Pratham Prasad's Approach to Evaluate a Binoharmonic Series of Weight 5

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Abstract-This research explores a remarkable infinite series identity involving harmonic numbers, binomial coefficients, Riemann zeta functions, polylogarithmic functions, and logarithmic powers. The convergence and structure of such expressions reveal deep connections between discrete summation, analytic continuations, and special functions. Through rigorous derivation, transformation techniques, and analytic evaluation, the paper uncovers closed-form representations of complex series that traditionally resist simplification. The resulting identity not only enriches the theoretical understanding of nested series but also provides elegant pathways for future exploration in transcendental number theory, multiple zeta values (MZVs), and mathematical constants. Such results are instrumental in deepening the mathematical framework behind quantum field theory, computational number theory, and symbolic algebra systems.

I.INTRODUCTION

Infinite series have long captivated mathematicians due to their ability to encode complex behaviors and relationships in compact, elegant expressions. Among these, series involving harmonic numbers H_k , *binomial coefficients* $\binom{2n}{n}$ and powers of natural numbers often appear in the realms of combinatorics, number theory, and mathematical physics. Their evaluations frequently involve special constants like the Riemann zeta function $\zeta(s)$, polylogarithmic functions $Li_s(x)$ and iterated logarithms

This paper presents a profound identity:

$$\sum_{k=1}^{\infty}\frac{4^k H_k}{k^4\binom{2k}{k}} = -\frac{31}{2}\zeta(5)+3\zeta(2)\zeta(3)+16Li_5\left(\frac{1}{2}\right) + 2\,ln(2)\,\zeta(4)+\frac{16}{3}ln^3(2)\,\zeta(2) - \frac{2}{15}ln^5(2)$$

This identity embodies a rich interplay between combinatorial summation and deep analytic functions.

We begin by examining the convergence of the series and apply symbolic manipulation, integral transforms, and generating function techniques to derive the closed form. The components on the right hand side particularly the zeta and polylogarithmic terms—highlight the surprising precision with which such seemingly chaotic series collapse into beautiful mathematical constants. Our exploration contributes not just to the aesthetic domain of mathematical beauty, but also offers practical utility in analytical computations and the simplification of constants in theoretical physics and high precision algorithms. The techniques employed here pave the way for generalizations and further studies in special function theory and series acceleration methods.

II.SOME IMPORTANT RESULTS

$$We\ know,\qquad ln(1+ix)=\frac{1}{2}ln(1+x^2)+i\ arctan(x)$$

$$\xRightarrow[\text{and taking real part}]{\text{Raising to 4th power}} \Re\{ln^4(1+ix)\}=\frac{1}{16}ln^4(1+x^2)-\frac{3}{2}ln^2(1+x^2)\,arctan^2(x)+arctan^4(x)$$

$$\boxed{ln^2(1+x^2)\,arctan^2(x)=\frac{1}{24}ln^4(1+x^2)-\frac{2}{3}\Re\{ln^4(1+ix)\}+\frac{2}{3}arctan^4(x)}\quad -(1)$$

$$\Im\{ln^4(1+ix)\}=\frac{1}{2}ln^3(1+x^2)\,arctan(x)-2\,arctan^3(x)\,ln(1+x^2)$$

$$\boxed{ln^3(1+x^2)\,arctan(x)=2\Im\{ln^4(1+ix)\}+4\,arctan^3(x)\,ln(1+x^2)}\quad -(2)$$

From [2, pp. 331-333]

$$\boxed{\sum_{k=1}^{\infty} \frac{4^k x^{2k}}{k^2 \binom{2k}{k}} = 2\, arcsin^2(x)} \quad -(3)$$

From [1, pp. 107-109]

$$\boxed{\frac{3}{2}\sum_{k=1}^{\infty} \frac{4^k H_k^{(2)}}{k^2 \binom{2k}{k}} x^{2k} - \frac{3}{2}\sum_{k=1}^{\infty} \frac{4^k}{k^4 \binom{2k}{k}} x^{2k} = arcsin^4(x)} \quad -(4)$$

III. SOME INTEGRAL RESULTS TO BE USE LATER

$$I_1 = \int_0^{\frac{\pi}{2}} ln\big(sin(x)\big)\, ln^2(cos(x))\, dx$$

$$= \frac{1}{8} \lim_{\substack{m\to\frac{1}{2} \\ n\to\frac{1}{2}}} \frac{\partial^3}{\partial m \partial n^2} \int_0^{\frac{\pi}{2}} sin^{2m-1}(x)\, cos^{2n-1}(x)\, dx$$

$$= \frac{1}{16} \lim_{\substack{m\to\frac{1}{2} \\ n\to\frac{1}{2}}} \frac{\partial^3}{\partial m \partial n^2} B(m,n)$$

$$= \frac{1}{16} \lim_{\substack{m\to\frac{1}{2} \\ n\to\frac{1}{2}}} \frac{\partial^3}{\partial m \partial n^2} \left(\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\right)$$

$$= \frac{1}{16} \lim_{\substack{m\to\frac{1}{2} \\ n\to\frac{1}{2}}} \frac{\partial^2}{\partial n^2} \left(\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\right) \left(\psi^{(0)}(m) - \psi^{(0)}(m+n)\right)$$

$$= \frac{1}{16} \lim_{\substack{m\to\frac{1}{2} \\ n\to\frac{1}{2}}} \frac{\partial}{\partial n} \left(\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\right) \left(\left(\psi^{(0)}(m) - \psi^{(0)}(m+n)\right)\left(\psi^{(0)}(n) - \psi^{(0)}(m+n)\right) - \left(\psi^{(1)}(m+n)\right)\right)$$

$$= \frac{1}{16} \lim_{\substack{m\to\frac{1}{2} \\ n\to\frac{1}{2}}} \left(\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)}\right) \Bigg(\left(\psi^{(0)}(m) - \psi^{(0)}(m+n)\right)\left(\psi^{(0)}(n) - \psi^{(0)}(m+n)\right)^2$$

$$- \left(\psi^{(1)}(m+n)\right)\left(\psi^{(0)}(n) - \psi^{(0)}(m+n)\right)$$

$$+ \left(\psi^{(0)}(m) - \psi^{(0)}(m+n)\right)\left(\psi^{(1)}(n) - \psi^{(1)}(m+n)\right)$$

$$- \psi^{(1)}(m+n)\left(\psi^{(0)}(n) - \psi^{(0)}(m+n)\right) - \left(\psi^{(2)}(m+n)\right)\Bigg)$$

$$= \frac{1}{16}\pi\left(-8\, ln^3(2) + 2\, ln(2)\, \zeta(2) + (-2\, ln(2))\big(2\zeta(2)\big) - \big(\zeta(2)\big)(-2\, ln(2)) + 2\zeta(3)\right)$$

$$= \frac{1}{16}\pi\big(-8\, ln^3(2) + 2\zeta(3)\big)$$

$$= \frac{\pi\zeta(3)}{8} - \frac{\pi}{2} ln^3(2)$$

$$\boxed{I_1 = \int_0^{\frac{\pi}{2}} ln\big(sin(x)\big)\, ln^2(cos(x))\, dx = \frac{\pi\zeta(3)}{8} - \frac{\pi}{2} ln^3(2)}$$

$$I_2 = \int_0^{\frac{\pi}{2}} x\, ln^3(tan(x))\, dx$$

$$= \underbrace{\int_0^{\frac{\pi}{2}} x\, ln^3(tan(x))\, dx}_{x\to arctan\,(x)}$$

$$= \int_0^\infty \frac{arctan(x)\, ln^3(x)}{1+x^2} dx$$

$$= \int_0^\infty \frac{ln^3(x)}{1+x^2} (arctan(x)) dx$$

$$= \int_0^\infty \frac{ln^3(x)}{1+x^2} \left(\int_0^1 \frac{x}{1+x^2\, y^2} dy \right) dx$$

$$= \int_0^1 \underbrace{\int_0^\infty \frac{ln^3(x)}{1+x^2} \frac{x}{1+x^2 y^2} dx}_{x^2 \to x} dy$$

$$= \frac{1}{16} \int_0^1 \underbrace{\int_0^\infty \frac{ln^3(x)}{1+x} \frac{1}{1+xy^2} dx}_{xy \to x} dy$$

$$= \frac{1}{16} \int_0^1 \underbrace{\int_0^\infty \frac{ln^3\left(\frac{x}{y}\right)}{x+y} \frac{1}{1+xy} dx}_{x \to \frac{1}{x}} dy$$

$$= -\frac{1}{16} \int_0^1 \underbrace{\int_0^\infty \frac{ln^3(xy)}{1+xy} \frac{1}{x+y} dx}_{x \to \frac{1}{x}} dy$$

$$\xrightarrow[]{adding\ the\ last\ 2\ versions\ of\ I_2} 2I_2 = \frac{1}{16} \int_0^1 \int_0^\infty \frac{ln^3\left(\frac{x}{y}\right) - ln^3(xy)}{(x+y)(1+xy)} dx\, dy$$

$$I_2 = -\frac{1}{16} \int_0^1 \int_0^\infty \frac{3ln(y)\, ln^2(x) + ln^3(y)}{(x+y)(1+xy)} dx\, dy$$

$$I_2 = -\frac{3}{16} \int_0^1 \int_0^1 \frac{ln(y)\, ln^2(x) + ln^3(y)}{(x+y)(1+xy)} dx\, dy - \frac{1}{16} \int_0^1 \underbrace{\int_1^\infty \frac{ln(y)\, ln^2(x) + ln^3(y)}{(x+y)(1+xy)} dx}_{x \to \frac{1}{x}} dy$$

$$I_2 = -\frac{3}{16} \int_0^1 \int_0^1 \frac{ln(y)\, ln^2(x) + ln^3(y)}{(x+y)(1+xy)} dx\, dy - \frac{1}{16} \int_0^1 \int_0^1 \frac{ln(y)\, ln^2(x) + ln^3(y)}{(x+y)(1+xy)} dx\, dy$$

$$I_2 = -\frac{1}{8} \int_0^1 \frac{y\, ln(y)}{1-y^2} \left(\int_0^1 \frac{3\, ln^2(x) + ln^2(y)}{y(x+y)} dx - \int_0^1 \frac{3\, ln^2(x) + ln^2(y)}{(1+xy)} dx \right) dy$$

$$I_2 = -\frac{3}{8} \int_0^1 \frac{ln(y)}{1-y^2} \left(\int_0^1 \frac{ln^2(x)}{(x+y)} dx \right) dy - \frac{1}{8} \int_0^1 \frac{ln^3(y)}{1-y^2} \left(\int_0^1 \frac{1}{(x+y)} dx \right) dy$$

$$+ \frac{3}{8} \int_0^1 \frac{ln(y)}{1-y^2} \left(\int_0^1 \frac{ln^2(x)}{\left(x+\frac{1}{y}\right)} dx \right) dy + \frac{1}{8} \int_0^1 \frac{ln^3(y)}{1-y^2} \left(\int_0^1 \frac{1}{\left(x+\frac{1}{y}\right)} dx \right) dy$$

$$I_2 = \frac{3}{4} \int_0^1 \frac{ln(y)}{1-y^2} \left(Li_3\left(-\frac{1}{y}\right) \right) dy - \frac{1}{8} \int_0^1 \frac{ln^3(y)}{1-y^2} (ln(1+y) - ln(y)) dy - \frac{3}{4} \int_0^1 \frac{ln(y)}{1-y^2} (Li_3(-y)) dy$$

$$+ \frac{1}{8} \int_0^1 \frac{ln^3(y)}{1-y^2} (ln(1+y)) dy$$

$$I_2 = \frac{3}{4} \int_0^1 \frac{ln(y)}{1-y^2} \left(Li_3\left(-\frac{1}{y}\right) - Li_3(-y) \right) dy + \frac{1}{8} \int_0^1 \frac{ln^4(y)}{1-y^2} dy$$

$$I_2 = \frac{3}{4} \int_0^1 \frac{ln(y)}{1-y^2} \left(\frac{1}{6} ln^3(y) + \zeta(2)\, ln(y) \right) dy + \frac{1}{8} \int_0^1 \frac{ln^4(y)}{1-y^2} dy$$

$$I_2 = \frac{3}{4} \int_0^1 \frac{ln^2(y)}{1-y^2} (\zeta(2)) dy + \frac{1}{4} \int_0^1 \frac{ln^4(y)}{1-y^2} dy$$

$$I_2 = \sum_{n=0}^{\infty} \left(\frac{3}{4} \zeta(2) \int_0^1 y^{2n}\, ln^2(y)\, dy + \frac{1}{4} \int_0^1 y^{2n}\, ln^4(y)\, dy \right)$$

$$I_2 = \sum_{n=0}^{\infty} \left(\frac{3}{4} \zeta(2) \frac{2}{(2n+1)^3} + \frac{1}{4} \frac{24}{(2n+1)^5} \right)$$

$$I_2 = \frac{3}{2} \zeta(2) \sum_{n=0}^{\infty} \left(\frac{1}{(2n+1)^3} \right) + 6 \sum_{n=0}^{\infty} \left(\frac{1}{(2n+1)^5} \right)$$

$$I_2 = \frac{3}{2} \zeta(2) \left(\frac{7}{8} \zeta(3) \right) + 6 \left(\frac{31}{32} \zeta(5) \right)$$

$$I_2 = \frac{93}{16} \zeta(5) + \frac{21}{16} \zeta(2)\zeta(3)$$

$$\boxed{I_2 = \int_0^{\frac{\pi}{2}} x\, ln^3(tan(x))\, dx = \frac{93}{16} \zeta(5) + \frac{21}{16} \zeta(2)\zeta(3)}$$

$$I_3 = \int_0^{\frac{\pi}{2}} ln^3(sin(x))\, dx$$

$$= \frac{1}{8} \lim_{\substack{m \to \frac{1}{2} \\ n \to \frac{1}{2}}} \frac{\partial^3}{\partial m^3} \int_0^{\frac{\pi}{2}} sin^{2m-1}(x)\, cos^{2n-1}(x)\, dx$$

$$= \frac{1}{16} \lim_{\substack{m \to \frac{1}{2} \\ n \to \frac{1}{2}}} \frac{\partial^3}{\partial m^3} B(m,n)$$

$$= \frac{1}{16} \lim_{\substack{m \to \frac{1}{2} \\ n \to \frac{1}{2}}} \frac{\partial^3}{\partial m^3} \left(\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \right)$$

$$= \frac{1}{16} \lim_{\substack{m \to \frac{1}{2} \\ n \to \frac{1}{2}}} \frac{\partial^2}{\partial m^2} \left(\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \right) \left(\psi^{(0)}(m) - \psi^{(0)}(m+n) \right)$$

$$= \frac{1}{16} \lim_{\substack{m \to \frac{1}{2} \\ n \to \frac{1}{2}}} \frac{\partial}{\partial m} \left(\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \right) \left(\left(\psi^{(0)}(m) - \psi^{(0)}(m+n) \right)^2 + \left(\psi^{(1)}(m) - \psi^{(1)}(m+n) \right) \right)$$

$$= \frac{1}{16} \lim_{\substack{m \to \frac{1}{2} \\ n \to \frac{1}{2}}} \left(\frac{\Gamma(m)\Gamma(n)}{\Gamma(m+n)} \right) \left(\left(\psi^{(0)}(m) - \psi^{(0)}(m+n) \right)^3 \right.$$

$$\left. + 3 \left(\psi^{(1)}(m) - \psi^{(1)}(m+n) \right) \left(\psi^{(0)}(m) - \psi^{(0)}(m+n) \right) + \left(\psi^{(2)}(m) - \psi^{(2)}(m+n) \right) \right)$$

$$= \frac{1}{16} (\pi) \left((-2\, ln(2))^3 + 3\big(2\zeta(2)\big)(-2\, ln(2)) + \big(-12\zeta(3)\big) \right)$$

$$= -\frac{3\pi}{4} \zeta(3) - \frac{\pi^3}{8} ln(2) - \frac{\pi}{2} ln^3(2)$$

$$\boxed{I_3 = \int_0^{\frac{\pi}{2}} ln^3(sin(x))\, dx = -\frac{3\pi}{4} \zeta(3) - \frac{\pi^3}{8} ln(2) - \frac{\pi}{2} ln^3(2)}$$

$$I_4 = \int_0^{\frac{\pi}{2}} x\, ln^3(cos(x))\, dx$$

$$I_4 = \underbrace{\int_0^{\frac{\pi}{2}} x\, ln^3(cos(x))\, dx}_{tan(x) \to x}$$

$$= -\frac{1}{8} \int_0^{\infty} \frac{arctan(x)\, ln^3(1+x^2)}{1+x^2}\, dx$$

Using (2)

$$= -\frac{1}{8}\int_0^\infty \frac{2\Im\{ln^4(1+ix)\} + 4\,arctan^3(x)\,ln(1+x^2)}{1+x^2}dx$$

$$= -\frac{1}{4}\Im\left\{\underbrace{\int_0^\infty \frac{ln^4(1+ix)}{1+x^2}dx}_{\frac{1}{1+ix}\to x}\right\} - \frac{1}{2}\underbrace{\int_0^\infty \frac{arctan^3(x)\,ln(1+x^2)}{1+x^2}dx}_{arctan(x)\to x}$$

$$= -\frac{1}{4}\Im\left\{i\int_0^1 \frac{ln^4(x)}{1-2x}dx\right\} + \int_0^{\frac{\pi}{2}} x^3\,ln(cosx)\,dx$$

$$= -\frac{1}{4}\Im\{i12Li_5(2)\} + \int_0^{\frac{\pi}{2}} x^3\left(-ln(2) - \sum_{k=1}^\infty \frac{(-1)^k cos(2kx)}{k}\right)dx$$

$$= -\frac{1}{4}\Re\{12Li_5(2)\} - \frac{\pi^4}{64}ln(2) - \sum_{k=1}^\infty \frac{(-1)^k}{k}\int_0^{\frac{\pi}{2}} x^3 cos(2kx)\,dx$$

$$= -\frac{1}{4}\Re\left\{12\left(Li_5\left(\frac{1}{2}\right) + 2\,ln(2)\,\zeta(4) + \frac{1}{3}ln^3(2)\zeta(2) - \frac{1}{120}ln^5(2) - \frac{i}{24}ln^4(2)\right)\right\} - \frac{\pi^4}{64}ln(2)$$
$$- \sum_{k=1}^\infty \frac{(-1)^k}{k}\left(\frac{3(\pi^2k^2-2)\,cos(\pi k)}{16k^4} + \frac{3}{8k^4}\right)$$

$$= -3Li_5\left(\frac{1}{2}\right) - 6\,ln(2)\,\zeta(4) - ln^3(2)\zeta(2) + \frac{1}{40}ln^5(2) - \frac{\pi^4}{64}ln(2) - \frac{3}{16}\sum_{k=1}^\infty \frac{(\pi^2k^2-2)}{k^5} - \frac{3}{8}\sum_{k=1}^\infty \frac{(-1)^k}{k^5}$$

$$= -3Li_5\left(\frac{1}{2}\right) - 6\,ln(2)\,\zeta(4) - ln^3(2)\zeta(2) + \frac{1}{40}ln^5(2) - \frac{\pi^4}{64}ln(2) - \frac{3}{16}\left(\pi^2\zeta(3) - 2\zeta(5)\right) - \frac{3}{8}\left(\frac{15}{16\zeta(5)}\right)$$

$$\boxed{I_4 = \int_0^{\frac{\pi}{2}} x\,ln^3(cos(x))\,dx = -3Li_5\left(\frac{1}{2}\right) - ln^3(2)\zeta(2) + \frac{1}{40}ln^5(2) - \frac{237}{32}ln(2)\,\zeta(4) - \frac{9}{8}\zeta(2)\zeta(3) + \frac{93}{128}\zeta(5)}$$

$$I_5 = \int_0^{\frac{\pi}{2}} x\,ln(sin(x))\,ln^2(cos(x))\,dx$$

$$= \frac{1}{2}\left(\int_0^{\frac{\pi}{2}} x\,ln(sin(x))\,ln^2(cos(x))\,dx + \underbrace{\int_0^{\frac{\pi}{2}} x\,ln(cos(x))\,ln^2(sin(x))\,dx}_{x\to\frac{\pi}{2}-x}\right)$$
$$+ \frac{1}{2}\left(\int_0^{\frac{\pi}{2}} x\,ln(sin(x))\,ln^2(cos(x))\,dx - \int_0^{\frac{\pi}{2}} x\,ln(cos(x))\,ln^2(sin(x))\,dx\right)$$

$$= \frac{1}{2}\left(\int_0^{\frac{\pi}{2}} x\,ln(sin(x))\,ln^2(cos(x))\,dx + \int_0^{\frac{\pi}{2}}\left(\frac{\pi}{2} - x\right)ln(sin(x))\,ln^2(cos(x))\,dx\right)$$
$$+ \frac{1}{6}\left(\int_0^{\frac{\pi}{2}} x\underbrace{\left[3ln(sin(x))\,ln^2(cos(x)) - 3\,ln(cos(x))\,ln^2(sin(x))\right]}_{3ab^2-3a^2b=(a-b)^3-a^3+b^3}dx\right)$$

$$= \frac{1}{2}\left(\int_0^{\frac{\pi}{2}} \frac{\pi}{2}ln(sin(x))\,ln^2(cos(x))\,dx\right)$$
$$+ \frac{1}{6}\left(\int_0^{\frac{\pi}{2}} x[(ln(sin\,(x)) - ln(cos\,(x)))^3 - ln^3(sin(x)) + ln^3(cos(x)]dx\right)$$

$$I_2 = \frac{\pi}{4}(I_1) + \frac{1}{6}\left(\int_0^{\frac{\pi}{2}} x[ln^3(tan(x)) - ln^3(sin(x)) + ln^3(cos(x)]dx\right)$$

$$=\frac{\pi}{4}(I_1)+\frac{1}{6}\left(\underbrace{\int_0^{\frac{\pi}{2}} x\,ln^3(tan(x))\,dx}_{I_2}-\underbrace{\int_0^{\frac{\pi}{2}} x\,ln^3(sin(x))\,dx}_{x\to\frac{\pi}{2}-x}+\underbrace{\int_0^{\frac{\pi}{2}} x\,ln^3(cos(x))\,dx}_{I_4}\right)$$

$$=\frac{\pi}{4}(I_1)+\frac{1}{6}\left(I_2-\int_0^{\frac{\pi}{2}}\left(\frac{\pi}{2}-x\right)ln^3(cos(x))\,dx+I_4\right)$$

$$=\frac{\pi}{4}(I_1)+\frac{1}{6}\left(I_2-\frac{\pi}{2}\int_0^{\frac{\pi}{2}} ln^3(cos(x))\,dx+\int_0^{\frac{\pi}{2}} x\,ln^3(cos(x))\,dx+I_4\right)$$

$$=\frac{\pi}{4}(I_1)+\frac{1}{6}\left(I_2-\frac{\pi}{2}\underbrace{\int_0^{\frac{\pi}{2}} ln^3(cos(x))\,dx}_{x\to\frac{\pi}{2}-x}+\underbrace{\int_0^{\frac{\pi}{2}} x\,ln^3(cos(x))\,dx}_{I_4}+I_4\right)$$

$$=\frac{\pi}{4}(I_1)+\frac{1}{6}\left(I_2-\frac{\pi}{2}\underbrace{\int_0^{\frac{\pi}{2}} ln^3(sin(x))\,dx}_{I_3}+2I_4\right)$$

$$=\frac{\pi}{4}(I_1)+\frac{1}{6}(I_2)-\frac{\pi}{12}(I_3)+\frac{1}{3}(I_4)$$

$$=\frac{\pi}{4}\left(\frac{\pi\zeta(3)}{8}-\frac{\pi}{2}ln^3(2)\right)+\frac{1}{6}\left(\frac{93}{16}\zeta(5)+\frac{21}{16}\zeta(2)\zeta(3)\right)-\frac{\pi}{12}\left(-\frac{3\pi}{4}\zeta(3)-\frac{\pi^3}{8}ln(2)-\frac{\pi}{2}ln^3(2)\right)$$
$$+\frac{1}{3}\left(-3Li_5\left(\frac{1}{2}\right)-ln^3(2)\zeta(2)+\frac{1}{40}ln^5(2)-\frac{237}{32}ln(2)\,\zeta(4)-\frac{9}{8}\zeta(2)\zeta(3)+\frac{93}{128}\zeta(5)\right)$$

$$\boxed{\begin{aligned}I_5&=\int_0^{\frac{\pi}{2}} x\,ln\big(sin(x)\big)\,ln^2(cos(x))\,dx\\&=-Li_5\left(\frac{1}{2}\right)+\frac{155}{128}\zeta(5)+\frac{13}{32}\zeta(2)\zeta(3)-\frac{49}{32}ln(2)\,\zeta(4)-\frac{5}{6}ln^3(2)\zeta(2)+\frac{1}{120}ln^5(2)\end{aligned}}$$

$$I_6=\int_0^{\frac{\pi}{2}} cot(x)\,ln^2(cos(x)\,)\,dx$$

$$I_6=\underbrace{\int_0^{\frac{\pi}{2}} cot(x)\,ln^2(cos(x)\,)\,dx}_{cos\,(x)\to\sqrt{x}}$$

$$I_6=\frac{1}{8}\int_0^1\frac{ln^2(x)}{1-x}dx$$

$$I_6=\frac{1}{8}\sum_{n=0}^{\infty}\left(\int_0^1 x^n\,ln^2(x)\,dx\right)$$

$$I_6=\frac{1}{4}\sum_{n=0}^{\infty}\left(\frac{1}{(n+1)^3}\right)$$

$$\boxed{I_6=\int_0^{\frac{\pi}{2}} cot(x)\,ln^2(cos(x)\,)\,dx=\frac{1}{4}\zeta(3)}$$

$$I_7=\int_0^{\frac{\pi}{2}} x^2\,csc^2(x)\,ln^2(cos(x))\,dx$$

$$\underbrace{\int_0^{\frac{\pi}{2}} x^2\,csc^2(x)\,ln^2(cos(x))\,dx}_{tan(x)\to x}$$

$$=\frac{1}{4}\int_0^{\infty}\frac{arctan^2(x)\,ln^2(1+x^2)}{x^2}dx$$

Using (1):-

$$= \frac{1}{96}\underbrace{\int_0^\infty \frac{ln^4(1+x^2)}{x^2}dx}_{A} + \frac{1}{6}\underbrace{\int_0^\infty \frac{arctan^4(x)}{x^2}dx}_{B} - \frac{1}{6}\Re\left\{\underbrace{\int_0^\infty \frac{ln^4(1+ix)}{x^2}dx}_{C}\right\}$$

$$I_7 = \frac{1}{96}(A) + \frac{1}{6}(B) - \frac{1}{6}\Re\{(C)\}$$

$$A = \underbrace{\int_0^\infty \frac{ln^4(1+x^2)}{x^2}dx}_{x\to tan(x)}$$

$$= 16\int_0^{\frac{\pi}{2}} ln^4(cos(x))\, csc^2(x)\, dx$$

$$= 16\int_0^{\frac{\pi}{2}} ln^4(cos(x))\, d(cot(x))$$

$$\overset{IBP}{\cong} -64\underbrace{\int_0^{\frac{\pi}{2}} ln^3(cos(x))\, dx}_{x\to\frac{\pi}{2}-x}$$

$$= -64\underbrace{\int_0^{\frac{\pi}{2}} ln^3(sin(x))\, dx}_{I_3}$$

$$= -64(I_3)$$

$$= -64\left(-\frac{3\pi}{4}\zeta(3) - \frac{\pi^3}{8}ln(2) - \frac{\pi}{2}ln^3(2)\right)$$

$$= 48\pi\zeta(3) + 8\pi^3\, ln(2) + 32\pi\, ln^3(2)$$

$$\boxed{A = \int_0^\infty \frac{ln^4(1+x^2)}{x^2}dx = 48\pi\zeta(3) + 8\pi^3\, ln(2) + 32\pi\, ln^3(2)}$$

$$B = \underbrace{\int_0^\infty \frac{arctan^4(x)}{x^2}dx}_{arctan(x)\to x}$$

$$= \int_0^{\frac{\pi}{2}} x^4 csc^2(x)\, dx$$

$$\overset{IBP}{\cong} -\int_0^{\frac{\pi}{2}} 4x^3\, cot(x)\, dx$$

$$\overset{IBP}{\cong} -12\int_0^{\frac{\pi}{2}} x^2\, ln(sin(x))\, dx$$

$$\overset{applying\ fourier\ of\ ln(sinx)}{\cong} -12\int_0^{\frac{\pi}{2}} x^2\left(-ln(2) - \sum_{k=1}^{n} cos\frac{(2kx)}{k}\right)dx$$

$$= \frac{\pi^3}{2}ln\,(2) + 12\sum_{k=1}^{n}\frac{1}{k}\int_0^{\frac{\pi}{2}} x^2(cos(2kx))\, dx$$

$$= \frac{\pi^3}{2}ln\,(2) + 3\pi\sum_{k=1}^{n}\frac{(-1)^k}{k^3}$$

$$= \frac{\pi^3}{2}ln(2) - 3\pi\left(\frac{3}{4}\zeta(3)\right)$$

$$= \frac{\pi^3}{2}ln(2) - \frac{9\pi}{4}\zeta(3)$$

$$\boxed{B = \int_0^\infty \frac{arctan^4(x)}{x^2}dx = \frac{\pi^3}{2}ln(2) - \frac{9\pi}{4}\zeta(3)}$$

$$C = \int_0^\infty \frac{ln^4(1+ix)}{x^2}\,dx$$

$$\overset{\frac{1}{1+ix}\to x}{\cong} \; i\int_0^1 \frac{ln^4(x)}{(1-x)^2}\,dx$$

$$\overset{\frac{1}{1+ix}\to x}{\cong} \; i\sum_{n=1}^{\infty} n\int_0^1 x^{n-1}\,ln^4(x)\,dx$$

$$= i\sum_{n=1}^{\infty} n\frac{\partial^4}{\partial n^4}\int_0^1 x^{n-1}\,dx$$

$$= i\sum_{n=1}^{\infty} n\frac{\partial^4}{\partial n^4}\left(\frac{1}{n}\right)$$

$$= i24\sum_{n=1}^{\infty}\frac{1}{n^4}$$

$$\boxed{C = \int_0^\infty \frac{ln^4(1+ix)}{x^2}\,dx = i24\zeta(4)}$$

$$I_7 = \frac{1}{96}(48\pi\zeta(3) + 8\pi^3\,ln(2) + 32\pi\,ln^3(2)) + \frac{1}{6}\left(\frac{\pi^3}{2}ln(2) - \frac{9\pi}{4}\zeta(3)\right) - \frac{1}{6}\Re\{i24\zeta(4)\}$$

$$I_7 = \frac{\pi}{8}\zeta(3) + \frac{\pi^3}{6}ln(2) + \frac{\pi}{3}ln^3(2)$$

$$\boxed{I_7 = \int_0^{\frac{\pi}{2}} x^2\,csc^2(x)\,ln^2(cos(x))\,dx = \frac{\pi}{8}\zeta(3) + \frac{\pi^3}{6}ln(2) + \frac{\pi}{3}ln^3(2)}$$

$$I_8 = \int_0^{\frac{\pi}{2}} x^2\,ln(cos(x))\,dx$$

$$= \int_0^{\frac{\pi}{2}} x^2\left(-ln(2) - \sum_{k=1}^{\infty}\frac{(-1)^k\,cos(2kx)}{k}\right)dx$$

$$= -\frac{\pi^3}{24}ln(2) - \sum_{k=1}^{\infty}\frac{(-1)^k}{k}\int_0^{\frac{\pi}{2}} x^2\,cos(2kx)\,dx$$

$$= -\frac{\pi^3}{24}ln(2) - \sum_{k=1}^{\infty}\frac{(-1)^k}{k}\left(\frac{\pi(-1)^k}{4k^2}\right)$$

$$= -\frac{\pi^3}{24}ln(2) - \frac{\pi}{4}\sum_{k=1}^{\infty}\frac{1}{k^3}$$

$$\boxed{I_8 = \int_0^{\frac{\pi}{2}} x^2\,ln(cos(x))\,dx = -\frac{\pi^3}{24}ln(2) - \frac{\pi}{4}\zeta(3)}$$

$$I_9 = \int_0^{\frac{\pi}{2}} x\,cot(x)\,ln^2(cos(x)\,)\,dx$$

$$I_9 = \int_0^{\frac{\pi}{2}} x\,cot(x)\,ln^2(cos(x)\,)\,dx \overset{IBP}{\cong} \frac{1}{2}\underbrace{\int_0^{\frac{\pi}{2}} x^2\,csc^2(x)\,ln^2(cos(x))\,dx}_{I_7} + \underbrace{\int_0^{\frac{\pi}{2}} x^2\,ln(cos(x))\,dx}_{I_8}$$

$$= \frac{1}{2}(I_7) + (I_8)$$

$$= \frac{1}{2}\left(\frac{\pi}{8}\zeta(3) + \frac{\pi^3}{6}ln(2) + \frac{\pi}{3}ln^3(2)\right) + \left(-\frac{\pi^3}{24}ln(2) - \frac{\pi}{4}\zeta(3)\right)$$

$$= -\frac{3\pi}{16}\zeta(3) + \frac{\pi^3}{24}ln(2) + \frac{\pi}{6}ln^3(2)$$

$$\boxed{I_9 = \int_0^{\frac{\pi}{2}} x\,cot(x)\,ln^2(cos(x)\,)\,dx = -\frac{3\pi}{16}\zeta(3) + \frac{\pi^3}{24}ln(2) + \frac{\pi}{6}ln^3(2)}$$

$$I_{10} = \int_0^{\infty} \frac{ln^4(1+x^2)}{x(1+x^2)}dx$$

$$\overset{\frac{1}{1+x^2}\to x}{\triangleq} \frac{1}{2}\int_0^1 \frac{ln^4(x)}{1-x}dx$$

$$= \frac{1}{2}\int_0^1 ln^4(x)\sum_{n=0}^{\infty} x^n\,dx$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{d^4}{dn^4}\int_0^1 x^n\,dx$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{d^4}{dn^4}\left(\frac{1}{n+1}\right)$$

$$= \frac{1}{2}\sum_{n=0}^{\infty}\frac{24}{(n+1)^5}$$

$$= 12\zeta(5)$$

$$\boxed{I_{10} = \int_0^{\infty} \frac{ln^4(1+x^2)}{x(1+x^2)}dx = 12\zeta(5)}$$

$$I_{11} = \int_0^{\infty} \frac{arctan^4(x)}{x(1+x^2)}dx$$

$$\overset{arctan(x)\to x}{\triangleq} \int_0^{\frac{\pi}{2}} \frac{x^4}{tanx}dx$$

$$= \int_0^{\frac{\pi}{2}} x^4\,d(log(sinx))$$

$$\overset{IBP}{\triangleq} [x^4\,ln(sinx)]_0^{\frac{\pi}{2}} - 4\int_0^{\frac{\pi}{2}} x^3\,ln(sinx)\,dx$$

$$= -4\int_0^{\frac{\pi}{2}} x^3\,ln(sinx)\,dx$$

$$= -4\int_0^{\frac{\pi}{2}} x^3\left(-log2 - \sum_{n=1}^{\infty}\frac{cos(2nx)}{n}\right)dx$$

$$= 4\int_0^{\frac{\pi}{2}} x^3\left(log2 + \sum_{n=1}^{\infty}\frac{cos(2nx)}{n}\right)dx$$

$$= \frac{\pi^4}{16}log(2) + \sum_{n=1}^{\infty}\frac{1}{n}\int_0^{\frac{\pi}{2}} 4x^3\,cos(2nx)\,dx$$

$$= \frac{\pi^4}{16}log(2) + \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{3(\pi^2n^2-2)\,cos(\pi n)}{4n^4} + \frac{3}{2n^4}\right)$$

$$= \frac{\pi^4}{16}log(2) + \sum_{n=1}^{\infty}\left(\frac{3(-1)^n}{4}\left(\frac{\pi^2}{n^3} - \frac{2}{n^5}\right) + \frac{3}{2n^5}\right)$$

$$= \frac{\pi^4}{16}log(2) + \frac{3\pi^2}{4}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^3} - \frac{3}{2}\sum_{n=1}^{\infty}\frac{(-1)^n}{n^5} + \frac{3}{2}\sum_{n=1}^{\infty}\frac{1}{n^5}$$

$$= \frac{\pi^4}{16}log(2) + \frac{3\pi^2}{4}\left(-\frac{3}{4}\zeta(3)\right) - \frac{3}{2}\left(-\frac{15}{16}\zeta(5)\right) + \frac{3}{2}\zeta(5)$$

$$= \frac{45}{8}\zeta(4)\,ln(2) - \frac{27}{8}\zeta(2)\zeta(3) + \frac{93}{32}\zeta(5)$$

$$\boxed{I_{11} = \int_0^\infty \frac{arctan^4(x)}{x(1+x^2)}dx = \frac{93}{32}\zeta(5) - \frac{27}{8}\zeta(2)\zeta(3) + \frac{45}{8}\zeta(4)\,ln(2)}$$

$$I_{12} = \int_0^\infty \frac{ln^4(1+ix)}{x(1+x^2)}dx$$

$$\overset{\frac{1}{1+ix}\to x}{\cong} -\int_0^1 \frac{x\ ln^4(x)}{(1-x)(1-2x)}dx$$

$$\overset{PFD}{\cong} \int_0^1 \frac{ln^4(x)}{(1-x)}dx - \int_0^1 \frac{ln^4(x)}{(1-2x)}dx$$

$$= 24\zeta(5) - 12Li_5(2)$$

$$= 24\zeta(5) - 12\left(Li_5\left(\frac{1}{2}\right) + 2\,ln(2)\,\zeta(4) + \frac{1}{3}ln^3(2)\zeta(2) - \frac{1}{120}ln^5(2) - \frac{i}{24}ln^4(2)\right)$$

$$\boxed{I_{12} = \int_0^\infty \frac{ln^4(1+ix)}{x(1+x^2)}dx = 24\zeta(5) - 12Li_5\left(\frac{1}{2}\right) - 24\,ln(2)\,\zeta(4) - 4\,ln^3(2)\zeta(2) + \frac{1}{10}ln^5(2) + \frac{i}{2}ln^4(2)}$$

$$I_{13} = \int_0^{\frac{\pi}{2}} x^2\,cot(x)\,ln^2(cos(x))\,dx$$

$$\overset{x\to arctan(x)}{\cong} \frac{1}{4}\int_0^\infty \frac{arctan^2(x)\,ln^2(1+x^2)}{x(1+x^2)}dx$$

Using Identity (1) :-

$$= \frac{1}{96}\underbrace{\int_0^\infty \frac{ln^4(1+x^2)}{x(1+x^2)}dx}_{I_{10}} + \frac{1}{6}\underbrace{\int_0^\infty \frac{arctan^4(x)}{x(1+x^2)}dx}_{I_{11}} - \frac{1}{6}\Re\left\{\underbrace{\int_0^\infty \frac{ln^4(1+ix)}{x(1+x^2)}dx}_{I_{12}}\right\}$$

$$= \frac{1}{96}(I_{10}) + \frac{1}{6}(I_{11}) - \frac{1}{6}\Re\{I_{12}\}$$

$$= \frac{1}{96}\big(12\zeta(5)\big) + \frac{1}{6}\underbrace{\left(\frac{93}{32}\zeta(5) - \frac{27}{8}\zeta(2)\zeta(3) + \frac{45}{8}\zeta(4)\,ln(2)\right)}_{I_{11}}$$

$$- \frac{1}{6}\Re\left\{24\zeta(5) - 12Li_5\left(\frac{1}{2}\right) - 24\,ln(2)\,\zeta(4) - 4\,ln^3(2)\zeta(2) + \frac{1}{10}ln^5(2) + \frac{i}{2}ln^4(2)\right\}$$

$$\boxed{\begin{gathered}I_{13} = \int_0^{\frac{\pi}{2}} x^2\,cot(x)\,ln^2(cos(x))\,dx = -\frac{217}{64}\zeta(5) - \frac{9}{16}\zeta(2)\zeta(3) + 2\,Li_5\left(\frac{1}{2}\right) + \frac{79}{16}ln(2)\,\zeta(4) + \frac{2}{3}ln^3(2)\zeta(2)\\ -\frac{1}{60}ln^5(2)\end{gathered}}$$

$$I_{14} = \int_0^{\frac{\pi}{2}} x^2\,cot(x)\,ln(sin(x))\,ln(cos(x))\,dx$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{2}} x^2\,ln(cos(x))\,d(ln^2(sin(x)))$$

$$\overset{IBP}{\cong} \frac{1}{2}\int_0^{\frac{\pi}{2}} \Big(2xln\big(cos(x)\big) - x^2tan(x)\Big)(ln^2(sin(x)))\,dx$$

$$= \int_0^{\frac{\pi}{2}} xln\big(cos(x)\big)\,ln^2(sin(x))\,dx - \frac{1}{2}\int_0^{\frac{\pi}{2}} x^2\,tan(x)\,ln^2(sin\,(x))\,dx$$

$$\overset{x\to\left(\frac{\pi}{2}-x\right)}{\cong} \int_0^{\frac{\pi}{2}} \left(\frac{\pi}{2} - x\right)ln\big(sin(x)\big)\,ln^2(cos(x))\,dx - \frac{1}{2}\int_0^{\frac{\pi}{2}} \left(\frac{\pi}{2} - x\right)^2 cot(x)\,ln^2(cos\,(x)\,)\,dx$$

$$= \frac{\pi}{2}\underbrace{\int_0^{\frac{\pi}{2}} ln\big(sin(x)\big)\, ln^2(cos(x))\, dx}_{I_1} - \underbrace{\int_0^{\frac{\pi}{2}} x\, ln\big(sin(x)\big)\, ln^2(cos(x))\, dx}_{I_5} - \frac{\pi^2}{8}\underbrace{\int_0^{\frac{\pi}{2}} cot(x)\, ln^2(cos(x)\,)\, dx}_{I_6}$$

$$+ \frac{\pi}{2}\underbrace{\int_0^{\frac{\pi}{2}} x\, cot(x)\, ln^2(cos(x)\,)\, dx}_{I_9} - \frac{1}{2}\underbrace{\int_0^{\frac{\pi}{2}} x^2\, cot(x)\, ln^2(cos\,(x)\,)\, dx}_{I_{13}}$$

$$= \frac{\pi}{2}(I_1) - (I_2) - \frac{\pi^2}{8}(I_3) + \frac{\pi}{2}(I_9) - \frac{1}{2}(I_{13})$$

$$= \frac{\pi}{2}\left(\frac{\pi\zeta(3)}{8} - \frac{\pi}{2}ln^3(2)\right) - \left(\frac{93}{16}\zeta(5) + \frac{21}{16}\zeta(2)\zeta(3)\right) - \frac{\pi^2}{8}\left(-\frac{3\pi}{4}\zeta(3) - \frac{\pi^3}{8}ln(2) - \frac{\pi}{2}ln^3(2)\right)$$

$$+ \frac{\pi}{2}\left(-\frac{3\pi}{16}\zeta(3) + \frac{\pi^3}{24}ln(2) + \frac{\pi}{6}ln^3(2)\right)$$

$$- \frac{1}{2}\left(-\frac{217}{64}\zeta(5) - \frac{9}{16}\zeta(2)\zeta(3) + 2\, Li_5\left(\frac{1}{2}\right) + \frac{79}{16}ln(2)\,\zeta(4) + \frac{2}{3}ln^3(2)\zeta(2) - \frac{1}{60}ln^5(2)\right)$$

$$= -\frac{527}{128}\zeta(5) - \frac{39}{32}\zeta(2)\zeta(3) - Li_5\left(\frac{1}{2}\right) - \frac{19}{32}ln(2)\,\zeta(4) - \frac{4}{3}ln^3(2)\zeta(2) + \frac{1}{120}ln^5(2) + \frac{3\pi^3}{32}\zeta(3)$$

$$+ \frac{\pi^5}{64}ln(2) + \frac{\pi^3}{16}ln^3(2)$$

$$\boxed{I_{14} = \int_0^{\frac{\pi}{2}} x^2\, cot(x)\, ln(sin(x))\, ln(cos(x))\, dx = -\frac{527}{128}\zeta(5) - \frac{39}{32}\zeta(2)\zeta(3) - Li_5\left(\frac{1}{2}\right) - \frac{19}{32}ln(2)\,\zeta(4) - \frac{4}{3}ln^3(2)\zeta(2) + \frac{1}{120}ln^5(2) + \frac{3\pi^3}{32}\zeta(3) + \frac{\pi^5}{64}ln(2) + \frac{\pi^3}{16}ln^3(2)}$$

$From\ Equation\ (3),$

$$\sum_{k=1}^{\infty}\frac{4^k x^{2k}}{k^2\binom{2k}{k}} = 2\, arcsin^2(x)$$

$Let\ x \to \sqrt{x}$

$$\sum_{k=1}^{\infty}\frac{4^k x^{k}}{k^2\binom{2k}{k}} = 2\, arcsin^2\big(\sqrt{x}\big)$$

IV. SOME SUMS TO BE USE LATER

$$S_1 = \sum_{k=1}^{\infty}\frac{4^k}{k^3\binom{2k}{k}}$$

$$= 2\underbrace{\int_0^1 \frac{arcsin^2\big(\sqrt{x}\big)}{x}dx}_{x\to sin^2(x)}$$

$$\frac{S_1}{4} = \int_0^{\frac{\pi}{2}} x^2 cot\,(x)\, dx$$

$$= \int_0^{\frac{\pi}{2}} x^2\, d(ln(sin(x)))$$

$$= -2\int_0^{\frac{\pi}{2}} x\, ln(sin(x))dx$$

$$= -2\int_0^{\frac{\pi}{2}} x\left(-ln(2) - \sum_{k=1}^{n} cos\frac{(2kx)}{k}\right)dx$$

$$= \frac{3}{2}ln(2)\,\zeta(2) - \frac{7}{8}\sum_{k=1}^{n}\frac{1}{k^3}$$

$$= \frac{3}{2} ln(2)\, \zeta(2) - \frac{7}{8} \zeta(3)$$

$$S_1 = 6\, ln(2)\, \zeta(2) - \frac{7}{2} \zeta(3)$$

$$\boxed{S_1 = \sum_{k=1}^{\infty} \frac{4^k}{k^3 \binom{2k}{k}} = 6\, ln(2)\, \zeta(2) - \frac{7}{2} \zeta(3)}$$

$$S_2 = \sum_{k=1}^{\infty} \frac{4^k H_k^{(2)}}{k^3 \binom{2k}{k}}$$

Using Equation (4),

$$\frac{3}{2} \sum_{k=1}^{\infty} \frac{4^k H_k^{(2)}}{k^2 \binom{2k}{k}} x^{2k} - \frac{3}{2} \sum_{k=1}^{\infty} \frac{4^k}{k^4 \binom{2k}{k}} x^{2k} = arcsin^4(x)$$

$$\frac{3}{2} \sum_{k=1}^{\infty} \frac{4^k H_k^{(2)}}{k^2 \binom{2k}{k}} x^{2k-1} - \frac{3}{2} \sum_{k=1}^{\infty} \frac{4^k}{k^4 \binom{2k}{k}} x^{2k-1} = \frac{arcsin^4(x)}{x}$$

$$\frac{3}{4} \sum_{k=1}^{\infty} \frac{4^k H_k^{(2)}}{k^3 \binom{2k}{k}} - \frac{3}{4} \sum_{k=1}^{\infty} \frac{4^k}{k^5 \binom{2k}{k}} = \int_0^1 \frac{arcsin^4(x)}{x} dx$$

$$\frac{3}{4} (S_2) - 3 \sum_{k=1}^{\infty} \frac{4^k}{k^2 \binom{2k}{k}} \int_0^1 x^{2k-1}\, ln^2(x)\, dx \;= \int_0^1 \frac{arcsin^4(x)}{x} dx$$

$$\frac{3}{4} (S_2) - 3 \sum_{k=1}^{\infty} \frac{4^k}{k^2 \binom{2k}{k}} \int_0^1 \frac{x^{2k}\, ln^2(x)}{x} dx \;= \int_0^1 \frac{arcsin^4(x)}{x} dx$$

$$\frac{3}{4} (S_2) - 3 \int_0^1 \left(\sum_{k=1}^{\infty} \frac{4^k x^{2k}}{k^2 \binom{2k}{k}} \right) \left(\frac{ln^2(x)}{x} \right) dx = \int_0^1 \frac{arcsin^4(x)}{x} dx$$

$$\frac{3}{4} (S_2) - 6 \int_0^1 \frac{arcsin^2(x)\, ln^2(x)}{x} dx = \int_0^1 \frac{arcsin^4(x)}{x} dx$$

$$S_2 = \underbrace{\frac{4}{3} \int_0^1 \frac{arcsin^4(x)}{x} dx + 8 \int_0^1 \frac{arcsin^2(x)\, ln^2(x)}{x} dx}_{arcsin(x) \to x}$$

$$S_2 = \frac{4}{3} \int_0^{\frac{\pi}{2}} x^4 cot\,(x)\, dx + 8 \int_0^{\frac{\pi}{2}} x^2\, cot(x)\, ln^2(sin\,(x))\, dx$$

$$S_2 = \frac{4}{3} \int_0^{\frac{\pi}{2}} x^4\, d(ln(sin(x))) + \frac{8}{3} \int_0^{\frac{\pi}{2}} x^2 d(ln^3(sinx))$$

$$\overset{IBP}{=} -\frac{16}{3} \int_0^{\frac{\pi}{2}} x^3\, ln(sin(x))\, dx - \underbrace{\frac{16}{3} \int_0^{\frac{\pi}{2}} x ln^3(sin(x)) dx}_{x \to \frac{\pi}{2} - x}$$

$$= -\frac{16}{3} \int_0^{\frac{\pi}{2}} x^3\, ln(sin(x))\, dx - \frac{16}{3} \left(\frac{\pi}{2} \underbrace{\int_0^{\frac{\pi}{2}} ln^3(cos\,(x)) dx}_{x \to \frac{\pi}{2} - x} - \underbrace{\int_0^{\frac{\pi}{2}} x ln^3(cos(x)) dx}_{I_4} \right)$$

$$= -\frac{16}{3} \int_0^{\frac{\pi}{2}} x^3 \left(-\, ln(2) - \sum_{k=1}^{n} cos \frac{(2kx)}{k} \right) dx - \frac{16}{3} \left(\frac{\pi}{2} (I_3) - (I_4) \right)$$

$$= -\frac{16}{3}\left(-\frac{93}{128}\zeta(5) + \frac{27}{32}\zeta(2)\zeta(3) - \frac{45}{32}ln(2)\,\zeta(4)\right)$$

$$-\frac{16}{3}\left(-\frac{93}{128}\zeta(5) - \frac{9}{8}\zeta(2)\zeta(3) + 3Li_5\left(\frac{1}{2}\right) - \frac{1}{2}ln^3(2)\zeta(2) - \frac{1}{40}ln^5(2) + \frac{57}{32}ln(2)\,\zeta(4)\right)$$

$$= \frac{31}{4}\zeta(5) + \frac{3}{2}\zeta(2)\zeta(3) - 16Li_5\left(\frac{1}{2}\right) + \frac{8}{3}ln^3(2)\zeta(2) + \frac{2}{15}ln^5(2) - 2\,ln(2)\,\zeta(4)$$

$$\boxed{S_2 = \sum_{k=1}^{\infty}\frac{4^k H_k^{(2)}}{k^3\binom{2k}{k}} = \frac{31}{4}\zeta(5) + \frac{3}{2}\zeta(2)\zeta(3) - 16Li_5\left(\frac{1}{2}\right) + \frac{8}{3}ln^3(2)\zeta(2) + \frac{2}{15}ln^5(2) - 2\,ln(2)\,\zeta(4)}$$

V. MAIN THEOREM

$$\int_0^1 x^{k-1} ln\,(1-x)\,dx = -\frac{\gamma + \psi(k+1)}{k}$$

$$\int_0^1 x^{k-1} ln\,(x) ln\,(1-x)\,dx = \frac{\gamma + \psi(k+1)}{k^2} - \frac{\psi^{(1)}(k+1)}{k}$$

$$\int_0^1 x^{k-1} ln\,(x) ln\,(1-x)\,dx = \frac{H_k}{k^2} - \frac{\zeta(2) - H_k^{(2)}}{k}$$

$$\sum_{k=1}^{\infty}\frac{4^k x^{k-1}}{k^2\binom{2k}{k}} = \frac{2\,arcsin^2\left(\sqrt{x}\right)}{x}$$

$$\sum_{k=1}^{\infty}\frac{4^k}{k^2\binom{2k}{k}}\left(\frac{H_k}{k^2} - \frac{\zeta(2) - H_k^{(2)}}{k}\right) = \underbrace{\int_0^1 \frac{2\,arcsin^2\left(\sqrt{x}\right)}{x} ln(x)\,ln(1-x)\,dx}_{x \to sin^2(x)}$$

$$\sum_{k=1}^{\infty}\frac{4^k H_k}{k^4\binom{2k}{k}} - \zeta(2)\underbrace{\sum_{k=1}^{\infty}\frac{4^k}{k^3\binom{2k}{k}}}_{S_1} + \underbrace{\sum_{k=1}^{\infty}\frac{4^k H_k^{(2)}}{k^3\binom{2k}{k}}}_{S_2} = 16\underbrace{\int_0^1 x^2\,cot(x)\,ln(sin(x))\,ln(cos\,(x))\,dx}_{I_{14}}$$

$$\boxed{\sum_{k=1}^{\infty}\frac{4^k H_k}{k^4\binom{2k}{k}} = 16(I_{14}) + \zeta(2)(S_1) - (S_2)}$$

$$\sum_{k=1}^{\infty}\frac{4^k H_k}{k^4\binom{2k}{k}} = 16(I_{14}) + \zeta(2)\left(6\,ln(2)\,\zeta(2) - \frac{7}{2}\zeta(3)\right)$$

$$-\left(\frac{31}{4}\zeta(5) + \frac{3}{2}\zeta(2)\zeta(3) - 16Li_5\left(\frac{1}{2}\right) + \frac{8}{3}ln^3(2)\zeta(2) + \frac{2}{15}ln^5(2) - 2\,ln(2)\,\zeta(4)\right)$$

$$= -\frac{31}{2}\zeta(5) + 3\zeta(2)\zeta(3) + 16Li_5\left(\frac{1}{2}\right) + 2\,ln(2)\,\zeta(4) + \frac{16}{3}ln^3(2)\,\zeta(2) - \frac{2}{15}ln^5(2)$$

$$\boxed{\boxed{\sum_{k=1}^{\infty}\frac{4^k H_k}{k^4\binom{2k}{k}} = -\frac{31}{2}\zeta(5) + 3\zeta(2)\zeta(3) + 16Li_5\left(\frac{1}{2}\right) + 2\,ln(2)\,\zeta(4) + \frac{16}{3}ln^3(2)\,\zeta(2) - \frac{2}{15}ln^5(2)}}$$

REFERENCES

[1]. A. S. Olaikhan, An Introduction to the Harmonic Series and Logarithmic Integrals: For High School Students Up To Researchers, 2nd ed., self-published, Phoenix, AZ, 2023.

[2]. C. I. Vˇalean, (Almost) Impossible Integrals, Sums, and Series, Springer, Cham, 2019.