Exploring Stereographic Projection: From Fundamentals to Advanced Applications

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Abstract**—Stereographic projection is a key mathematical tool for mapping three-dimensional data onto a two-dimensional plane. It is especially useful for polar regions and small-scale maps. In structural geology, it solves orientation problems, while in crystallography, it analyzes crystal structures and their relationships with crystallographic planes. It also aids in solving spatial problems in mathematics. Its versatility makes it indispensable across scientific disciplines, combining theoretical insights with practical applications. As a result, stereographic projection has become a vital technique in research and problem-solving, proving its value in both academic and real-world contexts**.

Index Terms**—Cartography, Crystallography, Geology, Photography, Planetary Science Riemann Sphere, Stereographic Projection.**

I. INTRODUCTION

This paper introduces stereographic projection and its diverse applications, highlighting its significance as one of the most widely used mapping techniques today. Particularly effective in polar regions and for small-scale maps, stereographic projection serves as a powerful tool across multiple disciplines. In structural geology, it aids in solving orientation problems, while in crystallography, it facilitates the analysis of crystal structures and their relationships with crystallographic planes. Its versatility extends to mathematics, where it provides solutions to spatial challenges. As a fundamental method for visualizing and interpreting spatial data, stereographic projection remains indispensable in both theoretical and applied sciences.

II. THE CONCEPT OF THE PROJECTION

The Earth is roughly spherical, while maps are displayed on flat surfaces like paper or screens, inevitably causing distortion when projecting a sphere onto a plane. Various map projections address this challenge, each suited to specific map sizes and purposes. These projections balance accuracy in shape, area, distance, or direction, ensuring the map meets its intended use while minimizing distortion for the region being represented.

III. HISTORY

The stereographic projection has a rich history, dating back to ancient times. It was known to Hipparchus, Ptolemy, and possibly even earlier to the Egyptians. Originally called the planisphere projection, it was first documented in Ptolemy's *Planisphaerium*, the oldest surviving text describing the method. One of its earliest and most significant uses was in creating celestial charts, a purpose for which the term "planisphere" is still used today.

During the 16th and 17th centuries, the equatorial aspect of the stereographic projection became popular for mapping the Eastern and Western Hemispheres. Maps like those by Gualterius Lud (1507), Jean Roze (1542), and Rumold Mercator (1595) are believed to have employed this projection. Interestingly, ancient astronomers like Ptolemy had already used this equatorial aspect for star charts.

The term "stereographic projection" was coined by François d'Aguilon in his 1613 work, *Opticorum libri sex*, which highlighted its usefulness for both philosophers and mathematicians. Later, in 1695, Edmond Halley, inspired by his interest in star charts, provided the first mathematical proof of the projection's conformal properties. He achieved this using calculus, a groundbreaking tool developed by his friend, Isaac Newton.

IV. DEFINATION

Riemann Sphere: The Riemann sphere is a unit sphere S^2 in three-dimensional space $\mathbb{R}^3$, defined as

$$S^2 = \{(x, y, z) \in \mathbb{R}^3 | x^2 + y^2 + z^2 = 1\}$$

The sphere is often placed such that the "north pole" is at (0,0,1) .

Complex Plane: The complex plane $\mathbb{C}$ consists of all complex numbers $\boldsymbol{z = x + iy}$, where x and y are real numbers.

Stereographic Projection: Stereographic Projection is a geometric mapping that projects points from the surface of a sphere onto a plane, typically the complex plane, by drawing a line from a designated projection point (commonly the north pole) through a point on the sphere and identifying where that line intersects the plane. This projection creates a one-to-one correspondence between all points on the sphere (excluding the projection point) and the plane, providing a powerful tool for visualizing and analyzing complex numbers and their properties.

V. METHOD

Let $\mathbb{C}$ is the complex plane in this method, we set up a correspondence between the points of the complex plane $\mathbb{C}$ and those of a sphere of radius $^1/_2$ with centre at $(0,0,1/2)$ tangent to this plane.

Let this line be $Z - axis\ of\ 3 - D$ Euclidean space in which a pointless coordinate (X , Y, Z).

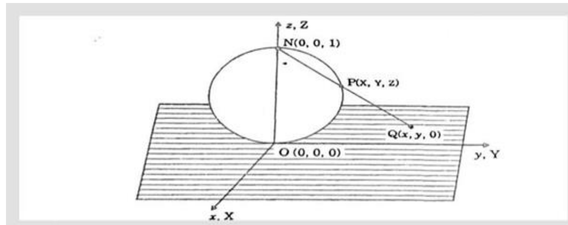
Consider the sphere S of radius ½ and centre at $(0,0, 1/2)$

$$S = \{(X, Y, Z) \in \mathbb{R}^3 : X^2 + Y^2 + (Z - ½)^2 = ¼\}$$

Let N (0,0,1) and O (0,0,0) denote the north pole and south pole of the sphere S respectively. The point $z = 0 + i.0$ coincides with the point O (0,0,0) of the complex plane b and that X and Y axes are the x and y axes respectively. Let $Q\ (x, y, 0)$ be any point in the plane $\mathbb{C}$. Corresponding to this point $Q\ (x, y, 0)$ on the complex plane, there exists a unique point on the sphere S. Through the points N and Q draw a straight-line NQ intersecting the sphere S at a point say $P\ (X, Y, Z)$. Then (X, Y, Z) is called the stereographic projection or image of $(x, y, 0)$ on the sphere . Now we see that there is a one-to-one correspondence between the points of and the points of S with one exception namely the north pole (0,0,1) . Let the north pole N of the sphere corresponds to the point at infinity and so we obtain a one-to-one correspondence between all the points of the sphere S on one hand. This sphere is called Rieman Sphere.

Let $Q\ (x, y, 0)$ (other than infinity) be any point on the complex plane and its image on sphere S be $P(X, Y, Z)$.

Points (0,0,1) , (X, Y, Z) and $Q(x, y, 0)$ are collinear



$$(X - 0)/(x - 0) = (Y - 0)/(y - 0) = (Z - 1)/(0 - 1)$$

$$=> \quad X/x = Y/y = 1 - Z \quad \ldots\ldots\ldots (1)$$

$$=> \quad x = X/1 - Z \ and\ y = Y/1 - Z \quad \ldots\ldots\ldots (2)$$

$$=> \quad x^2 + y^2 = (X^2 + Y^2)/(1 - Z)^2$$

$$=> \quad = \{¼ - (Z - ½)^2\} / (1 - Z)^2$$

$$x^2 + y^2 = Z\,(1 - Z)\,/(1 - Z)^2 \ldots\ldots\ldots (3)$$

$As\ any\ point\ P\ (X, Y, Z) on\ sphere\ is$
$different\ from\ N$ (0,0,1)

$$Z \neq 1 \quad => \quad 1 - Z \neq 0$$

$$EQ\ (3)\ becomes\ x^2 + y^2 = \frac{Z}{1 - Z}$$

$$Or \quad (x^2 + y^2)(1 - Z) = Z$$

$$Or \quad x^2 + y^2 - Z(x^2 + y^2) = Z$$

$$Or \quad x^2 + y^2 = Z(1 + x^2 + y^2)$$

$$Z = (x^2 + y^2)/(1 + x^2 + y^2)$$

$Here$ $1 - Z = (1 - x^2 + y^2)/(1 + x^2 + y^2)$

$Z = 1/(1 + x^2 + y^2)$ ……….(4)

$Using$ (4) in (1) $we\ get$ ………. $X/x = 1/(1 + x^2 + y^2)$,

$Y/y = 1/(1 + x^2 + y^2)$ $X = x/(1 + x^2 + y^2)$,

$Y = y/(1 + x^2 + y^2)$

$$Z = (x^2 + y^2)/(1 + x^2 + y^2).$$

Thus, image of point $(x, y, 0)$ on the complex plane is the point (X, Y, Z) on the sphere S, where

$$X = x/(1 + x^2 + y^2)\ , Y = y/(1 + x^2 + y^2)\ , and\ \ Z = (x^2 + y^2)/(1 + x^2 + y^2)$$

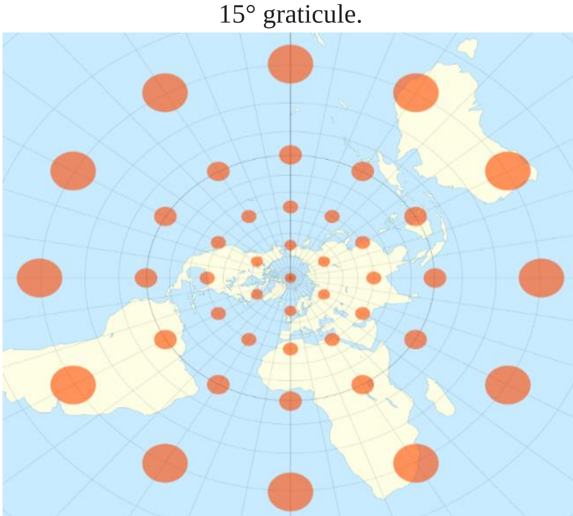
VI. APPLICATION

1. Cartography:

The fundamental problem of cartography is that no map from the sphere to the plane can accurately represent both angles and areas. In general, area-preserving map projections are preferred for statistical applications, while angle-preserving (conformal) map projections are preferred for navigation.

Stereographic projection falls into the second category. When the projection is centered at the Earth's north or south pole, it has additional desirable properties: It sends meridians to rays emanating from the origin and parallels to circles centered at the origin.

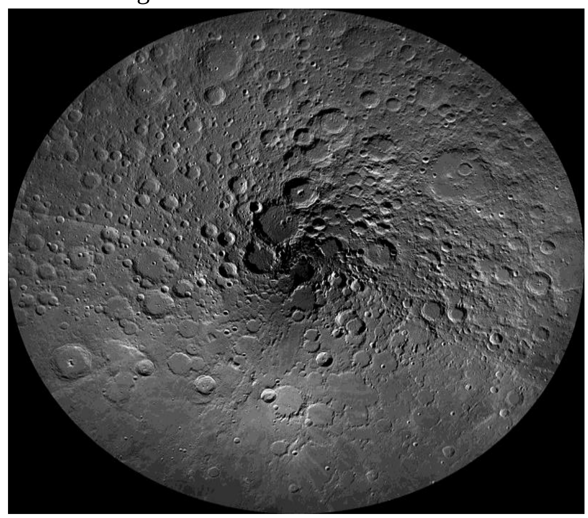
Stereographic projection of the world north of 30°S. 15° graticule.



The stereographic projection with Tissot's indicatrix of deformation.

2.Planetary science:

The stereographic is the only projection that maps all circles on a sphere to circles on a plane. This property is valuable in planetary mapping where craters are typical features. The set of circles passing through the point of projection have unbounded radius, and therefore degenerate into lines.

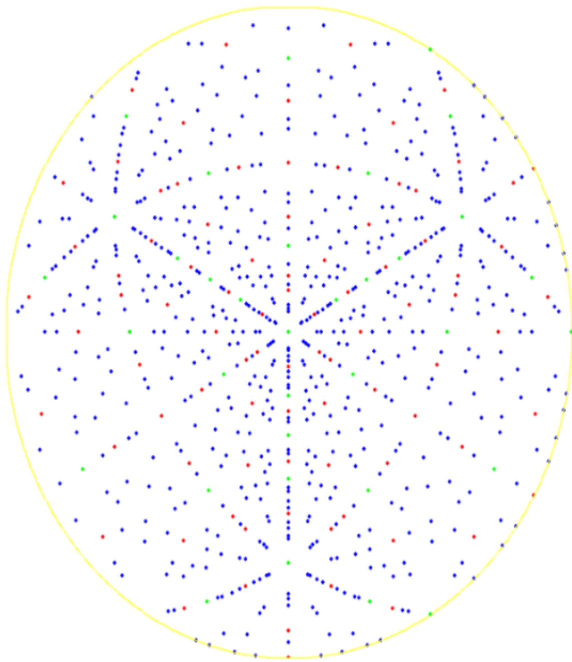


Stereographic Projection of the Moon (≥60°N)

3. Crystallography:

In crystallography, understanding the orientations of crystal axes and faces in three-dimensional space is crucial, particularly when interpreting X-ray and electron diffraction patterns. These orientations can be visualized by intersecting crystal axes and the poles of crystal planes with the northern hemisphere and then plotting them using a stereographic projection. Such a plot, known as a pole figure, provides a clear representation of the crystal's geometric structure.

In electron diffraction, Kikuchi line pairs emerge as bands that highlight the intersection between lattice plane traces and the Ewald sphere. These lines offer a practical way to experimentally access a crystal's stereographic projection. Additionally, model Kikuchi maps in reciprocal space and fringe visibility maps, used for analyzing bend contours in direct space, serve as valuable tools for navigating orientation space when studying crystals in a transmission electron microscope. These maps act as guides, helping researchers explore and interpret the complex orientation relationships within crystalline materials.

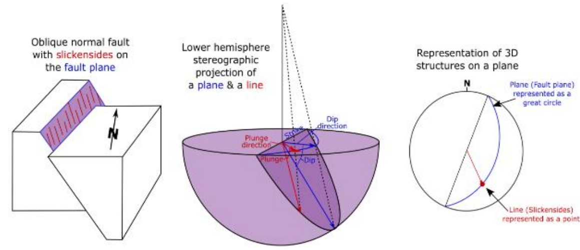


A crystallographic pole figure for the diamond lattice in [111] direction

4. Geology:

In structural geology, researchers focus on the orientations of planes and lines for various purposes. For instance, the foliation of a rock, which is a planar feature, often includes a linear feature known as lineation. Similarly, a fault plane, another planar feature, may contain linear elements such as slickensides.

The orientations of these lines and planes, observed at different scales, can be plotted using methods similar to those described in the visualization of lines and planes section. As in crystallography, planes are typically represented by their poles. However, unlike crystallography, the southern hemisphere is used for plotting in structural geology because the geological features of interest are located beneath the Earth's surface. In this context, the stereographic projection is often referred to as the equal-angle lower-hemisphere projection. Additionally, the equal-area lower-hemisphere projection, defined by the Lambert azimuthal equal-area projection, is also employed, particularly when the plot will undergo further statistical analysis, such as density contouring. This approach ensures accurate representation and facilitates detailed examination of geological structures.



Stereographic Projection of Fault Plane and Slickenside Lineation

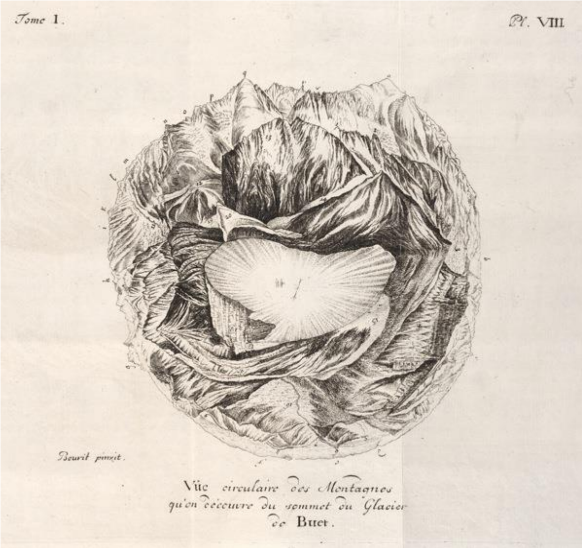
5. Photography:

Some fisheye lenses utilize a stereographic projection to capture wide-angle views. Unlike traditional fisheye lenses that employ an equal-area projection, stereographic lenses preserve the shape of objects near the edges and produce less curvature in straight lines. However, these lenses are generally more costly to produce. Software tools like Panotools enable the automatic conversion of images from an equal-area fisheye projection to a stereographic projection.

The stereographic projection has a long history in mapping spherical panoramas, dating back to Horace Bénédict de Saussure's work in 1779. Depending on the projection's center, this technique can create distinctive effects, such as the "little planet" effect (when the center is the nadir) or the "tube" effect (when the center is the zenith).

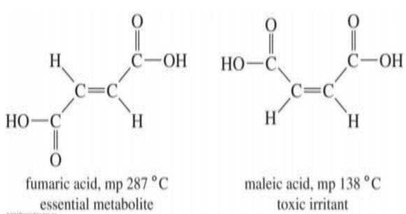
The preference for stereographic projections in panorama mapping, compared to other azimuthal projections, stems from their conformal nature, which ensures better preservation of shapes and angles. This characteristic makes them particularly appealing for applications where maintaining accurate proportions and minimizing distortion are important.

Stereographic Projection of Last Supper Sculpture – Esino Lario, 2016



From Horace-Benedict de Saussure's *Voyage dans les Alpes* (1779-96, Pl. 8)

6. Chemistry:
In chemistry, stereographic projection is used to study chiral molecules. Stereochemistry, also known as 3D chemistry, focuses on the spatial arrangements of atoms in molecules. It plays a crucial role in understanding stereochemical problems across various branches, including organic, inorganic, biological, and supramolecular chemistry. In organic chemistry, even slight variations in spatial arrangement can lead to significant effects on molecular properties and reactions.
For Example



VI. CONCLUSIONS

Stereographic projection serves as a powerful mathematical tool for research, offering valuable insights and applications. Historically, it originated as a practical method for solving astronomical and navigational problems. Over time, it has become indispensable in studying the optical properties of crystals and addressing structural challenges. Its versatility continues to make it a significant tool in various fields of mathematical and scientific research.

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