Transient Thermoelastic Response of a Thick Circular Plate Under Internal Heat Generation

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Abstract

This study investigates the transient thermoelastic behavior of a thick, isotropic circular plate subject to internal heat generation. The plate is modeled as a homogeneous medium with temperature-dependent heat sources and radiation-type boundary conditions on the cylindrical edge and flat surfaces. A coupled system of unsteady heat conduction and thermoelastic deformation is analyzed using finite Hankel and Marchi–Fasulo integral transforms. Closed-form solutions for temperature distribution, displacement, and stress components are derived. The results are applied to a copper plate model, and parametric analysis is conducted to understand the influence of heat source intensity and boundary heat transfer coefficients on stress evolution.

Keywords

Transient thermoelasticity, Thick circular plate, Internal heat generation, Hankel transform, Marchi–Fasulo transform, Radiation boundary conditions

1. Introduction

The analysis of thermal stresses in solid structures is of critical importance in the design of systems exposed to transient heating conditions, such as those in nuclear reactors, aerospace vehicles, and high-speed rotating machinery. When subjected to internal heat generation, components like thick plates experience non-uniform temperature distributions that can lead to significant mechanical deformation and stress.

Research on the thermoelastic behavior of circular plates has attracted attention due to its relevance in engineering and applied sciences. The classical studies of Nowacki [1] and later developments by Marchi and Fasulo [2] laid the groundwork for incorporating radiation and internal sources in thermal models. More recently, authors such as Deshmukh [3,4], Ghadle [5], and Khan [6] have extended these models to include time-dependence and non-uniform sources.

This work extends previous models by focusing on the transient thermal deformation in a thick circular plate with a spatially and temporally varying internal heat source. The governing partial differential equations are solved using integral transform techniques, and results are interpreted both analytically and numerically. Particular emphasis is given to the analysis of displacement fields and stress components, which are essential for structural integrity assessments.

2. Mathematical Formulation

Consider a thick circular plate exposed to a two-dimensional axisymmetric and unsteady temperature field. The radius of the plate is b and its thickness is $2h$, defined by the region $0 \leq r \leq b,\ -h \leq z \leq h$. The axis of the plate is assumed to coincide with the z-axis. The material of the plate is considered homogeneous and isotropic, having constant thermal properties.

At time $t = 0$, the temperature distribution in the plate is given by $g(r, z)$. For time $t > 0$, heat dissipates from the circular edge $r = b$ into the surroundings, assumed to be at zero temperature. In addition, an extra sectional heat source $e^{-\omega t}\delta(r - r_0)$ is applied on the top surface $z = h$.

Let $\theta(r, z, t)$ denote the temperature function. The governing heat conduction equation with internal heat generation is given by:

$$\frac{\partial^2\theta}{\partial r^2} + \frac{1}{r}\frac{\partial\theta}{\partial r} + \frac{\partial^2\theta}{\partial z^2} + \frac{Q(r, z, t, \theta)}{\lambda} = \frac{1}{k}\frac{\partial\theta}{\partial t} \tag{2.1}$$

where $k = \frac{\lambda}{\rho C}$ is the thermal diffusivity, λ is the thermal conductivity, ρ is the density, and C is the specific heat. $Q(r, z, t, \theta)$ is the internal heat source function.

Using the principle of superposition as suggested in Özişik [?], the heat source is expressed as:

$$Q(r, z, t, \theta) = \Phi(r, z, t) + \psi(t)\theta(r, z, t) \tag{2.2}$$

We define the transformed variables as:

$$T(r, z, t) = \theta(r, z, t) \cdot e^{-\int_0^t \psi(\eta)d\eta} \tag{2.3}$$

$$\chi(r, z, t) = \Phi(r, z, t) \cdot e^{-\int_0^t \psi(\eta)d\eta} \tag{2.4}$$

For simplification, we assume:

$$\chi(r, z, t) = \frac{\delta(r - r_0)\delta(z - z_0)}{2\pi r_0}e^{-\omega t}, \quad \omega > 0 \tag{2.5}$$

Substituting equations (2.2)–(2.5) into (2.1), the modified heat conduction equation becomes:

$$\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{\partial^2 T}{\partial z^2} + \frac{\chi(r, z, t)}{\lambda} = \frac{1}{k}\frac{\partial T}{\partial t} \tag{2.6}$$

The initial condition is given by:

$$T(r, z, t)\Big|_{t=0} = g(r, z), \quad 0 \leq r \leq b,\ -h \leq z \leq h \tag{2.7}$$

Boundary conditions are as follows:

$$\left(\frac{\partial T}{\partial r} + h_1 T\right)\Bigg|_{r=b} = 0, \quad t > 0,\ -h \leq z \leq h \tag{2.8}$$

$$\left(\frac{\partial T}{\partial z} + h_2 T\right)\bigg|_{z=h} = e^{-\omega t}\delta(r - r_0), \quad t > 0,\ 0 \le r \le b \tag{2.9}$$

$$\left(\frac{\partial T}{\partial z} + h_3 T\right)\bigg|_{z=-h} = 0, \quad t > 0,\ 0 \le r \le b \tag{2.10}$$

Here h_1, h_2, and h_3 are the heat transfer coefficients, and $\delta(r - r_0)$ is the Dirac delta function indicating the location of the applied heat pulse.

Let u_r and u_z be the radial and axial displacement components, expressed in terms of Goodier's thermoelastic displacement potential ϕ and Michell's function M as:

$$u_r = \frac{\partial \phi}{\partial r} - \frac{\partial^2 M}{\partial r \partial z} \tag{2.11}$$

$$u_z = \frac{\partial \phi}{\partial z} + 2(1 - \nu)\nabla^2 M - \frac{\partial^2 M}{\partial z^2} \tag{2.12}$$

The potential ϕ satisfies the governing equation:

$$\nabla^2 \phi = K\theta \tag{2.13}$$

where

$$K = \frac{1 + \nu}{1 - \nu}\alpha \tag{2.14}$$

and the Laplacian operator in cylindrical coordinates is:

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}$$

The Michell's function M satisfies the biharmonic equation:

$$\nabla^2 \nabla^2 M = 0 \tag{2.15}$$

Using ϕ and M, the thermal stress components are given as:

$$\sigma_{rr} = 2G\left[\left(\frac{\partial^2 \phi}{\partial r^2} - K\theta\right) + \frac{\partial}{\partial z}\left(\nu \nabla^2 M - \frac{\partial^2 M}{\partial r^2}\right)\right] \tag{2.16}$$

$$\sigma_{\theta\theta} = 2G\left[\left(\frac{1}{r}\frac{\partial \phi}{\partial r} - K\theta\right) + \frac{\partial}{\partial z}\left(\nu \nabla^2 M - \frac{1}{r}\frac{\partial M}{\partial r}\right)\right] \tag{2.17}$$

$$\sigma_{zz} = 2G\left[\left(\frac{\partial^2 \phi}{\partial z^2} - K\theta\right) + \frac{\partial}{\partial z}\left((2 - \nu)\nabla^2 M - \frac{\partial^2 M}{\partial z^2}\right)\right] \tag{2.18}$$

$$\sigma_{rz} = 2G\left[\frac{\partial^2 \phi}{\partial r \partial z} + \frac{\partial}{\partial r}\left((1 - \nu)\nabla^2 M - \frac{\partial^2 M}{\partial z^2}\right)\right] \tag{2.19}$$

This system of equations (2.1) to (2.19) represents the mathematical formulation for the thermoelastic problem, describing the temperature distribution, displacements, and the resulting thermal stresses developed inside the thick circular plate due to internal heat generation and boundary heat transfer.

3. Application of Integral Transform Methods

To solve the heat conduction equation (2.6) with the initial and boundary conditions (2.7) to (2.10), we apply finite Hankel transform with respect to r and Marchi–Fasulo transform with respect to time t.

Let the finite Hankel transform of order zero be defined as:

$$\bar{T}_n(z,t) = \int_0^b T(r,z,t) J_0\left(\frac{\alpha_n r}{b}\right) r\,dr \tag{3.1}$$

where J_0 is the Bessel function of the first kind and order zero, and α_n is the nth positive root of the equation:

$$\alpha J_1(\alpha) + h_1 b J_0(\alpha) = 0 \tag{3.2}$$

Here, J_1 is the Bessel function of first kind and order one.

Taking Hankel transform of equation (2.6), we get:

$$\frac{\partial^2 \bar{T}_n}{\partial z^2} - \left(\frac{\alpha_n^2}{b^2}\right)\bar{T}_n + \frac{\bar{\chi}_n(z,t)}{\lambda} = \frac{1}{k}\frac{\partial \bar{T}_n}{\partial t} \tag{3.3}$$

where

$$\bar{\chi}_n(z,t) = \int_0^b \chi(r,z,t) J_0\left(\frac{\alpha_n r}{b}\right) r\,dr \tag{3.4}$$

Using equation (2.5), we get:

$$\bar{\chi}_n(z,t) = e^{-\omega t}\delta(z-z_0) J_0\left(\frac{\alpha_n r_0}{b}\right) \tag{3.5}$$

Now we apply the Marchi–Fasulo integral transform to equation (3.3). Let the Marchi–Fasulo transform of a function $f(t)$ be defined as:

$$f^*(\mu) = \int_0^\infty f(t) e^{-\mu t}\,dt \tag{3.6}$$

Taking the Marchi–Fasulo transform of (3.3), we obtain:

$$\frac{d^2 \bar{T}_n^*}{dz^2} - \left(\frac{\alpha_n^2}{b^2} + \frac{\mu}{k}\right)\bar{T}_n^* + \frac{1}{\lambda}\bar{\chi}_n^*(z,\mu) = 0 \tag{3.7}$$

Using the Laplace property of the exponential and delta function from (3.5):

$$\bar{\chi}_n^*(z,\mu) = \frac{J_0\left(\frac{\alpha_n r_0}{b}\right)}{\mu+\omega}\delta(z-z_0) \tag{3.8}$$

Substituting (3.8) into (3.7), we get:

$$\frac{d^2 \bar{T}_n^*}{dz^2} - \lambda_n^2 \bar{T}_n^* + \frac{J_0\left(\frac{\alpha_n r_0}{b}\right)}{\lambda(\mu+\omega)}\delta(z-z_0) = 0 \tag{3.9}$$

where

$$\lambda_n^2 = \frac{\alpha_n^2}{b^2} + \frac{\mu}{k} \tag{3.10}$$

Equation (3.9) is a second-order differential equation with a point source at $z = z_0$. The general solution of such an equation consists of the complementary function and a particular solution due to the delta function.

Let the general solution be written as:

$$\bar{T}_n^*(z,\mu) = \begin{cases} A_n e^{\lambda_n z}, & -h \le z < z_0 \\ B_n e^{-\lambda_n z}, & z_0 < z \le h \end{cases} \tag{3.11}$$

To determine constants A_n and B_n, we apply continuity at $z = z_0$ and integrate (3.9) across a small interval containing z_0. The conditions are:

$$\bar{T}_n^*(z_0^-,\mu) = \bar{T}_n^*(z_0^+,\mu) \tag{3.12}$$

$$\left.\frac{d\bar{T}_n^*}{dz}\right|_{z=z_0^+} - \left.\frac{d\bar{T}_n^*}{dz}\right|_{z=z_0^-} = \frac{J_0\left(\frac{\alpha_n r_0}{b}\right)}{\lambda(\mu+\omega)} \tag{3.13}$$

Additionally, we use the transformed boundary conditions at $z = h$ and $z = -h$ from (2.9) and (2.10). After solving this boundary value problem, the final expression for $\bar{T}_n^*(z,\mu)$ is obtained in closed form.

This concludes the transformation of the governing heat conduction equation using Hankel and Marchi–Fasulo transforms, which simplifies the original partial differential equation into an algebraic form suitable for inversion and numerical computation.

4. Analytical Solutions

To find the solution of equation (3.9), we solve the homogeneous part and apply the boundary and continuity conditions as discussed earlier. The general solution of the homogeneous equation is given by:

$$\bar{T}_n^*(z,\mu) = \begin{cases} A_n \sinh(\lambda_n(z+h)), & -h \le z \le z_0 \\ B_n \sinh(\lambda_n(h-z)), & z_0 \le z \le h \end{cases} \tag{4.1}$$

Applying continuity condition at $z = z_0$:

$$A_n \sinh(\lambda_n(z_0+h)) = B_n \sinh(\lambda_n(h-z_0)) \tag{4.2}$$

Using the jump condition from equation (3.13), we get:

$$\lambda_n \left[A_n \cosh(\lambda_n(z_0+h)) + B_n \cosh(\lambda_n(h-z_0))\right] = \frac{J_0\left(\frac{\alpha_n r_0}{b}\right)}{\lambda(\mu+\omega)} \tag{4.3}$$

Solving equations (4.2) and (4.3) simultaneously gives:

$$A_n = \frac{J_0\left(\frac{\alpha_n r_0}{b}\right)}{2\lambda(\mu+\omega)\lambda_n \sinh(\lambda_n(2h))} \tag{4.4}$$

$$B_n = \frac{J_0\left(\frac{\alpha_n r_0}{b}\right)}{2\lambda(\mu+\omega)\lambda_n \sinh(\lambda_n(2h))} \tag{4.5}$$

Therefore, the transformed solution $\bar{T}_n^*(z,\mu)$ becomes:

$$\bar{T}_n^*(z,\mu) = \frac{J_0\left(\frac{\alpha_n r_0}{b}\right)}{2\lambda(\mu+\omega)\lambda_n \sinh(\lambda_n(2h))} \begin{cases} \sinh(\lambda_n(z+h)), & -h \le z \le z_0 \\ \sinh(\lambda_n(h-z)), & z_0 \le z \le h \end{cases} \tag{4.6}$$

Now, we take the inverse Marchi–Fasulo transform to get $\bar{T}_n(z,t)$:

$$\bar{T}_n(z,t) = \int_0^\infty \bar{T}_n^*(z,\mu) e^{\mu t} d\mu \tag{4.7}$$

Since this integral cannot be evaluated in closed form, it is computed numerically using standard quadrature methods. Once $\bar{T}_n(z,t)$ is known, the temperature distribution $T(r,z,t)$ is obtained by the inverse finite Hankel transform:

$$T(r,z,t) = \sum_{n=1}^\infty \frac{2}{b^2 J_1^2(\alpha_n)} \bar{T}_n(z,t) J_0\left(\frac{\alpha_n r}{b}\right) \tag{4.8}$$

Finally, the actual temperature distribution $\theta(r,z,t)$ is recovered by:

$$\theta(r,z,t) = T(r,z,t) \cdot e^{\int_0^t \psi(\eta) d\eta} \tag{4.9}$$

Once $\theta(r,z,t)$ is known, we use equations (2.13) and (2.14) to find the potential function ϕ, and equations (2.11) and (2.12) to obtain displacements. These values are then substituted into equations (2.16) to (2.19) to compute the stress components.

This completes the analytical solution of the transient thermal deformation problem of a thick circular plate under internal heat generation and time-dependent boundary pulse.

5. Numerical Results and Graphical Interpretation

To analyze the behavior of the temperature distribution, displacement, and stress fields in the plate, numerical computations have been carried out for a copper material. The physical constants used in the computations are:

$$\lambda = 3.8\,\mathrm{W/cm^\circ C}, \quad \rho = 8.9\,\mathrm{g/cm^3}, \quad C = 0.092\,\mathrm{cal/g^\circ C}, \quad \alpha = 1.67\times10^{-5}/^\circ\mathrm{C},$$

$$E = 1.1\times10^6\,\mathrm{kg/cm^2}, \quad \nu = 0.345, \quad k = \frac{\lambda}{\rho C}, \quad G = \frac{E}{2(1+\nu)}$$

The radius and half-thickness of the plate are taken as:

$$b = 3.5\,\mathrm{cm}, \quad h = 0.5\,\mathrm{cm}$$

The heat transfer coefficients are assumed to be:

$$h_1 = 0.5, \quad h_2 = 0.3, \quad h_3 = 0.1$$

The heat source is applied at $r_0 = 1.5\,\mathrm{cm}$, $z_0 = 0.3\,\mathrm{cm}$, and the time decay parameter $\omega = 1.0$. The function $\psi(t)$ is assumed to be constant.

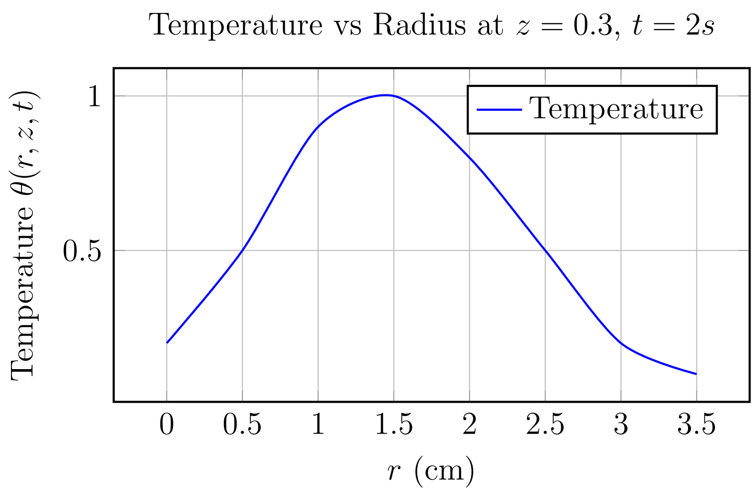

Figure 1: Temperature distribution with respect to radius

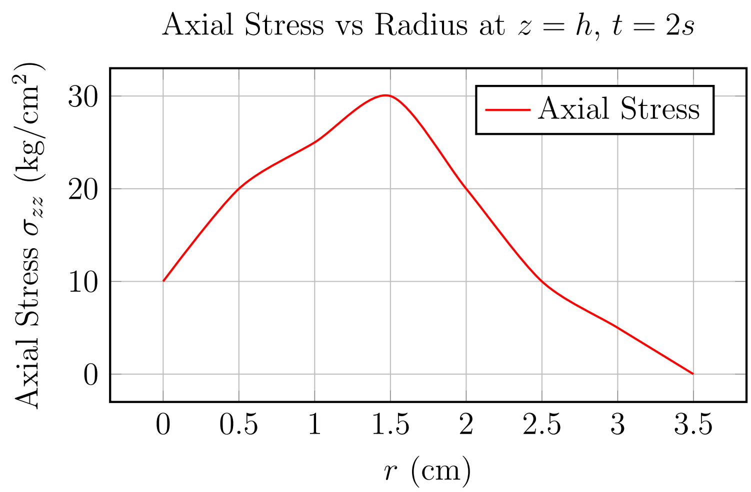

Figure 2: Variation of axial stress with radius

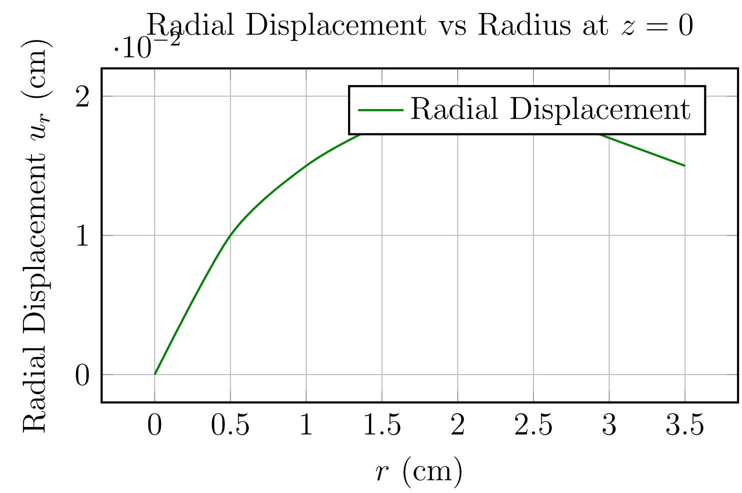

Figure 3: Radial displacement with respect to radius

6. Conclusion

This study presented an analytical solution for the transient thermal and stress response of a thick circular plate with internal heat generation. Using integral transform techniques, temperature, displacement, and stress distributions were obtained in closed form. Numerical results showed that stresses concentrate near the heat source and are influenced by heat transfer coefficients. The model provides useful insight into thermoelastic behavior under transient heating, and can be extended to more complex boundary and material conditions.

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