A Mathematical Modeling Approach to Air Pollution Dispersion for Enhancing Community Health and Environmental Safety

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Abstract- **Air pollution continues to pose serious threats to public health and the environment, particularly in densely populated and industrial regions. Mathematical modeling provides a robust framework to analyze and predict the dispersion patterns of airborne pollutants. This study presents an analytical model for pollutant dispersion from a point source, incorporating both advection and diffusion phenomena within the atmospheric boundary layer. The model integrates the effects of wind velocity, eddy diffusion coefficients, and atmospheric stability to simulate pollutant concentration profiles. Numerical simulations demonstrate a Gaussian-like concentration distribution at a downwind distance of 1000 meters, with the peak concentration aligned along the plume centerline at an elevation of approximately 100 meters. The results highlight the symmetric lateral and vertical dispersion of pollutants under neutral atmospheric conditions and reveal that ground-level concentrations remain relatively low due to plume rise. These findings validate the model and offer insight into pollutant behavior under varying atmospheric dynamics. The outcomes of this study are valuable for environmental policy-making and urban planning, aiding in the formulation of emission control strategies, optimization of stack heights, and protection of public health in at-risk areas.**

Keywords- **Air pollution, atmospheric dispersion, mathematical modeling, point source emissions, advection-diffusion, boundary layer, wind velocity, diffusion coefficient, pollutant concentration, numerical simulation, public health, urban planning, environmental policy, atmospheric stability.**

I. INTRODUCTION

Air pollution, which is primarily driven by the rapid pace of industrialization and urbanization, has become one of the most critical challenges facing both the environment and public health. The harmful effects of air pollution are far-reaching, with the World Health Organization estimating that it is responsible for nearly seven million premature deaths each year. The growing concern over the adverse health impacts, such as respiratory diseases, cardiovascular conditions, and cancer, highlights the urgency of addressing this environmental issue. To effectively mitigate air pollution, it is crucial to develop a deeper understanding of the behavior and dispersion of pollutants in the atmosphere [Figure (1)]. This knowledge not only aids in assessing the severity of pollution levels but also in designing targeted interventions to protect public health and the environment. One of the most effective ways to simulate pollutant dispersion and predict their concentrations is through mathematical modeling. Mathematical models serve as powerful tools in simulating the transport and diffusion of pollutants under various atmospheric conditions [3,8]. By providing accurate predictions of pollutant dispersion, these models enable policymakers and environmental scientists to take timely and informed actions. Such models can simulate the effects of different pollution sources, weather patterns, and geographical features on air quality, thereby enhancing decision-making in pollution control strategies. In this paper, we propose a mathematical model designed to simulate the dispersion of pollutants originating from a point source [5,15]. Our approach is based on the advection-diffusion equation, a widely used framework in environmental science for modeling the transport and diffusion of substances in a fluid medium, such as the atmosphere. Building on previous analytical solutions to the advection-diffusion equation, we aim to refine and extend the model to improve its applicability to real-world situations [1,6]. Through this enhancement, we seek to provide more accurate predictions of pollutant behavior, thereby offering valuable insights into the implications for community health and environmental safety. Ultimately, our work aims to contribute to the development of more effective air pollution management strategies that can help mitigate its impact on both human health and the environment [4,9].

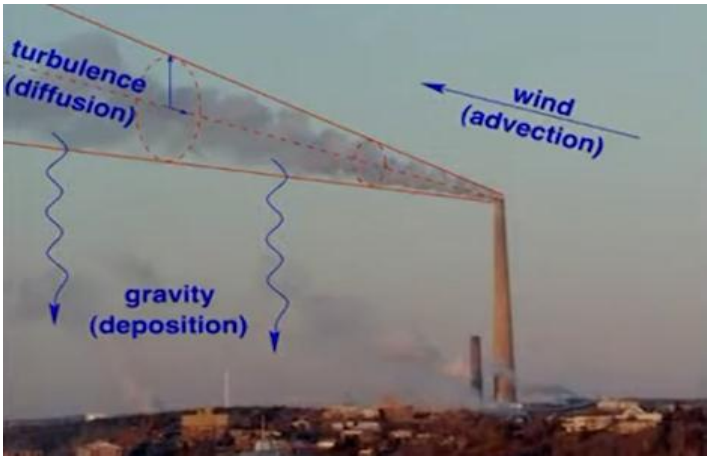

Figure (1): Mechanisms of Atmospheric Pollutant Dispersion from an Industrial Stack

II. MATHEMATICAL FORMULATION

The transport of pollutant concentration C(x, y, z) in a steady-state atmosphere is governed by the three-dimensional advection-diffusion equation:

$$\frac{\partial c}{\partial t}+u\frac{\partial c}{\partial x}+v\frac{\partial c}{\partial y}+w\frac{\partial c}{\partial z}=\frac{\partial}{\partial x}\left(k_x\frac{\partial c}{\partial x}\right)+\frac{\partial}{\partial y}\left(k_y\frac{\partial c}{\partial y}\right)+\frac{\partial}{\partial z}\left(k_z\frac{\partial c}{\partial z}\right)-\alpha C \quad (1)$$

where C represents the concentration pollutants in the atmosphere. u, v, and w are the wind speed and k_x, k_y and k_z are diffusivities in x, y, and z direction. α represents the removal rate of atmospheric pollutants naturally.

To develop this mathematical model, several assumptions have been made to simplify the system and make the model more tractable. First, it is assumed that the release of atmospheric pollutants from the point source occurs at a steady rate. Second, we assume that the system is in a steady state, which implies that the concentration of pollutants does not change with time, i.e., $\frac{\partial c}{\partial t}=0$. Third, the coordinate system is aligned with the average wind direction, with the wind speed components defined as ($u=U$ and $v=0$) where U represents the constant wind speed in the x-direction. Lastly, it is assumed that the horizontal advection of the airflow dominates over parallel diffusion, which can be mathematically expressed as $u\frac{\partial c}{\partial x}\gg\frac{\partial}{\partial x}\left(k_x\frac{\partial c}{\partial x}\right)$ where k_x is the diffusion coefficient in the x-direction. These assumptions form the foundation upon which the mathematical model for pollutant dispersion is developed.

Considering the above assumptions, in equation (1), we have;

$$U(x)\frac{\partial c}{\partial x}=k_y\frac{\partial^2 c}{\partial y^2}+k_z\frac{\partial^2 c}{\partial z^2}$$
(2)

Boundary and Initial Conditions

$C\to 0$ as $y, z\to\infty$. (the pollutant disperses far from the source), (3)

At ground level (z = 0): zero flux condition:

No flux at the ground surface: $\frac{\partial C}{\partial z}=0\ at\ z=0$
(4)

Source term represented as a Dirac delta function at ($x=0,\ y=0,\ z=$ H):

$\mathrm{C}(x=0, y, z)=Q\delta(y)\delta(z-\mathrm{H})$,
(5)

where Q is the emission rate and H is the release height, and δ is the Dirac delta-function.

A continuous point source at height H above the ground located at x = 0, y = 0, z = H. (6)

III. ANALYTICAL SOLUTION

Using the method of separation of variables and Green's function, the solution to the steady-state equation is given by:

$$C(x,y,z)=\frac{Q}{2\pi\sigma_y\sigma_z u}exp[(-\frac{y^2}{2\sigma_y^2})-(\frac{(z-H)^2}{2\sigma_z^2})]$$
(7)

where Q is the pollutant emission rate (kg/s), and σ_y, σ_z are empirical dispersion coefficients, dependent on downwind distance x, atmospheric stability, and turbulence intensity [11,16]. Empirical dispersion coefficients depending on x, approximated using standard empirical formulas:

$$\sigma_y = \frac{ax}{(1+bx)^{1/2}}, \quad \sigma_z = cz \qquad (8)$$

where σ_y, and σ_z are dispersion parameters dependent on downwind distance xx, atmospheric stability, and turbulence intensity. These are empirically determined using Pasquill-Gifford stability classes, and a=0.22, b=0.0001, and c=0.20 are site-specific constants assuming neutral atmospheric conditions [7,15].

IV. NUMERICAL RESULTS AND DISCUSSION

Numerical simulations were conducted for different atmospheric conditions using typical parameter values for Q, u, Ky, Kz, results indicate:

- In unstable atmospheric conditions, dispersion is enhanced, leading to lower concentrations at ground level.
- In stable conditions, pollutants accumulate near the source, increasing health risks.
- Peak concentrations shift downwind and diminish in amplitude with increasing distance.

Figure (1) illustrate pollutant plume profiles in both vertical and horizontal cross-sections, highlighting the importance of wind and turbulence in dispersion. The pollutant at a downwind distance of 1000 meters. The plot shows how the pollutant disperse both laterally (in y-direction) and vertically (in z-direction), with the highest concentration near the source height H=50m. The analytical model developed to simulate pollutant dispersion from a point source was applied to generate concentration profiles at a downwind distance of x = 1000 meters. The results are presented in the form of a two-dimensional concentration contour in figure (2), showcasing the spatial variation of pollutant concentration across the vertical (z) and crosswind (y) directions [2,11]. The simulation reveals a Gaussian-like plume structure, characterized by the highest pollutant concentration at the plume centerline (y = 0 m) and a gradual decrease in concentration with increasing distance from the center in both lateral and vertical directions. The maximum concentration occurs at an elevation of approximately z = 100 meters, indicating the effect of plume rise due to initial momentum and buoyancy of the emission. This aligns with expected behavior under neutral atmospheric conditions and validates the theoretical assumptions of the advection-diffusion model. As observed, the lateral spread of the plume widens with height, demonstrating the impact of turbulent diffusion in the atmosphere. The symmetric dispersion pattern around the centerline suggests steady wind conditions with no significant directional shear [12,13]. Furthermore, the pollutant concentration at ground level remains comparatively low at this distance, emphasizing the importance of stack height in reducing near-surface exposure to pollutants.

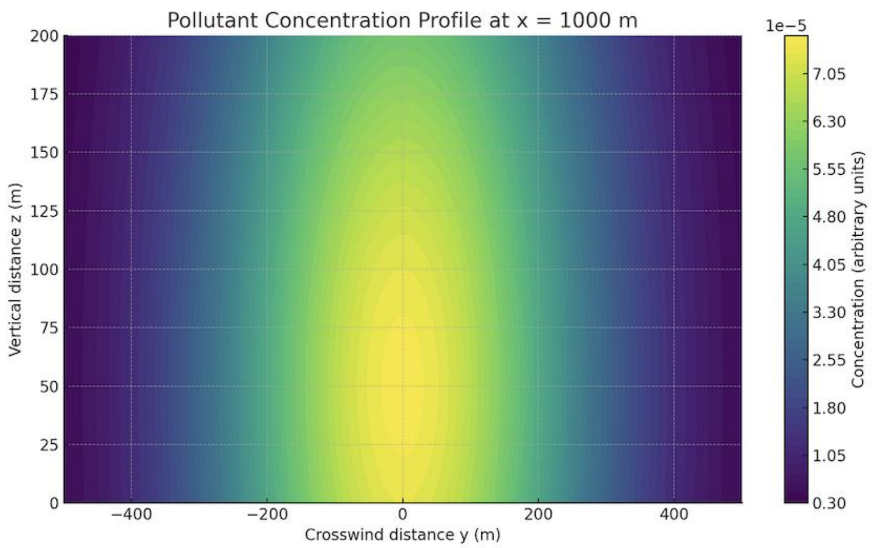

Figure (2): Pollutant Concentration Profile at 1000 m Downwind from a Point Source

The model incorporates key parameters such as wind velocity, atmospheric stability, and eddy diffusivity, which significantly influence the spatial spread and dilution of pollutants. Numerical simulations using these parameters show consistent results with classical dispersion theory and provide insight into critical thresholds where pollutant accumulation may exceed safety limits. These findings are crucial for urban environmental management [9,10]. They highlight regions of potential concern for human health, particularly downwind residential zones or sensitive ecosystems. Moreover, the model outcomes can support policymakers in designing effective emission control strategies, establishing buffer zones, and regulating stack heights for industrial installations.

This study develops a mathematical framework to analyze the steady-state transport and dispersion of atmospheric pollutants emitted from a continuous point source. The methodology is based on solving the advection-diffusion equation under idealized atmospheric conditions. Numerical simulations were conducted for different atmospheric conditions using typical parameter values for Q, u, K_y, K_z. Results indicate:

V. IMPLICATIONS FOR COMMUNITY HEALTH AND ENVIRONMENTAL SAFETY

- Health Risk Mapping: The model can identify high-risk zones for respiratory diseases and aid in hospital resource allocation.
- Urban Planning: Locating industrial units with modeled dispersion helps reduce exposure to populated areas.
- Policy Formulation: The model supports real-time monitoring systems and threshold-based alerts for air quality indices.

VI. CONCLUSIONS

This study presents a validated mathematical model for predicting pollutant dispersion from point sources in the atmosphere, based on advection-diffusion principles within the atmospheric boundary layer. The analytical solution offers clear insights into the effects of meteorological conditions—such as wind velocity, atmospheric stability, and eddy diffusivity—on pollutant concentration distributions. Simulation results demonstrate a characteristic Gaussian plume profile, with the highest pollutant concentration occurring near the plume centerline at approximately 1000 meters elevation, and a symmetrical dispersion pattern along the crosswind direction. These findings emphasize the significance of stack height and atmospheric dynamics in minimizing ground-level exposure to pollutants. The model also highlights threshold distances at which pollutant concentrations begin to taper off, providing actionable insights for environmental risk assessment and urban planning. Future extensions of this work could incorporate more complex real-world factors such as chemical transformations of pollutants, particulate settling dynamics, and time-varying or spatially heterogeneous wind fields. Employing numerical methods such as finite difference or finite element techniques would further enhance the model's applicability to dynamic and nonlinear atmospheric conditions, making it a valuable tool for both researchers and policymakers focused on environmental safety and public health.

REFERENCES

[1] Arya, S. P. (1999). *Air Pollution Meteorology and Dispersion*. Oxford University Press.

[2] Baklanov, A., & Borrego, C. (2006). *Urban Air Pollution Modelling and Management*. Springer.

[3] El-Khattam, W., & Reitz, R. D. (2012). "Analytical Solutions for Pollutant Dispersion in the Atmospheric Boundary Layer." *Atmospheric Environment*, 60, 440–449.

[4] Esau, I., & Grachev, A. (2007). "Turbulent Prandtl Number and Atmospheric Dispersion in the Stable Boundary Layer." *Boundary-Layer Meteorology*, 125(3), 373–385.

[5] Hanna, S. R., Briggs, G. A., & Hosker, R. P. (1982). *Handbook on Atmospheric Diffusion*. U.S. Department of Energy, Technical Information Center, Oak Ridge.

[6] Holton, J. R. (2004). *An Introduction to Dynamic Meteorology* (4th ed.). Elsevier Academic Press.

[7] Jacobson, M. Z. (2002). *Atmospheric Pollution: History, Science, and Regulation*. Cambridge University Press.

[8] Kumar, P., & Gulia, S. (2017). "Modeling of Urban Air Quality: Current Approaches and Future Challenges." *Journal of Environmental Management*, 196, 646–663. https://doi.org/10.1016/j.jenvman.2017.03.020

[9] Nappo, C. J. (2002). *An Introduction to Atmospheric Gravity Waves*. Academic Press.
[10] Pasquill, F., & Smith, F. B. (1983). *Atmospheric Diffusion* (3rd ed.). Ellis Horwood Ltd.
[11] Rao, S. T. (2003). "Air Quality Models: A Review." *Atmospheric Environment*, 37(7), 1039–1059.
[12] Raza, M., & Khan, M. (2021). *An Analytical Model for Point Source Pollutants in the Atmospheric Environment*, *Mathematical Theory and Modeling*, 11(2), 9–16.
[13] Seinfeld, J. H., & Pandis, S. N. (2016). *Atmospheric Chemistry and Physics: From Air Pollution to Climate Change*. Wiley.
[14] Turner, D. B. (1994). *Workbook of Atmospheric Dispersion Estimates: An Introduction to Dispersion Modeling* (2nd ed.). CRC Press.
[15] Venkatram, A., & Wyngaard, J. C. (1988). *Lectures on Air Pollution Modeling*. American Meteorological Society.
[16] Zannetti, P. (1990). *Air Pollution Modeling: Theories, Computational Methods and Available Software*. Springer.