = 1$.

To find the dominating set of the following interval graph with edge colors we will introduce the new table namely vertices mapping with the edge colors shortly denoted with $C_e - V$ table as follows,

V / C_e	1	2	3	4	5
v_1	v_2	-	-	-	-
v_2	v_1	v_4	v_3	-	-
v_3	v_4	v_5	v_2	-	-
v_4	v_3	v_2	v_5	-	-
v_5	v_6	v_3	v_4	v_7	-
v_6	v_7	v_5	v_4	v_3	v_9
v_7	v_8	v_9	v_{10}	v_5	v_6
v_8	v_7	v_{10}	v_9	v_6	-
v_9	v_{10}	v_7	v_8	-	-
v_{10}	v_9	v_8	v_7	-	-

Now we will find the dominating set by using the above $C_e - V$ table as follows,

Step 1:

For the vertex v_1, the set of adjacent vertices with the respective edge colors is $\{v_2\}$, we consider the vertex v_2 which was the highest numbered adjacent vertex of v_1 into the dominating set, i.e., Dominating set = $\{v_2\}$

Step 2:

The vertex v_2 itself exists in the dominating set, so can't consider any set of adjacent vertices for the dominating set.

Step 3:

For the vertices v_3, v_4 the vertex v_2 exists in the set of adjacent vertices with the respective edge colors.

Step 4:

For the vertex v_5 the vertex v_3 not exists in the set of adjacent vertices with the respective edge colors so we take the highest numbered adjacent vertex v_7 into the dominating set.

i.e., Dominating set = $\{v_2, v_7\}$

Step 5:

For the vertices v_6, v_8, v_9, v_{10} the vertex v_7 exists in the set of adjacent vertices with the respective colors, and for the vertex v_7 itself exists in the dominating set.

Then the dominating set is Dominating set = {v_2, v_7} and the domination number $\gamma(G)=2$ and also the chromatic index of the graph G is $\chi^{/}(G)=5$

Now the vertex induced sub graph V\D is as follows,

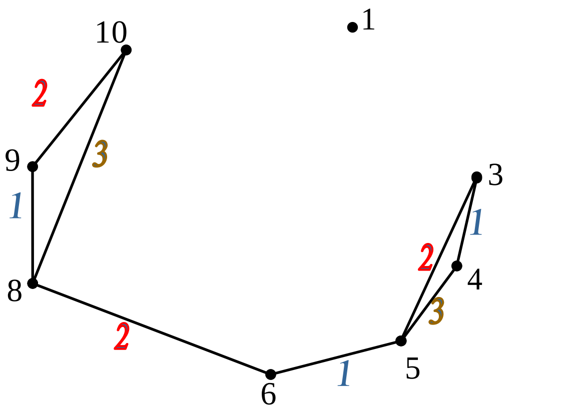

Fig.4: Induced sub graph *V/D* with the edge colors

The chromatic index of the induced sub graph V/D is $\chi^{/}(V/D)=3$

Then clearly, we can establish the relationship for the domination number of the graph G and for the chromatic index of the vertex induced sub graph G as

$\chi^{/}(G)=\gamma(G)+\chi^{/}(V/D)$

Hence the theorem is proved.

3. Theorem:

Let G be an interval graph with edge coloring and with the vertices {v_1, v_2, ……, v_n} corresponding to an interval family $I=\{i_1,i_2,.....,i_n\}$. If v_i, v_j, v_k are any three vertices in the interval graph and the vertex v_k which belongs to the dominating set doesn't intersects the second vertex of its succeeding vertices and the other vertex of the dominating set doesn't intersects the second vertex of its preceding vertices then the chromatic index of the graph G satisfied the equality with domination and with the chromatic index of the induced sub graph of the graph G, whenever the dominating set produced by the C_e – V table.

Proof:

Let G be an interval graph with edge coloring and with the vertices {v_1, v_2, ……, v_n} corresponding to an interval family $I=\{i_1,i_2,.....,i_n\}$. If v_i, v_j, v_k are any three vertices in the interval graph and the vertex v_k which belongs to the dominating set doesn't intersects the second vertex of its succeeding vertices and the other vertex of the dominating set doesn't intersects the second vertex of its preceding vertices, if in both the cases the vertices intersects there will not be any change in the chromatic index of the graph G and in the dominating set with domination number but it reflects the chromatic index of the induced sub graph of the graph G as it increases for taking the colors to its edges then it leads to chromatic index of the graph G less than the addition of the domination number with the chromatic index of the induced sub graph of G and doesn't gives the equality whenever the dominating set produced by the C_e – V table.

Illustration:

The interval graph *G* with the edge colors is as follows,

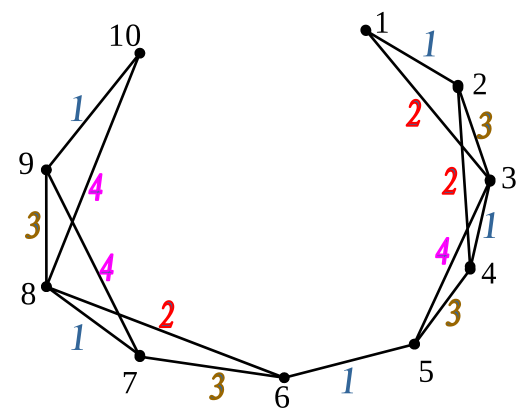

Fig.5: Interval graph *G* with the edge colors

In this Interval graph *G* the dominating set is {3, 8}.

The edge colors as follows from the above interval graph *G*, that no two edges sharing the same color.

$C(v_1,v_2)=1$, $C(v_1,v_3)=2$,
$C(v_2,v_3)=3$, $C(v_2,v_4)=2$,
$C(v_3,v_4)=1$, $C(v_3,v_5)=4$,
$C(v_4,v_5)=3$, $C(v_5,v_6)=1$,
$C(v_6,v_7)=3$, $C(v_6,v_8)=2$,
$C(v_7,v_8)=1$, $C(v_7,v_9)=4$,
$C(v_7,v_{10})=5$, $C(v_8,v_9)=3$,
$C(v_8,v_{10})=4$, $C(v_9,v_{10})=1$

To find the dominating set of the following interval graph with edge colors we will introduce the new table namely vertices mapping with the edge colors shortly denoted with C_e – V table as follows,

V / C_e	1	2	3	4	5
v_1	v_2	v_3	-	-	-
v_2	v_1	v_4	v_3	-	-
v_3	v_4	v_1	v_2	v_5	-
v_4	v_3	v_2	v_5	v_6	-
v_5	v_6	v_7	v_4	v_3	-
v_6	v_5	v_8	v_7	v_4	-
v_7	v_8	v_5	v_6	v_9	-
v_8	v_7	v_6	v_9	v_{10}	-
v_9	v_{10}	-	v_8	v_7	-
v_{10}	v_9	-	-	v_8	-

Now we will find the dominating set by using the above $C_e - V$ table as follows,

Step 1:

For the vertex v_1, the set of adjacent vertices with the respective edge colors is $\{v_2, v_3\}$, we consider the vertex v_3 which was the highest numbered adjacent vertex of v_1 into the dominating set,i.e., Dominating set = $\{v_3\}$

Step 2:

For the vertex v_2 , the set of adjacent vertices with the respective edge colors consists the vertex v_3

Step 3:

The vertex v_3 itself exists in the dominating set

Step 4:

For the vertices v_4 and v_5 the dominating vertex v_3 exists in the set of adjacent vertices with the respective edge colors.

Step 5:

For the vertex v_6 the highest numbered adjacent vertex was v_8 and the dominating vertex v_3 not exists in the set of adjacent vertices with the respective edge colors.

Then the dominating set is Dominating set = $\{v_3, v_8\}$ and the domination number $\gamma(G) = 2$ and also the chromatic index of the graph G is $\chi^{/}(G) = 4$

Now the vertex induced sub graph V\D is as follows,

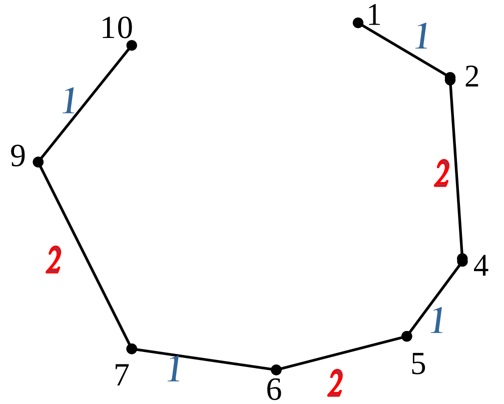

Fig.6: Interval graph *G* with the edge colors

The chromatic index of the induced sub graph V/D is

$\chi^{/}(V/D) = 2$

Then clearly we can establish the relationship for the domination number of the graph G and for the chromatic index of the vertex induced sub graph G as,

$$\chi^{/}(G) = \gamma(G) + \chi^{/}(V/D)$$

Hence the theorem is proved.

REFERENCES

[1] P. Dankelmann and R. C. Laskar. Factor domination and minimum degree. Discrete Math., 262(1-3):113-119, 2003.

[2] L. W. Beineke, R. J. Wilson, On the edge-chromatic number of a graph, Discrete Mathematics, 5, (1973), P: 15-20

[3] H. L. Bodlaender, On the complexity of some coloring games, Lecture Notes in Computer Science, 484, (1991), P: 30-40

[4] L. Cai, X. Zhu, Game Chromatic Index of k-degenerate Graphs, J. Graph Theory, 36, (2001), P: 144-155

[5] G. Chartrand, L. Lesniak, Graphs & Digraphs, Chapman & Hall / CRC, (2000)