c-Distance and common fixed-point theorem in cone metric spaces

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Abstract- We introduced the notion of c-distance in a cone metric space and used it to establish a new common fixed point theorem by using the distance. This result generalizes several known fixed point theorems and has potential applications in functional analysis and optimization problems

Keyword: cone metric space, common fixed point, c-distance.

1. INTRODUCTION

Since Huang and Zhang [1] introduced the concept of a cone metric space, numerous researchers have established fixed point theorems in both normal and non-normal cone metric spaces (see [2-17] and related references). In the study of Shenghua Wanga and Baohua Guoa [20] they introduce a novel concept called c-distance in cone metric spaces, which serves as a cone-based counterpart to the ω-distance proposed by Kada et al. [18]. Utilizing this new distance function, we prove a common fixed point theorem within the framework of cone metric spaces.

2. PRELIMINARIES

Definition 2.1: Let E be a real Banach Spaces. A subset P of E is called a cone if and only if

a. P is closed ,non empty and $p \neq 0$
b. $a, b \in R, a, b \geq 0$ and $x, y \in P$ imply $ax + by \in P$
c. $P \cap (-P) = \{0\}$

Given a cone $P \subset E$ we define the partial ordering $\preccurlyeq$ with respect to P by

$$x \preccurlyeq y \text{ if and only if } y - x \in P.$$

We write $x \prec y$ to denote that $x \preccurlyeq y$ but $x \neq y$, while $x << y$ will stand for $y - x \in int.p$

Definition 2.2: Let X be a nonempty set. Suppose the mapping $d: X \times X \to E$ satiafies the following condition:

(i). $0 \prec d(x, y)\ \forall\, x, y \in X$ with $x \neq y$ and $d(x, y) = 0 \Leftrightarrow x = y$
(ii). $d(x, y) = d(y, x)\ ,\ \forall x, y \in X$
(iii). $d(x, y) \preccurlyeq d(x, y) + d(x, y),\ \forall x, y \in X$

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definition 2.3: Let (X, d) be a cone metric space. Let $\{x_n\}$ be a sequence in X and $x \in X$.

(i). For all $c \in E$ with $0 \ll c$, if there exists a positive integer N such that
$$d(x_n, x) \ll c \text{ for all } n > N,$$
then $\{x_n\}$ is said to be convergent to x and the point x is the limit of $\{x_n\}$.
We denote this by $x_n \to$ x.

(ii). For all $c \in E$ with $0 \ll c$,ifthere exists a positive integer N such that

$$d(x_n, x_m\} \ll c..\text{ for all } m,n > N,$$

then $\{x_n\}$ is called a Cauchy sequence in X.

(iii). A cone metric space (X,d) is said to be complete if every Cauchy sequence in X is convergent.

Lemma 2.4:[1] Let (X,d) be a cone metric space, P be a normal cone with normal constant K. Let $\{x_n\}$ be a sequence in X. If $\{x_n\}$ converges to x and $\{x_n\}$ converges to y, then $x = y$. That is the limit of $\{x_n\}$ is unique.

Definition 2.5:[20] Let (X,d) be a cone metric space. Then the mapping $q: X \times X \to E$ is called a $c - distance$ on X if the following are satisfied:

(q_1). $0 \preccurlyeq q(x,y)\ \forall\, x,y \in X$

(q_2). $q(x,z) \preccurlyeq q(y,x) + q(y,z),\ \forall\, x,y,z \in X$

(q_3). For all $x \in X$, if $q(x,y_n) \preccurlyeq u$ for some $u = u_x \in P$ and all $n \geq 1$, then $q(x,y) \preccurlyeq u$ whenever $\{y_n\}$ is a sequence in X converging to a point $y \in X$.

(q_4). For all $c \in E$ with $0 \ll c$, there exists $e \in E$ with $0 \ll e$ such that $q(z,x) \ll e$ $q(z,y) \ll e$ imply $d(x,y) \ll c$.

Example 2.6 [20]. Let (X,d) be a cone metric space and P be a normal cone. Put $q(x,y) = d(x,y)$ for all $x,y \in X$. Then q is a $c - distance$. In fact, (q_1) and (q_2) are immediate. Lemma 2.4, shows that (q_3) holds. Let $c \in E$ with $0 \ll c$ be given and put $e = \frac{c}{2}$. Suppose that $q(z,x) \ll e$ and $q(z,y) \ll e$.Then $d(x,y) = q(x,y) \preccurlyeq q(x,z) + q(z,y) \ll e + e = c$.This shows that q satisfies (q_4) and hence q is a $c - distance$.

Lemma 2.5[20] Let (X,d) be a cone metric space and q be a $c - distance$ on X. Let $\{x_n\}$ be a sequence in X. Suppose that $\{u_n\}$ is a sequence in P converging to 0. If $q(x_n, x_m) \preccurlyeq u_n$ for all $m > n$, then $\{x_n\}$ is a Cauchy sequence in X.

Proof. Let $c \in E$ with $0 \ll c$.Thenthereexists $\delta > 0$ such that $c - x \in intP$ for any $x \in P$ with $\|x\| < \delta$. Since $\{u_n\}$ converges to 0, there exists a positive integer N such that

$$\|u_n\| < \delta \text{ for all } n \geq N \text{ and}$$

so $c - u_n \in intP$, i.e., $u_n \ll c$ for all $n \geq N$. By the hypothesis $q(x_n, x_m) \preccurlyeq u_n \ll$ c for all $m > n$ with $n \geq N$.This implies that

$q(x_n, x_{n+1}) \preccurlyeq u_n \ll$ c and $q(x_n, x_{m+1}) \preccurlyeq u_n \ll$ c for all $m > n$ with $n > N$.

From(q_4) with $e = c$ it follows that $q(x_{n+1}, x_{m+1}) \ll$ c for all $m > n$ with $n > N$. By the definition of Cauchy sequence, we conclude that $\{x_n\}$ is a Cauchy sequence. This completes the proof.

3. Main Results

Theorem 3.1: Let X a non empty set, (X,d) be a Cone metric space over a Banach Space E with a normal cone $P \subset E$, and $F,G: X \to X$ be mappings. Assume that the following conditions satisfying:

(i). The cone P is normal with normal constant N.

(ii). There exist constants $\alpha \geq 0, \beta \geq 0$ such that $\frac{(2\alpha+\beta)}{(1-\alpha-\beta)} \in (0,1)$

(iii). The mapping $F, and\ G$ satisfy a generalized Banach- Kannan Type contraction for all $x,y \in X$;

$$q\big(F(x),F(y)\big) \preccurlyeq \alpha[q(Gx,Fy) + q(Fx,Gx)] + \beta\left\{\frac{q(Gx,Fy)}{1 + \{q(Fx,Gy)\}^2}\right\} \quad ...(3.1)$$

Suppose that the range of G contains the range of F and G(X) is a complete subspace of X. if F and G satisfy

$$inf\{\|q(Fx,y)\| + \|q(Gx,y)\| + \|q(Gx,Fx)\| : x \in X\} > 0$$

For all $y \in X$ with $y \neq Fy$ or $y \neq Gy$, then F and G have a common fixed point in X.
Where $\preccurlyeq$ is the partial ordering induced by the cone P,
Proof:
Let $x_0 \in X$ be an arbitrary point. Since $F(X) \subset G(X)$, there exists an $x_1 \in X$ such that

$$Fx_0 = Gx_1$$

By induction, a sequence x_n can chosen such that
$x_{n-1} = Fx_n = Gx_{n+1}$, $n = 0,1,2,...$, By (3.1) for any natural number n, we have

$$\begin{aligned}
q(Gx_n, Gx_{n+1}) &= q(Fx_{n-1}, Fx_n) \\
&\preccurlyeq \alpha[q(Gx_{n-1}, Fx_n) + q(Fx_{n-1}, Gx_{n-1})] + \beta\left\{\frac{q(Gx_{n-1}, Fx_n)}{1 + \{q(Fx_{n-1}, Gx_n)\}^2}\right\} \\
&\preccurlyeq \alpha[q(Gx_{n-1}, Gx_{n+1}) + q(Gx_n, Gx_{n-1})] + \beta\left\{\frac{q(Gx_{n-1}, Gx_{n+1})}{1 + \{q(Gx_n, Gx_n)\}^2}\right\} \\
&\preccurlyeq \alpha[q(Gx_{n-1}, Gx_n) + q(Gx_n, Gx_{n+1}) + q(Gx_n, Gx_{n-1})] \\
&\qquad + \beta\left\{\frac{q(Gx_{n-1}, Gx_n) + q(Gx_n, Gx_{n+1})}{1}\right\} \\
&\preccurlyeq \alpha[2q(Gx_{n-1}, Gx_n)] + \alpha q(Gx_n, Gx_{n+1}) + \beta\{q(Gx_{n-1}, Gx_n) + q(Gx_n, Gx_{n+1})\} \\
&\preccurlyeq (2\alpha + \beta)q(Gx_{n-1}, Gx_n) + (\alpha + \beta)q(Gx_n, Gx_{n+1})
\end{aligned}$$

$$q(Gx_n, Gx_{n+1}) \preccurlyeq (2\alpha + \beta)q(Gx_{n-1}, Gx_n) + (\alpha + \beta)q(Gx_n, Gx_{n+1})$$

$$(1 - \alpha - \beta)q(Gx_n, Gx_{n+1}) \preccurlyeq (2\alpha + \beta)q(Gx_{n-1}, Gx_n)$$

$$q(Gx_n, Gx_{n+1}) \preccurlyeq \frac{(2\alpha + \beta)}{(1 - \alpha - \beta)} q(Gx_{n-1}, Gx_n)$$

So,

$$q(Gx_n, Gx_{n+1}) \preccurlyeq k\, q(Gx_{n-1}, Gx_n) \quad , \quad n = 1,2,3 \ ...$$

Where $k = \frac{(2\alpha+\beta)}{(1-\alpha-\beta)} \in (0,1)$ By induction, we get

$$q(Gx_n, Gx_{n+1}) \preccurlyeq k^n\, q(Gx_0, Gx_1) \qquad ...(3.2)$$

Let m, n with $m > n$ be arbitrary integers. From (3.2) and (q_2) it follows that

$$q(Gx_n, Gx_m) \preccurlyeq q(Gx_n, Gx_{n+1}) + q(Gx_{n+1}, Gx_{n+2}) + \ .... + q(Gx_{m-1}, Gx_m)$$

$$q(Gx_n, Gx_m) \preccurlyeq k^n\, q(Gx_0, Gx_1) + k^{n+1}\, q(Gx_0, Gx_1) + \ .... + k^{m-1}\, q(Gx_0, Gx_1)$$

$$q(Gx_n, Gx_m) \preccurlyeq \{k^n + k^{n+1} + \ .... + k^{m-1}\}\, q(Gx_0, Gx_1)$$

$$q(Gx_n, Gx_m) \preccurlyeq \sum_{i=n}^{m-1} k^i\, q(Gx_0, Gx_1)$$

$$q(Gx_n, Gx_m) \preccurlyeq \sum_{i=n}^{\infty} k^i\, q(Gx_0, Gx_1)$$

$$q(Gx_n, Gx_m) \preccurlyeq \frac{k^n}{1-k}\, q(Gx_0, Gx_1) \qquad ...(3.3)$$

By using Lemma 2,5, we conclude that the sequence $\{Gx_n\}$ is a Cauchy sequence in X.
Since $G(X)$ is complete, there exists some point $y \in G(X)$such that $Gx_n \to y, n \to \infty$. By (3.3) and q_3

$$q(Gx_n, y) \preccurlyeq \frac{k^n}{1-k}\, q(Gx_0, Gx_1), \quad n = 0,1,2, ... \qquad ...(3.4)$$

Since P is a normal cone with normal constant K, From (3.4) it follows that

$$\|q(Gx_n, y)\| \leq \frac{Kk^n}{1-k}\, \|q(Gx_0, Gx_1)\|, \quad n = 0,1,2, ... \qquad ...(3.5)$$

From (3.3) we have

$$\|q(Gx_n, Gx_m)\| \leq \frac{Kk^n}{1-k}\, \|q(Gx_0, Gx_1)\|, \quad n = 0,1,2, ...$$

For all $m > n$. In particular, we have

$$\|q(Gx_n, Gx_{n+1})\| \leq \frac{Kk^n}{1-k} \|q(Gx_0, Gx_1)\|, \quad n = 0,1,2, \ldots$$

For all $n = 0,1, \ldots$.
Suppose that $y \neq Gy$ or $y \neq Fy$. Then by Hypothesis (3.5) and (3.6) we have

$$0 < inf\{\|q(Fx, y)\| + \|q(Gx, y)\| + \|q(Gx, Fx)\|: x \in X\}$$
$$\leq inf\{\|q(Fx_n, y)\| + \|q(Gx_n, y)\| + \|q(Gx_n, Fx_n)\|: n \geq 1\}$$
$$= inf\{\|q(Gx_{n+1}, y)\| + \|q(Gx_n, y)\| + \|q(Gx_n, Gx_{n+1})\|: n \geq 1\}$$
$$\leq inf\left\{\frac{Kk^n}{1-k} \|q(Gx_1, Gx_0)\| + \|q(Gx_1, Gx_0)\| + \|q(Gx_1, Gx_0)\|: n \geq 1\right\} = 0$$

This is a contradiction. Hence, $y = Gy = Fy$. This completes the proof.

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