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Testing Function Spaces and Operation Transform Formulae for Sumudu-Laplace Transform

A. N. Rangari[1], R. S. Pawar[2] and V. D. Sharma [3]
[1]*Department of Mathematics, Adarsh College, Dhamangaon Rly.- 444709 (M.S), India.*
[2]*Department of Mathematics, SVKM's Dwarkadas J. Sanghavi College of Engineering, Mumbai, India.*
[3]*Department of Mathematics, Arts, Commerce and Science College, Amravati- 444606(M.S), India.*

Abstract: Integral transforms are a critical class of mathematical techniques used to simplify and solve complex problems in a wide range of engineering and scientific disciplines. The *Laplace Transform****, which converts differential equations into algebraic equations in the complex frequency domain, is especially useful for solving*** *initial value problems* ***(IVPs). It is widely used in*** *control theory****,*** *signal processing****,*** *electronics****, and*** *mechanical systems****, where systems are modeled by ordinary differential equations (ODEs). This transform is particularly effective for analyzing the behavior of dynamic systems, such as electrical circuits, vibrations, and systems governed by transient conditions.***

The *Sumudu Transform* ***is an emerging alternative that, like the Laplace transform, can convert complex differential equations into simpler forms for easier solving. However, it offers distinct advantages in certain problem scenarios: Effective for Boundary Value Problems, Applications in Control Theory and Electrical Engineering,*** *Efficiency in Numerical solutions****.***

The present paper gives the various testing function spaces which are the backbone of distribution theory. And by using these testing function spaces we have described the operation transform formulae of Sumudu-Laplace transform which will be useful for solving various differential and integral equations.

Keyword: Differential operator, Generalized function, Laplace Transform, shifting & Scaling Property, Sumudu Transform, Sumudu-Laplace Transform, Testing function space,

I. INTRODUCTION

Sumudu and Laplace transform showed their remarkable impact in the field of sciences and engineering. Sumudu transform has its application in signal processing, heat transfer, Fluid dynamics [1]. Vashi J. and M G Timol had discussed the application of Sumudu and Laplace transform in the area of physics followed by the application to electric circuit [14]. Osman M. and Bashir M.A presented solution of partial differential equation with variable coefficients using double Sumudu transform [4]. Watugala, G. presented applied Sumudu transform to solve differential equations and control engineering problems [5].

Inderdeep Singh, Nizamul
Haque Ansari, Gurpreet Singh had applied Sumudu Transform technique for solving partial differential equation arising in liquid drop pattern [6]. Laplace Transform already proved its phenomenal application in various field such as mathematical sciences, physics, Engineering problems, as this transform is effectively used to solve ordinary differential equations, Partial differential equation, integro-differential equation and many more.

In our previous papers we have introduced a combination of Sumudu and Laplace transform and as Combining Sumudu and Laplace transform is a great tool for solving differential equation, fractional differential equation in various field, serving Computer friendly approach while studying Viscoelasticity, diffusion and more.

In the present paper, Various testing function are described in section 2. In section 3, definition of distributional Sumudu-Laplace Transform is defined. Operators on the space $SL^{\beta}_{a,c,\alpha}$ is defined and Shifting and Scaling property for Sumudu Transform are proved in the section 4, Shifting and scaling property for Laplace transform is proved in section 5, Some theorems on Differential operator are proved in section 6, Some results on differential operator are proved in section 7, Lastly Conclusion are given in Section 8.

Notations and terminologies are as per Zemanian [11], [12].

II TESTING FUNCTION SPACES FOR SUMUDU-LAPLACE TRANSFORM:

2.1. THE SPACE $SL_{a,c,\alpha}$

Let I be the open set in $R_+ \times R_+$ *and* E_+ denotes the class of infinitely differentiable function defined on I, the space $SL_{a,c,\alpha}$ is given by ,

$$SL_{a,c,\alpha} = \Big\{\phi : \phi \in E_+ \,/\, \gamma_{a,c,q,l}\ \phi(x,z) = \sup_{\substack{0<x<\infty \\ 0<z<\infty}} \left|K_{a,b}(x) K_{c,d}(z) D_x^l\ D_z^q\ \phi(x,z)\right| \le C_{lq} A^a a^{a\alpha} \Big\}$$

Where the constants A and C_{lq} depend on the testing function ϕ .

Also, where $K_{a,b}(x) = \begin{cases} e^{ax}, & 0 \le x < \infty \\ e^{bx}, & -\infty < x < 0 \end{cases}$ and $K_{c,d}(z) = \begin{cases} e^{cz}, & 0 \le x < \infty \\ e^{dz}, & -\infty < x < 0 \end{cases}$ are the kernels for testing function space of Sumudu and Laplace Transform respectively.

2.2. THE SPACE $SL_{a,c}^{\beta}$

The Space $SL_{a,c}^{\beta}$ is given by,

$$SL_{a,c}^{\beta} = \Big\{\phi : \phi \in E_+ \,/\, \sigma_{a,c,q,l}\ \phi(x,z) = \sup_{\substack{0<x<\infty \\ 0<z<\infty}} \left|K_{a,b}(x) K_{c,d}(z) D_x^l D_z^q\ \phi(x,z)\right| \le C_{aq}\ B^l l^{l\beta} \Big\}$$

Where the constants B and C_{aq} depend on the testing function ϕ .

2.3. THE SPACE $SL_{a,c,\alpha}^{\beta}$

The Space $SL_{a,c,\alpha}^{\beta}$ is given by,

$$SL_{a,c,\alpha}^{\beta} = \Big\{\phi : \phi \in E_+ \,/\, \rho_{a,c,q,l}\ \phi(x,z) = \sup_{\substack{0<x<\infty \\ 0<z<\infty}} \left|K_{a,b}(x) K_{c,d}(z) D_x^l D_z^q\ \phi(x,z)\right| \le CA^a a^{a\alpha} B^l l^{l\beta} \Big\}$$

2.4. THE SPACE $SL_{a,c,\gamma}$

The space $SL_{a,c,\gamma}$ is given by,

$$SL_{a,c,\gamma} = \Big\{\phi : \phi \in E_+ \,/\, \xi_{a,c,q,l}\ \phi(x,z) = \sup_{\substack{0<x<\infty \\ 0<z<\infty}} \left|K_{a,b}(x) K_{c,d}(z) D_x^l\ D_z^q\ \phi(x,z)\right| \le C_{al}\ A^q q^{q\gamma} \Big\}$$

Following the order of the above spaces, we have now defined the subspaces of each of the above spaces, which are used for defining the strict inductive limits of these spaces.

2.5. THE SPACE $SL_{a,c,\alpha,m}$

The space $SL_{a,c,\alpha,m}$ is given by,

$$SL_{a,c,\alpha,m} = \Big\{\phi : \phi \in E_+ \,/\, \gamma_{a,c,q,l}\ \phi(x,z) = \sup_{\substack{0<x<\infty \\ 0<z<\infty}} \left|K_{a,b}(x) K_{c,d}(z) D_x^l\ D_z^q\ \phi(x,z)\right| \le C_{lq\delta}(m+\delta)^a a^{a\alpha} \Big\}$$

for any $\delta > 0$, where 'm' is the constant depending on the function ϕ .

2.6. THE SPACE $SL_{a,c}^{\beta,n}$

The Space $SL_{a,c}^{\beta,n}$ is given by

$$SL_{a,c}^{\beta,n} = \Big\{\phi : \phi \in E_+ \,/\, \sigma_{a,c,q,l}\ \phi(x,z) = \sup_{\substack{0<x<\infty \\ 0<z<\infty}} \left|K_{a,b}(x) K_{c,d}(z) D_x^l D_z^q\ \phi(x,z)\right| \le C_{aq\varepsilon}(n+\varepsilon)^l l^{l\beta} \Big\}$$

for any $\varepsilon > 0$, where 'n' is the constant depending on the function ϕ .

2.7. THE SPACE $SL_{a,c,\alpha,m}^{\beta,n}$

The Space is defined by combining the conditions 2.5 and 2.6 as,

$$SL_{a,c,\alpha,m}^{\beta,n} = \Big\{\phi : \phi \in E_+ \,/\, \rho_{a,c,q,l}\ \phi(x,z) = \sup_{\substack{0<x<\infty \\ 0<z<\infty}} \left|K_{a,b}(x) K_{c,d}(z) D_x^l D_z^q\ \phi(x,z)\right| \le C_{\delta\varepsilon}(m+\delta)^a (n+\varepsilon)^l a^{a\alpha} l^{l\beta} \Big\}$$

for any $\delta > 0$, $\varepsilon > 0$, and for given $m > 0$ and $n > 0$.

2.8. THE SPACE $SL_{a,c,\gamma,p}$

The space $SL_{a,c,\gamma,p}$ is given by,

$$SL_{a,c,\gamma,p} = \Big\{\phi : \phi \in E_+ \,/\, \xi_{a,c,q,l}\ \phi(x,z) = \sup_{\substack{0<x<\infty \\ 0<z<\infty}} \left|K_{a,b}(x) K_{c,d}(z) D_x^l D_z^q\ \phi(x,z)\right| \le C_{alr}\,(p+r)^q q^{q\gamma}\Big\}$$

for any $r > 0$, where 'p' is the constant depending on the function ϕ.

Analogous to the spaces 2.1, 2.2. 2.3, and 2.4 we have defined the spaces of the functions which have domain I_2. These spaces are called negative spaces.

2.9. THE SPACE $S^{\upsilon}L_{a,c,\alpha}$

The Space $S^{\upsilon}L_{a,c,\alpha}$ is given by

$$SL^{\upsilon}_{a,c,\alpha} = \Big\{\phi : \phi \in E_- \,/\, i_{a,c,q,l}\ \phi(x,z) = \sup_{\substack{-\infty<x<0 \\ 0<z<\infty}} \left|K_{a,b}(-x) K_{c,d}(z) D_x^l D_z^q\ \phi(x,z)\right| \le C_{lq} A^a a^{a\alpha}\Big\}$$

The smooth function $\phi(x,z)$ defined on I_2 is in $S^{\upsilon}L_{a,c,\alpha}$, if $\phi^{\upsilon}(x,z) = \phi(-x,z)$ is in $SL_{a,c,\alpha}$.

2.10. THE SPACE $S^{\upsilon}L^{\beta}_{a,c}$

The Space $S^{\upsilon}L^{\beta}_{a,c}$ is given by,

$$S^{\upsilon}L^{\beta}_{a,c} = \Big\{\phi : \phi \in E_- \,/\, j_{a,c,q,l}\ \phi(x,z) = \sup_{\substack{-\infty<x<0 \\ 0<z<\infty}} \left|K_{a,b}(-x) K_{c,d}(z) D_x^l D_z^q\ \phi(x,z)\right| \le C_{aq} B^l l^{l\beta}\Big\}$$

2.11. THE SPACE $S^{\upsilon}L^{\beta}_{a,c,\alpha}$

Combining the conditions of 2.9 and 2.10 we get the space

$$S^{\upsilon}L^{\beta}_{a,c,\alpha} = \Big\{\phi : \phi \in E_- \,/\, \mu_{a,c,q,l}\ \phi(x,z) = \sup_{\substack{-\infty<x<0 \\ 0<z<\infty}} \left|K_{a,b}(-x) K_{c,d}(z) D_x^l D_z^q\ \phi(x,z)\right| \le C\, A^a a^{a\alpha} B^l l^{l\beta}\Big\}$$

Where the constants A, B, C depend on the testing function ϕ.

2.12. THE SPACE $S^{\upsilon}L_{a,c,\gamma}$

The Space $S^{\upsilon}L_{a,c,\gamma}$ is given by

$$S^{\upsilon}L_{a,c,\gamma} = \Big\{\phi : \phi \in E_- \,/\, \lambda_{a,c,q,l}\ \phi(x,z) = \sup_{\substack{0<x<\infty \\ -\infty<z<0}} \left|K_{a,b}(x) K_{c,d}(-z) D_x^l D_z^q\ \phi(x,z)\right| \le C_{al}\, A^q q^{q\gamma}\Big\}$$

Unless specified otherwise, the spaces introduced in 2.1 through 2.12 will henceforth be considered equipped with their natural Hausdorff locally convex topologies to be denoted respectively by

$T_{a,c,\alpha}$, $T^{\beta}_{a,c}$, $T^{\beta}_{a,c,\alpha}$, $T_{a,c,\gamma}$, $T_{a,c,\alpha,m}$, $T^{\beta,n}_{a,c}$, $T^{\beta,n}_{a,c,\alpha,m}$, $T_{a,c,\gamma,p}$, $T^{\upsilon}_{a,c,\alpha}$,

$T^{\upsilon,\beta}_{a,c}$, $T^{\upsilon,\beta}_{a,c,\alpha}$ and $T^{\upsilon}_{a,c,\gamma}$.

These topologies are respectively generated by the total families of semi norms.

$\{\gamma_{a,c,q,l}\}$, $\{\sigma_{a,c,q,l}\}$, $\{\rho_{a,c,q,l}\}$, $\{\xi_{a,c,q,l}\}$, $\{\gamma_{a,c,q,l}\}$, $\{\sigma_{a,c,q,l}\}$, $\{\rho_{a,c,q,l}\}$, $\{\xi_{a,c,q,l}\}$, $\{i_{a,c,q,l}\}$, $\{j_{a,c,q,l}\}$, $\{\mu_{a,c,q,l}\}$ and $\{\lambda_{a,c,q,l}\}$

III DISTRIBUTIONAL GENERALIZED SUMUDU-LAPLACE TRANSFORM (SLT)

For $f(x,z) \in SL^{*\beta}_{a,c,\alpha}$, where $SL^{*\beta}_{a,c,\alpha}$ is the dual space of $FL^{\beta}_{a,c,\alpha}$. It contains all distributions of compact support. The distributional Sumudu-Laplace transform is a function of $f(x,z)$ is defined as,

$$SL\{f(x,z)\} = F(k,s) = \langle f(x,z), \phi(x,z,k,s)\rangle \quad (3.1)$$

where $\phi(x,z,k,s) = \frac{1}{k} e^{-\left(\frac{x}{k}+sz\right)}$ and for each fixed $x(0<x<\infty)$, $z(0<z<\infty)$. Also $k>0$ $s>0$. The right-hand side of (3.1) has a sense as an application of $f(x,z) \in SL^{*\beta}_{a,c,\alpha}$ to $\phi(x,z,k,s) \in SL^{\beta}_{a,c,\alpha}$.

IV OPERATORS ON THE SPACE $SL^{\beta}_{a,c,\alpha}$

4.1 Proposition: (Shifting Property for Sumudu Transform)

If $\phi(x,z) \in SL^{\beta}_{a,c,\alpha}$ and ξ is any fixed real number then $\phi(x+\xi,z) \in SL^{\beta}_{a,c,\alpha}$, $x+\xi>0$ and $\phi(x+\xi,z) \in S^{\upsilon}L^{\beta}_{a,c,\alpha}$, $x+\xi<0$.

Proof: Consider,

$$\rho_{a,c,q,l}\ \phi(x+\xi,z) =$$

$$\sup_I \left|K_{a,b}(x)K_{c,d}(z)D_x^l D_z^q\ \phi(x+\xi,z)\right|$$

$$= \sup_I \left| e^{ax}\ e^{cz}\ D_x^l\ D_z^q\ \phi(x+\xi,z)\right|$$

$$= \sup_I \left| e^{a(x'-\xi)}\ e^{cz}\ D_{x'}^l\ D_z^q\ \phi(x',z)\right|$$

where $x+\xi = x' \Rightarrow x = x'-\xi$

$$\leq CA^a a^{a\alpha} B^l l^{l\beta}$$

Thus, $\phi(x+\xi,z) \in SL^{\beta}_{a,c,\alpha}$, *for* $x+\xi>0$

Similarly, it can be show that $\phi(x+\xi,z) \in S^{v}L^{\beta}_{a,c,\alpha}$, $x+\xi<0$.

4.2 Proposition:

The Translation (Shifting) operator $\xi : \phi(x,z) \to \phi(x+\xi,z)$ is a topological automorphism on $SL^{\beta}_{a,c,\alpha}$ *for* $x+\xi>0$ and it is a topological isomorphism from $SL^{\beta}_{a,c,\alpha}$ *onto* $S^{v}L^{\beta}_{a,c,\alpha}$ *for* $x+\xi<0$.

4.3 Proposition: (Scaling property for Sumudu Transform)

If $\phi(x,z) \in SL^{\beta}_{a,c,\alpha}$ and $p>0$, strictly positive number then $\phi(px,z) \in SL^{\beta}_{a,c,\alpha}$.

Proof: Consider

$$\rho_{a,c,q,l}\ \phi(px,z) = \sup_I \left|K_{a,b}(x)K_{c,d}(z)D_x^l D_z^q\ \phi(px,z)\right|$$

$$= \sup_I \left| e^{ax}\ e^{cz}\ D_x^l\ D_z^q\ \phi(px,z)\right|$$

Put $px = X \ \therefore x = \frac{X}{p}$

$$= \sup_I \left| e^{\frac{a}{p}X}\ e^{cz}\ D_X^l\ D_z^q\ \phi(X,z)\right|$$

$$= M \sup_I \left| e^{aX}\ e^{cz}\ D_x^l\ D_z^q\ \phi(X,z)\right| ,$$

where M is a constant depending on p.

$$\leq M\ C\ A^a a^{a\alpha} B^l l^{l\beta}$$

$$\leq C'\ A^a a^{a\alpha} B^l l^{l\beta}, \text{where } C' = MC$$

Thus, $\phi(px,z) \in SL^{\beta}_{a,c,\alpha}$, $p>0$

4.4 Proposition:

If $\phi(x,z) \in SL^{\beta}_{a,c,\alpha}$ *and* $p>0$ then the scaling operator $R : SL^{\beta}_{a,c,\alpha} \to SL^{\beta}_{a,c,\alpha}$ defined by $R\phi = \psi$, *where* $\psi(x,z) = \phi(px,z)$ is a topological auto morphism.

V. SHIFTING AND SCALING PROPERTY FOR LAPLACE TRANSFORM.

5.1. Proposition: (Shifting property for Laplace Transform)

If $\phi(x,z) \in SL^{\beta}_{a,c,\alpha}$ and η is any fixed real number then $\phi(x,z+\eta) \in SL^{\beta}_{a,c,\alpha}$, $z+\eta>0$ and $\phi(x,z+\eta) \in S^{v}L^{\beta}_{a,c,\alpha}$, $z+\eta<0$.

Proof: Consider,

$$\rho_{a,c,q,l}\ \phi(x,z+\eta) =$$

$$\sup_I \left| K_{a,b}(x) K_{c,d}(z) D_x^l D_z^q \; \phi(x, z+\eta) \right|$$

$$= \sup_I \left| \; e^{ax} \; e^{cz} \; D_x^l \; D_z^q \; \phi(x, z+\eta) \right|$$

$$= \sup_I \left| \; e^{ax} \; e^{c(z'-\eta)} \; D_x^l \; D_{z'}^q \; \phi(x, z') \right|$$

where $z+\eta = z' \quad \therefore \; z = z' - \eta$

$$\le CA^a a^{a\alpha} B^l l^{l\beta}$$

Thus, $\phi(x, z+\eta) \in SL_{a,c,\alpha}^{\beta}$, *for* $z+\eta > 0$

Similarly, it can be show that $\phi(x, z+\eta) \in S^v L_{a,c,\alpha}^{\beta}$, $z+\eta < 0$.

5.2 Proposition: The Translation (Shifting) operator $\xi : \phi(x, z) \to \phi(x, z+\eta)$ is a topological automorphism on $SL_{a,c,\alpha}^{\beta}$ *for* $z+\eta > 0$ and it is a topological isomorphism from $SL_{a,c,\alpha}^{\beta}$ *onto* $S^v L_{a,c,\alpha}^{\beta}$ *for* $z+\eta < 0$.

5.3 Proposition: (Scaling property for Laplace Transform)

If $\phi(x, z) \in SL_{a,c,\alpha}^{\beta}$ and $p > 0$, strictly positive number then $\phi(x, pz) \in SL_{a,c,\alpha}^{\beta}$.

Proof: Consider

$$\rho_{a,c,q,l} \; \phi(x, pz) =$$

$$\sup_I \left| K_{a,b}(x) K_{c,d}(z) D_x^l D_z^q \; \phi(x, pz) \right|$$

$$= \sup_I \left| \; e^{ax} \; e^{cz} \; D_x^l \; D_z^q \; \phi(x, pz) \right|$$

Put $pz = Z \; \therefore z = \dfrac{Z}{p}$

$$= \sup_I \left| \; e^{ax} \; e^{\frac{c}{p}Z} \; D_x^l \; D_Z^q \; \phi(x, Z) \right|$$

$$= N \sup_I \left| \; e^{ax} \; e^{cZ} \; D_x^l \; D_Z^q \; \phi(x, Z) \right| ,$$

where N is a constant depending on p.

$$\le N \, C \, A^a a^{a\alpha} B^l l^{l\beta}$$

$$\le C' \, A^a a^{a\alpha} B^l l^{l\beta}, \textit{where } C' = NC$$

Thus, $\phi(x, pz) \in SL_{a,c,\alpha}^{\beta}$, $p > 0$.

5.4 Proposition:

If $\phi(x, z) \in SL_{a,c,\alpha}^{\beta}$ *and* $p > 0$ then the scaling operator $R : SL_{a,c,\alpha}^{\beta} \to SL_{a,c,\alpha}^{\beta}$ defined by $R\phi = \psi$, *where* $\psi(x,z) = \phi(x, pz)$ is a topological auto morphism.

5.5 Proposition:

If $\phi(x, z) \in SL_{a,c,\alpha}^{\beta}$ and ξ and η are any fixed real number then

$$\phi(x+\xi, z+\eta) \in SL_{a,c,\alpha}^{\beta}, \; x+\xi > 0 \; \textit{and} \; z+\eta > 0$$

also

$$\phi(x+\xi, z+\eta) \in S^v L_{a,c,\alpha}^{\beta}, \; x+\xi < 0 \; \textit{and} \; z+\eta < 0.$$

Proof:- Proof is simple, hence omitted.

5.6 Proposition:

If $\phi(x, z) \in SL_{a,c,\alpha}^{\beta}$ and $p > 0$, strictly positive number then $\phi(px, qz) \in SL_{a,c,\alpha}^{\beta}$.

Proof:- Proof is simple, hence omitted.

VI. DIFFERENTIAL OPERATOR S-TYPE

6.1 Theorem:-

The operator $\phi(x, z) \to D_x \, \phi(x, z)$ is defined on the space $SL_{a,c,\alpha}^{\beta}$ and the transforms this space into itself.

Proof: Let $\phi(x, z) \in SL_{a,c,\alpha}^{\beta}$.

If $D_x \, \phi(x, z) = \psi(x,z)$, we have

$$\rho_{a,c,q,l} \; \psi(x, z) =$$

$$\sup_I \left| K_{a,b}(x) K_{c,d}(z) D_x^l D_z^q \; \psi(x, z) \right|$$

$$= \sup_I \left| \; e^{ax} \; e^{cz} \; D_x^l \; D_z^q \; D_x \phi(x,z) \right|$$

$$= \sup_I \left| \; e^{ax} \; e^{cz} \; D_x^{l+1} \; D_z^q \; \phi(x,z) \right|$$

$\leq C\, A^a a^{a\,\alpha} B^{l+1} (l+1)^{(l+1)\beta}$

$\psi(x,z) \in SL^{\beta}_{a,c,\alpha}$ i.e $D_x\, \phi(x,z) \in SL^{\beta}_{a,c,\alpha}$

Or
$\rho_{a,c,q,l}\left(D_x\phi(x,z)\right) = \rho_{a,c,q,l+1}\left(\phi(x,z)\right), a,c,q,l \in N_0$

6.2 Theorem

The operator $\phi(x,z) \to D_z\, \phi(x,z)$ is defined on the space $SL_{a,c,\gamma}$ and the transforms this space into itself.

Proof: Let $\phi(x,z) \in SL_{a,c,\gamma}$.

If $D_z\, \phi(x,z) = \psi(x,z)$

We have, $\xi_{a,c,q,l}\ \psi(x,z) =$

$$\sup_I \left|K_{a,b}(x) K_{c,d}(z) D_x^l D_z^q\ \psi(x,z)\right|$$
$$= \sup_I \left|\ e^{ax}\ e^{cz}\ D_x^l\ D_z^q\ D_z\phi(x,z)\right|$$
$$= \sup_I \left|\ e^{ax}\ e^{cz}\ D_x^l\ D_z^{q+1}\ \phi(x,z)\right|$$

$$\leq C_{al}\ A^{q+1} (q+1)^{(q+1)\gamma},$$

$a,q,l = 0,1,2,........$

Therefore, $\psi(x,z) \in SL_{a,c,\gamma}$ i.e
$D_z\, \phi(x,z) \in SL_{a,c,\gamma}$

$$\xi_{a,c,q,l}\ \ D_Z\, \phi(x,z) = \xi_{a,c,q+1,l}\ \ \phi(x,z)$$

Note that some operators can be defined for the space $SL_{a,c,\alpha}$.

6.3 Proposition:

The Differential operator S-type $S : \phi(x,z) \to D_x\, \phi(x,z)$ is a topological auto morphism on $SL^{\beta}_{a,c,\alpha}$

6.4 Proposition:

The Differential operator L-type $L : \phi(x,z) \to D_z\, \phi(x,z)$ is a topological auto morphism on $SL_{a,c,\gamma}$

VII. RESULTS ON DIFFERENTIAL OPERATOR D

7.1 Theorem:

For $m = (m_1, m_2)$ where $m_1,\ m_2 = 0,1,2.....$, If $\phi(x,z) \in SL^{\beta}_{a,c,\alpha}$, then $\psi(x,z) \in SL^{\beta}_{a,c,\alpha}$ where, $\psi(x,z) = D^m\, \phi(x,z)$. Further the mapping $\Delta = D^m\phi : \phi \to \psi$ is one-one, linear and continuous.

Proof: For $\psi(x,z) \in SL^{\beta}_{a,c,\alpha}$

$$\rho_{a,c,q,l}\ \ \psi(x,z) = \sup_I \left|K_{a,b}(x) K_{c,d}(z) D_x^l D_z^q\ \psi(x,z)\right|$$

$$= \sup_I \left|\ e^{ax}\ e^{cz}\ D_x^l\ D_z^q\ \psi(x,z)\right|$$

$$= \sup_I \left|\ e^{ax}\ e^{cz}\ D_x^l\ D_z^q\ D^m\phi(x,z)\right| ,$$

where, $m = (m_1, m_2) = (1,1)$

$$= \sup_I \left|\ e^{ax}\ e^{cz}\ D_x^{l+1}\ D_z^{q+1}\ \phi(x,z)\right|$$

$$\leq C A^a a^{a\,\alpha} B^l l^{l\beta} \qquad (7.1.1)$$

Thus $\psi(x,z) \in SL^{\beta}_{a,c,\alpha}$ if
$\phi(x,z) \in SL^{\beta}_{a,c,\alpha}$.

It is obviously linear.

It is injective for, if $D^m\phi = 0$ then $\phi = C = Cons\tan t$.

If $C = 0$ then $\phi = 0$ and D is injective. But if $C \neq 0$ then

$$\sup_I \left|\ e^{ax}\ e^{cz}\ D_x^l\ D_z^q\ C\right| = \sup_I \left|\ e^{ax}\ e^{cz}\ C\right|,$$
for $l = q = 0$.

As the right-hand side is not bounded, we conclude that $\phi(x,z) \notin SL^{\beta}_{a,c,\alpha}$,

Which is a contradiction.

Hence 'C' must be zero and therefore $\phi = 0$.

For continuity we observe from equation (7.1.1) that,

$$\rho_{a,c,q,l}\ D^m\phi(x,z) \le M\ \rho_{a,c,q,l}\ \phi(x,z),$$

where M is some constant. Thus, the theorem is proved.

7.2 Theorem

For $\sigma \in R$ and $\phi(x,z) \in SL^{\beta}_{a,c-\sigma,\alpha}$, $E(\phi) = \psi(x,z) = e^{-\sigma z}\phi(x,z) \in SL^{\beta}_{a,c,\alpha}$.

Proof: Let $\phi(x,z) \in SL^{\beta}_{a,c-\sigma,\alpha}$

Consider, $\rho_{a,c,q,l}\ \psi(x,z) =$

$$\sup_I \left| K_{a,b}(x) K_{c,d}(z) D^l_x D^q_z\ \psi(x,z) \right|$$

$$= \sup_I \left| e^{ax}\ e^{cz}\ D^l_x\ D^q_z\ \psi(x,z) \right|$$

$$= \sup_I \left| e^{ax}\ e^{cz}\ D^l_x\ D^q_z\ e^{-\sigma z}\ \phi(x,z) \right|$$

$$= \sup_I \left| e^{ax}\ e^{(c-\sigma)z}\ D^l_x\ D^q_z\ \phi(x,z) \right|$$

$$\le CA^a a^{a\alpha} B^l l^{l\beta}$$

Thus $\psi(x,z) \in SL^{\beta}_{a,c,\alpha}$ if $\phi(x,z) \in SL^{\beta}_{a,c-\sigma,\alpha}$.

In the view of this Proposition, we have the exponential multiplier operator

$E : SL^{\beta}_{a,c-\sigma,\alpha} \to SL^{\beta}_{a,c,\alpha}$ is a topological isomorphism.

7.3 Theorem

For $\tau \in R$ and $\phi(x,z) \in SL^{\beta}_{a-\tau,c,\alpha}$, $E(\phi) = \psi(x,z) = e^{-\tau x}\phi(x,z) \in SL^{\beta}_{a,c,\alpha}$.

Proof: Let $\phi(x,z) \in SL^{\beta}_{a-\tau,c,\alpha}$,

Consider,

$$\rho_{a,c,q,l}\ \psi(x,z) = \sup_I \left| K_{a,b}(x) K_{c,d}(z) D^l_x D^q_z\ \psi(x,z) \right|$$

$$= \sup_I \left| e^{ax}\ e^{cz}\ D^l_x\ D^q_z\ \psi(x,z) \right|$$

$$= \sup_I \left| e^{ax}\ e^{cz}\ D^l_x\ D^q_z\ e^{-\tau x}\ \phi(x,z) \right|$$

$$= \sup_I \left| e^{(a-\tau)x}\ e^{cz}\ D^l_x\ D^q_z\ \phi(x,z) \right|$$

$$\le CA^a a^{a\alpha} B^l l^{l\beta}$$

Thus $\psi(x,z) \in SL^{\beta}_{a,c,\alpha}$ if $\phi(x,z) \in SL^{\beta}_{a-\tau,c,\alpha}$.

In the view of this Proposition, we have the exponential multiplier operator

$E : SL^{\beta}_{a-\tau,c,\alpha} \to SL^{\beta}_{a,c,\alpha}$ is a topological isomorphism.

VIII. CONCLUSIONS

Testing function space is the backbone of distribution theory. So, we described various testing function spaces. In this paper, the shifting and Scaling Property of Sumudu & Laplace transform are described, some results on differential operator are proved with the help of testing function space.

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