Nirmala Uphill Indices of Graphs

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Abstract: **In this study, we introduce the Nirmala uphill and modified Nirmala uphill indices and their corresponding exponentials of a graph. Furthermore, we compute these indices for some standard graphs, wheel graphs, gear graphs and helm graphs.**

Key words: Graph, modified Nirmala index, Nirmala index.

I. INTRODUCTION

In this paper, G denotes a finite, simple, connected graph, $V(G)$ and $E(G)$ denote the vertex set and edge set of G. The degree $d_G(u)$ of a vertex u is the number of vertices adjacent to u.

The uphill domination is introduced in (Deering, 2013).

A u-v path P in G is a sequence of vertices in G, starting with u and ending at v, such that consecutive vertices in P are adjacent, and no vertex is repeated. A path $\pi = v_1, v_2, \ldots v_{k+1}$ in G is a uphill path if for every i, $1 \le i \le k$, $d_G(v_i) \le d_G(v_{i+1})$.

A vertex v is uphill dominates a vertex u if there exists an uphill path originated from u to v. The uphill neighborhood of a vertex v is denoted by $N_{up}(v)$ and defined as: $N_{up}(v) = \{u: v \text{ uphill dominates } u\}$. The uphill degree $d_{up}(v)$ of a vertex v is the number of uphill neighbors of v, see (Kulli, 2025).

Recently, the F-uphill index was studied in (Kulli, 2025) and the Sombor uphill index was studied in (Kulli, 2025).

The Nirmala index was introduced in (Kulli, 2021) and it is defined as

$$N(G) = \sum_{uv \in E(G)} \sqrt{d_G(u) + d_G(v)}.$$

Motivated by the definition of Nirmala index, we introduce the Nirmala uphill index of a graph and it is defined as

$$NU(G) = \sum_{uv \in E(G)} \sqrt{d_{up}(u) + d_{up}(v)}.$$

Considering the Nirmala uphill index, we introduce the Nirmala uphill exponential of a graph G and defined it as

$$NU(G,x) = \sum_{uv \in E(G)} x^{\sqrt{d_{up}(u) + d_{up}(v)}}.$$

We define the modified Nirmala uphill index of a graph G as

$$^{m}NU(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_{up}(u) + d_{up}(v)}}.$$

Considering the modified Nirmala uphill index, we introduce the modified Nirmala uphill exponential of a graph G and defined it as

$$^{m}NU(G,x) = \sum_{uv \in E(G)} x^{\frac{1}{\sqrt{d_{up}(u) + d_{up}(v)}}}.$$

Recently, the Nirmala leap index was studied in (Majhi et al, 2023), the Nirmala index was studied in (Kulli et al, 2023) and the delta Nirmala index was studied in (Kulli, 2023).

In this paper, the Nirmala uphill index, modified Nirmala uphill index and their corresponding exponentials of certain graphs are computed.

II. RESULTS FOR SOME STANDARD GRAPHS

Proposition 1. Let G be r-regular with n vertices and $r \geq 2$. Then

$$NU(G) = \frac{nr\sqrt{n-1}}{\sqrt{2}}.$$

Proof: Let G be an r-regular graph with n vertices and $r \geq 2$ and $\frac{nr}{2}$ edges. Then $d_{up}(v) = n-1$ for every v in G.

$$NU(G) = \sum_{uv \in E(G)} \sqrt{d_{up}(u) + d_{up}(v)}$$

$$= \frac{nr}{2}\sqrt{(n-1)+(n-1)}$$

$$= \frac{nr\sqrt{n-1}}{\sqrt{2}}.$$

Corollary 1.1. Let C_n be a cycle with $n \geq 3$ vertices. Then

$$NU(C_n) = n\sqrt{2}\sqrt{n-1}.$$

Corollary 1.2. Let K_n be a complete graph with $n \geq 3$ vertices. Then

$$NU(K_n) = \frac{n(n-1)\sqrt{n-1}}{\sqrt{2}}.$$

Proposition 2. Let G be r-regular with n vertices and $r \geq 2$. Then

$$NU(G,x) = \frac{nr}{2} x^{\sqrt{2(n-1)}}.$$

Proof: Let G be an r-regular graph with n vertices and $r \geq 2$ and $\frac{nr}{2}$ edges. Then $d_{up}(v) = n-1$ for every v in G.

$$NU(G,x) = \sum_{uv \in E(G)} x^{\sqrt{d_{up}(u) + d_{up}(v)}}$$

$$= \frac{nr}{2} x^{\sqrt{(n-1)+(n-1)}}$$

$$= \frac{nr}{2} x^{\sqrt{2(n-1)}}.$$

Corollary 2.1. Let C_n be a cycle with $n \geq 3$ vertices. Then

$$NU(C_n, x) = nx^{\sqrt{2(n-1)}}.$$

Corollary 2.2. Let K_n be a complete graph with $n \geq 3$ vertices. Then

$$NU(K_n, x) = \frac{n(n-1)}{2} x^{\sqrt{2(n-1)}}.$$

Proposition 3. Let G be r-regular with n vertices and $r \geq 2$. Then

$$^{m}UN(G) = \frac{nr}{2\sqrt{2}(n-1)}.$$

Proof: Let G be an r-regular graph with n vertices and $r \geq 2$ and $\frac{nr}{2}$ edges. Then $d_{up}(v) = n-1$ for every v in G.

$$^{m}NU(G) = \sum_{uv \in E(G)} \frac{1}{\sqrt{d_{up}(u) + d_{up}(v)}}$$

$$= \frac{nr}{2} \frac{1}{\sqrt{(n-1)+(n-1)}}$$

$$= \frac{nr}{2\sqrt{2(n-1)}}.$$

Corollary 3.1. Let C_n be a cycle with $n \geq 3$ vertices. Then

$$^{m}NU(C_n) = \frac{n}{\sqrt{2(n-1)}}.$$

Corollary 3.2. Let K_n be a complete graph with $n \geq 3$ vertices. Then

$$^{m}NU(K_n) = \frac{n\sqrt{(n-1)}}{2\sqrt{2}}.$$

Proposition 4. Let G be r-regular with n vertices and $r \geq 2$. Then

$$^{m}NU(G) = \frac{nr}{2} x^{\frac{1}{\sqrt{2(n-1)}}}.$$

Proof: Let G be an r-regular graph with n vertices and $r \geq 2$ and $\frac{nr}{2}$ edges. Then $d_{up}(v) = n-1$ for every v in G.

$$^{m}NU(G,x)=\sum_{uv\in E(G)} x^{\frac{1}{\sqrt{d_{up}(u)+d_{up}(v)}}}$$

$$=\frac{nr}{2}x^{\frac{1}{\sqrt{(n-1)+(n-1)}}}$$

$$=\frac{nr}{2}x^{\frac{1}{\sqrt{2(n-1)}}}.$$

Corollary 4.1. Let C_n be a cycle with $n\geq 3$ vertices. Then

$$^{m}NU(C_n,x)=nx^{\frac{1}{\sqrt{2(n-1)}}}.$$

Corollary 4.2. Let K_n be a complete graph with $n\geq 3$ vertices. Then

$$^{m}NU(K_n,x)=\frac{n(n-1)}{2}x^{\frac{1}{\sqrt{2(n-1)}}}$$

Proposition 5. Let P be a path with $n\geq3$ vertices. Then

$$NU(P)=2\sqrt{2n-5}+(n-3)\sqrt{2(n-3)}.$$

Proof: Let P be a path with $n\geq3$ vertices. Clearly, P has two types of edges based on the uphill degree of end vertices of each edge as follows:

$E_1=\{uv\in E(P)\mid d_{up}(u)=n-2,\ d_{up}(v)=n-3\}$, $|E_1|=2$.

$E_2=\{uv\in E(P)\mid d_{up}(u)=d_{up}(v)=n-3\}$, $|E_2|=n-3$.

$$NU(P)=\sum_{uv\in E(P)}\sqrt{d_{up}(u)+d_{up}(v)}$$

$$=2\sqrt{(n-2)+(n-3)}+(n-3)\sqrt{(n-3)+(n-3)}$$

$$=2\sqrt{2n-5}+(n-3)\sqrt{2(n-3)}.$$

Proposition 6. Let P be a path with $n\geq3$ vertices. Then

$$NU(P,x)=2x^{\sqrt{2n-5}}+(n-3)x^{\sqrt{2(n-3)}}.$$

Proof: We obtain

$$NU(P,x)=\sum_{uv\in E(P)}x^{\sqrt{d_{up}(u)+d_{up}(v)}}$$

$$=2x^{\sqrt{(n-2)+(n-3)}}+(n-3)x^{\sqrt{(n-3)+(n-3)}}$$

$$=2x^{\sqrt{2n-5}}+(n-3)x^{\sqrt{2(n-3)}}.$$

Proposition 7. Let P be a path with $n\geq3$ vertices. Then

$$^{m}NU(P)=\frac{2}{\sqrt{2n-5}}+\frac{\sqrt{(n-3)}}{\sqrt{2}}.$$

Proof: We obtain

$$^{m}NU(P)=\sum_{uv\in E(P)}\frac{1}{\sqrt{d_{up}(u)+d_{up}(v)}}$$

$$=\frac{2}{\sqrt{(n-2)+(n-3)}}+\frac{(n-3)}{\sqrt{(n-3)+(n-3)}}$$

$$=\frac{2}{\sqrt{2n-5}}+\frac{\sqrt{(n-3)}}{\sqrt{2}}.$$

Proposition 8. Let P be a path with $n\geq3$ vertices. Then

$$^{m}NU(P,x)=2x^{\frac{1}{\sqrt{2n-5}}}+(n-3)x^{\frac{1}{\sqrt{2(n-3)}}}.$$

Proof: We obtain

$$^{m}NU(P,x)=\sum_{uv\in E(P)}x^{\frac{1}{\sqrt{d_{up}(u)+d_{up}(v)}}}$$

$$=2x^{\frac{1}{\sqrt{(n-2)+(n-3)}}}+(n-3)x^{\frac{1}{\sqrt{(n-3)+(n-3)}}}$$

$$=2x^{\frac{1}{\sqrt{2n-5}}}+(n-3)x^{\frac{1}{\sqrt{2(n-3)}}}.$$

III. RESULTS FOR WHEEL GRAPHS

Let W_n be a wheel with $n+1$ vertices and $2n$ edges, $n\geq4$. Then there are two types of edges based on the uphill degree of end vertices of each edge as follows:

$E_1=\{uv\in E(W_n)\mid d_{up}(u)=0,\ d_{up}(v)=n\}$, $|E_1|=n$.

$E_2=\{uv\in E(W_n)\mid d_{up}(u)=d_{up}(v)=n\}$, $|E_2|=n$.

Theorem 1. Let W_n be a wheel with n+1vertices and 2n edges, n≥4. Then

$$NU(W_n) = (1+\sqrt{2})n\sqrt{n}.$$

Proof. We deduce

$$NU(W_n) = \sum_{uv\in E(W_n)} \sqrt{d_{up}(u)+d_{up}(v)}$$
$$= n\sqrt{0+n} + n\sqrt{n+n}$$
$$= (1+\sqrt{2})n\sqrt{n}.$$

Theorem 2. Let W_n be a wheel with n+1vertices and 2n edges, n≥4. Then

$$NU(W_n,x) = nx^{\sqrt{n}} + nx^{\sqrt{2n}}.$$

Proof. We obtain

$$NU(W_n,x) = \sum_{uv\in E(W_n)} x^{\sqrt{d_{dn}(u)+d_{dn}(v)}}$$
$$= nx^{\sqrt{0+n}} + nx^{\sqrt{n+n}}$$
$$= nx^{\sqrt{n}} + nx^{\sqrt{2n}}.$$

Theorem 3. Let W_n be a wheel with n+1vertices and 2n edges, n≥4. Then

$${}^{m}NU(W_n) = \left(1+\frac{1}{\sqrt{2}}\right)\sqrt{n}.$$

Proof. We deduce

$${}^{m}NU(W_n) = \sum_{uv\in E(W_n)} \frac{1}{\sqrt{d_{up}(u)+d_{up}(v)}}$$
$$= \frac{n}{\sqrt{0+n}} + \frac{n}{\sqrt{n+n}}$$
$$= \left(1+\frac{1}{\sqrt{2}}\right)\sqrt{n}.$$

Theorem 4. Let W_n be a wheel with n+1vertices and 2n edges, n≥4. Then

$${}^{m}UN(W_n,x) = nx^{\frac{1}{n}} + nx^{\frac{1}{\sqrt{2n}}}.$$

Proof. We obtain

$${}^{m}NU(W_n,x) = \sum_{uv\in E(W_n)} x^{\frac{1}{\sqrt{d_{up}(u)+d_{up}(v)}}}$$
$$= nx^{\frac{1}{\sqrt{0+n}}} + nx^{\frac{1}{\sqrt{n+n}}}$$
$$= nx^{\frac{1}{\sqrt{n}}} + nx^{\frac{1}{\sqrt{2n}}}.$$

IV. RESULTS FOR DEAR GRAPHS

A bipartite wheel graph is a graph obtained from W_n with n+1 vertices adding a vertex between each pair of adjacent rim vertices and this graph is denoted by G_n and also called as a gear graph. Clearly, $|V(G_n)| = 2n+1$ and $|E(G_n)| = 3n$. A gear graph G_n is depicted in Figure 1.

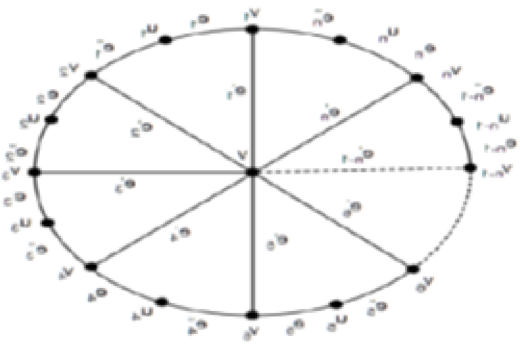

Figure 1. Gear graph G_n

Let G_n be a gear graph with 3n edges, n≥4. Then there are two types of edges based on the uphill degree of end vertices of each edge as follows:

$E_1 = \{u \in E(G_n) \mid d_{up}(u) = 1, d_{up}(v) = 0\}, | E_1 | = n.$
$E_2 = \{u \in E(G_n) \mid d_{up}(u) = 1, , d_{up}(v) = 3\}, | E_2 | = 2n.$

Theorem 5. Let G_n be a gear graph with 2n+1vertices, n≥3. Then the uphill Sombor index of G_n is

$$NU(G_n) = 5n.$$

Proof: We deduce

$$NU(G_n) = \sum_{uv\in E(G_n)} \sqrt{d_{up}(u)+d_{up}(v)}$$
$$= n\sqrt{1+0} + 2n\sqrt{1+3}$$
$$= 5n.$$

Theorem 6. Let G_n be a gear graph with 2n+1vertices, n≥3. Then the uphill Sombor exponential of G_n is

$$NU(G_n,x) = nx^1 + 2nx^2.$$

Proof: We deduce

$$NU(G_n,x) = \sum_{uv\in E(G_n)} x^{\sqrt{d_{up}(u)+d_{up}(v)}}$$
$$= nx^{\sqrt{1+0}} + 2nx^{\sqrt{1+3}}$$
$$= nx^1 + 2nx^2.$$

Theorem 7. Let G_n be a gear graph with $2n+1$vertices, $n\geq 3$. Then the modified uphill Sombor index of G_n is

$$^{m}NU(G_n) = 2n.$$

Proof: We deduce

$$^{m}NU(G_n) = \sum_{uv\in E(G_n)} \frac{1}{\sqrt{d_{up}(u)+d_{up}(v)}}$$

$$= \frac{n}{\sqrt{1+0}} + \frac{2n}{\sqrt{1+3}}$$

$$= 2n.$$

Theorem 8. Let G_n be a gear graph with $2n+1$vertices, $n\geq 3$. Then the modified uphill Sombor exponential of G_n is

$$^{m}NU(G_n,x) = nx^{1} + 2nx^{\frac{1}{2}}.$$

Proof: We deduce

$$^{m}NU(G_n,x) = \sum_{uv\in E(G_n)} x^{\frac{1}{\sqrt{d_{up}(u)+d_{up}(v)}}}$$

$$= nx^{\frac{1}{\sqrt{1+0}}} + 2nx^{\frac{1}{\sqrt{1+3}}}$$

$$= nx^{1} + 2nx^{\frac{1}{2}}.$$

V. RESULTS FOR HELM GRAPHS

The helm graph H_n is a graph obtained from W_n (with $n+1$ vertices) by attaching an end edge to each rim vertex of W_n. Clearly, $|V(H_n)| = 2n+1$ and $|E(H_n)| = 3n$. A graph H_n is shown in Figure 2.

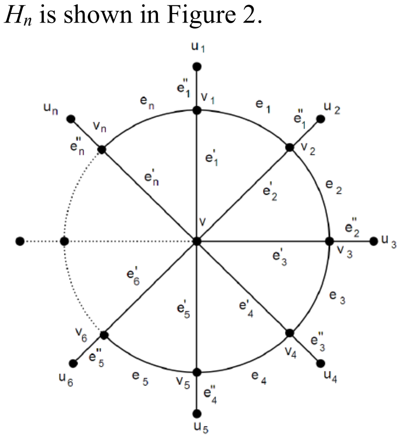

Figure 2. Helm graph H_n

Let H_n be a helm graph with $3n$ edges, $n\geq 3$. Then H_n has three types of the uphill degree of edges as follows:

$E_1 = \{uv \in E(H_n) \mid d_{up}(u) = n+1, d_{up}(v) = n\}$. $|E_1| = n$.

$E_2 = \{uv \in E(H_n) \mid d_{up}(u) = d_{up}(v) = n\}$. $|E_2| = n$.

$E_3 = \{uv \in E(H_n) \mid d_{up}(u) = n, d_{up}(v) = 0\}$. $|E_3| = n$.

Theorem 9. Let H_n be a helm graph with $2n+1$ vertices, $n\geq 3$. Then the uphill Sombor index of H_n is

$$NU(H_n) = n\sqrt{2n+1} + (\sqrt{2}+1)n\sqrt{n}.$$

Proof: We obtain

$$NU(H_n) = \sum_{uv\in E(H_n)} \sqrt{d_{up}(u)+d_{up}(v)}$$

$$= n\sqrt{(n+1)+n} + n\sqrt{n+n} + n\sqrt{n+0}$$

$$= n\sqrt{2n+1} + (\sqrt{2}+1)n\sqrt{n}.$$

Theorem 10. Let H_n be a helm graph with $2n+1$vertices, $n\geq 3$. Then the uphill Sombor exponential of H_n is

$$NU(H_n,x) = nx^{\sqrt{2n+1}} + nx^{\sqrt{2n}} + nx^{\sqrt{n}}.$$

Proof: We deduce

$$NU(H_n,x) = \sum_{uv\in E(H_n)} x^{\sqrt{d_{up}(u)+d_{up}(v)}}$$

$$= nx^{\sqrt{(n+1)+n}} + nx^{\sqrt{n+n}} + nx^{\sqrt{n+0}}$$

$$= nx^{\sqrt{2n+1}} + nx^{\sqrt{2n}} + nx^{\sqrt{n}}.$$

Theorem 11. Let H_n be a helm graph with $2n+1$vertices, $n\geq 3$. Then the modified uphill Sombor index of H_n is

$$^{m}NU(H_n) = \frac{n}{\sqrt{2n+1}} + \left(\frac{1}{\sqrt{2}} + 1\right)\sqrt{n}.$$

Proof: We deduce

$$^{m}NU(H_n) = \sum_{uv\in E(H_n)} \frac{1}{\sqrt{d_{up}(u)+d_{up}(v)}}$$

$$= \frac{n}{\sqrt{(n+1)+n}} + \frac{n}{\sqrt{n+n}} + \frac{n}{\sqrt{n+0}}$$

$$= \frac{n}{\sqrt{2n+1}} + \left(\frac{1}{\sqrt{2}} + 1\right)\sqrt{n}.$$

Theorem 12. Let H_n be a helm graph with $2n+1$vertices, $n\geq 3$. Then the modified uphill Sombor exponential of H_n is

$$ {}^{m}NU(H_n,x) = nx^{\frac{1}{\sqrt{2n+1}}} + nx^{\frac{1}{\sqrt{2n}}} + nx^{\frac{1}{\sqrt{n}}}. $$

Proof: We deduce

$$ {}^{m}NU(H_n,x) = \sum_{uv\in E(H_n)} x^{\frac{1}{\sqrt{d_{up}(u)+d_{up}(v)}}} $$

$$ = nx^{\frac{1}{\sqrt{(n+1)+n}}} + nx^{\frac{1}{\sqrt{n+n}}} + nx^{\frac{1}{\sqrt{n+0}}} $$

$$ = nx^{\frac{1}{\sqrt{2n+1}}} + nx^{\frac{1}{\sqrt{2n}}} + nx^{\frac{1}{\sqrt{n}}}. $$

VI. CONCLUSION

In this study, the Nirmala uphill and modified Nirmala uphill indices are introduced. Also the Nirmala uphill index, modified Nirmala uphill index and their corresponding exponentials of certain graphs are determined.

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