A Note on Max-Radial Subdivision Number of a Graph

M. Mathan[1], M. Bhuvaneshwari[2], Selvam Avadayappan[3]
[1]*Department of Mathematics Sri Vidhya College of Arts and Science, Virudhunagar, India*
[2,3]*Research Department of Mathematics, VHN Senthikumara Nadar College (Autonomous), (Affiliated to Madurai Kamaraj University) Virudhunagar, India*

***Abstract*: The *Max-Radial Subdivision Number* $sd_{\partial_R}(G)$ of a graph G is the minimum number of edges that must be subdivided (where an edge can be subdivided at most once) in order to increase the Max-Radial number. The Max-Radial subdivision number $sd_{\partial_R}(G) = \infty$, if the subdivision of edges, the Max-Radial number does not increase. In this paper, we determine the Max-Radial subdivision number of some standard graphs. Also we discuss existence theorems and realization theorems of this parameter.**

Keywords: Radius, Max-Radial number, ∂R-set, Max-Radial subdivision number.

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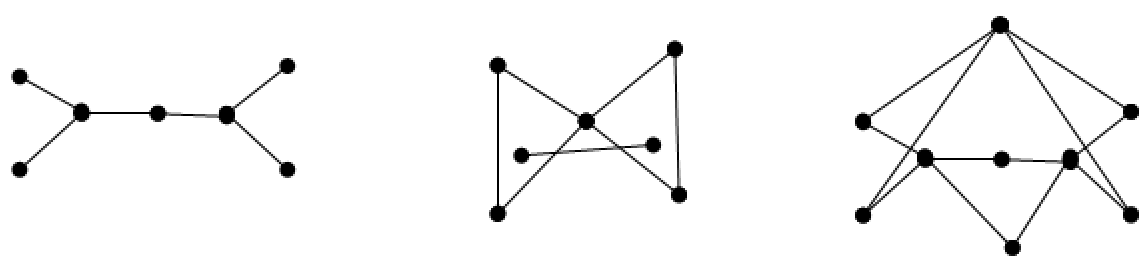
1 INTRODUCTION

Let $G(V, E)$ be a simple connected graph of order $|V(G)| = n$. For other notation and terminology, we follow [2, 6]. The *distance* $d(u, v)$ between two vertices u and v in G is the length of a shortest path joining them. The *eccentricity* $e(v)$ of a vertex v in a connected graph G is the maximum distance between the vertices v and u for all u in G. The *radius* $rad(G)$ is the minimum eccentricity of the vertices. The *diameter*$d(G)$ is the maximum eccentricity of the vertices. For further reference on distance in graphs, one can refer [3]. Two vertices of a graph are said to be *radial* to each other if the distance between them is equal to the radius of the graph. The *radial graph* of a graph G, denoted by $R(G)$, has the vertex set as in G and two vertices are adjacent in $R(G)$ if and only if they are radial in G. If G is disconnected, then two vertices are adjacent in $R(G)$ if they belong to different components of G. A graph G is called a *radial graph* if $R(H) = G$ for some graph H. Further details on radial graph one can refer [1,8]. Let $G(V, E)$ be a graph and let $V_i = \{v \epsilon V / deg(v) = i\}$. The degree splitting graph $DS(G)$ is obtained from G, by adding a new vertex w_i for each V_i such that $|V_i| \geq 2$ and joining w_i to each vertex of V_i. For example, a graph G and its radial graph $R(G)$, degree splitting graph $DS(G)$ are shown in Figure 1.1.



G: $R(G)$: $DS(G)$:

Figure 1.1

For further reading on max-radial number one can refer [9, 10] discussed. For a graph $G(V, E)$, the S-radial set, $B_R(S)$, is defined for any set $S \subseteq V$, as the set of vertices $u \in V \setminus S$ which are at a distance of radius of G from some vertex $v \in S$. The Max-radial number of G, $\partial_R(G)$, is the parameter which is defined as $\max_S\{|B_R(S)| - |S|\}$. For any graph G, $\partial_R(G)$ varies between 0 and n - 2. In [10], the Max-Radial subdivision number for some special graphs has been determined.

In this paper we study the effect of subdividing an edge on the Max-Radial number of a graph. An edge $uv \in E$ is subdivided if the edge uv is deleted, but a new vertex x (called a subdivision vertex) is added, along with two new edges: ux and xv. We restrict ourselves to subdivide an edge at most once, that is no edge incident to a subdivision vertex can be subdivided. The *Max-Radial subdivision number* $sd_{\partial_R}(G)$ is defined as the minimum number of edges that must be subdivided in order to create a graph G' for which $\partial_R(G') > \partial_R(G)$. The Max-Radial subdivision number $sd_{\partial_R}(G) = \infty$ does not rise in the event that there is an edge subdivision.

From the definition of Max-Radial number in graphs, the following observation and theorems have been proved in [9].

2 BASIC THEOREMS

Observation 2.1 For any connected graph G of order n, $0 \leq \partial_R(G) \leq n-2$.

Theorem 2.2 For any graph G, $\partial_R(G) = 0$ if and only if $R(G) \cong F$ where F is 1-factor.

Theorem 2.3 For any graph G of order n, $\partial_R(G) = n-2$ if and only if G contains a full vertex.

Theorem 2.4 For any given natural number m, there exists a graph G such that $\partial_R(G) = |Cen(G)| = m$.

Theorem 2.5 For any positive integer m, there exists a graph G such that $\partial_R(G) = \chi(G) = m$.

Lemma 2.6 [7] For any graph G, $B_R(X) = Ɓ_{R(G)}(X)$ for all $X \subseteq V(G)$, where $Ɓ_{R(G)}(X)$ is the boundary of X in R(G), the radial graph G.

Theorem 2.7 [7] For any graph G, $\partial_R(G) = \partial(R(G))$.

3 MAIN RESULT

In this section, we determine the Max-radial subdivision number $sd_{\partial_R}(G)$ of some standard graphs and existence theorems.

Observation 3.1

The Max-Radial number, Max-Radial subdivision number of a graph G are not comparable. That is, $\partial_R(G)$ and $sd_{\partial_R}(G)$ are not comparable.

For example, (i) if $G \cong K_4$, $\partial_R(G) < sd_{\partial_R}(G)$,(ii) if $G \cong K_5$, $\partial_R(G) = sd_{\partial_R}(G)$, (iii) if $G \cong K_6$, $\partial_R(G) > sd_{\partial_R}(G)$.

Proposition 3.2 For any path $P_n (n \geq 2)$, $sd_{\partial_R}(P_n) = \begin{cases} 1 & if\ n\ is\ even \\ \infty & if\ n\ is\ odd. \end{cases}$

Proof Let $v_1, v_2 \dots v_n$ be the vertices of a path $P_n (n \geq 2)$.

Case(i) n is even

Since n is even. Then $\partial_R(P_n) = 0$. Now, we subdivide an edge v_1v_2 in P_n, we have an odd path P_{n+1}. Therefore, $\partial_R(P_{n+1}) = 1$. Thus $sd_{\partial_R}(P_n) = 1$.

Case(ii) n is odd

Since n is odd. Then $\partial_R(P_n) = 1$. Now we subdivide an edge v_1v_2 in P_n, we have an even path P_{n+1}, $\partial_R(P_{n+1}) = 0$. Therefore, $\partial_R(P_n) > \partial_R(P_{n+1})$. Again, we subdivide the edges v_1v_2 and v_2v_3 in P_n, $\partial_R(P_{n+2}) = 1$. Therefore, $\partial_R(P_n) = \partial_R(P_{n+1})$. Therefore, we subdivide any edge in P_n, the Max-Radial number does not increase. Hence $sd_{\partial_R}(P_n) = \infty$.

Proposition 3.3 For any even cycle $C_n (n \geq 2)$, $sd_{\partial_R}(C_n) = 1$.

Proof Let $v_1, v_2 \dots v_n$ be the vertices of an even cycle $C_n (n \geq 2)$. Then $\partial_R(C_n) = 0$. Now, we subdivide an edge v_1v_2 in C_n, the resulting graph C_{n+1} is an odd cycle.Therefore, $\partial_R(G') = \left\lfloor \frac{n+1}{3} \right\rfloor$, $\partial_R(C_n) < \partial_R(C_{n+1})$. Thus $sd_{\partial_R}(C_n) = 1$.

Proposition 3.4 For any odd cycle $C_n (n > 3)$,

$$sd_{\partial_R}(C_n) = \begin{cases} 2 & if\ \ n \equiv 1,2\ (mod\ 3) \\ 4 & if\ \ n \equiv 0\ (mod\ 3). \end{cases}$$

Proof Let $v_1v_2, \dots, v_{n-1}v_n$ be the vertices of odd cycle $C_n (n \geq 3)$, $\partial_R(C_n) = \left\lfloor \frac{n}{3} \right\rfloor$.

Case(i) $n \equiv 1,2\ (mod 3)$

Now we subdivide an edge v_1v_2 in C_n, the Max-Radial number of the resulting graph $G' \cong C_{n+1}$ as 0. Therefore, $\partial_R(C_n) > \partial_R(G')$. Next we subdivide the edges v_1v_2, v_2v_3 in C_n, we get the Max-Radial number of the resulting graph $G'' \cong C_{n+2}$ as $\left\lfloor \frac{n+1}{3} \right\rfloor$. Therefore, $\partial_R(C_n) < \partial_R(G'')$. Hence $sd_{\partial_R}(G) = 2$.

Case(ii) $n \equiv 0\ (mod 3)$

Now we subdivide an edge v_1v_2 in C_n, the Max-Radial number of the resulting graph $G' \cong C_{n+1}$ as 0. Therefore, $\partial_R(C_n) > \partial_R(G')$. Next, we subdivide the edges v_1v_2, v_2v_3 in C_n, we get the Max-Radial number of the resulting graph $G'' \cong C_{n+2}$ as $\left\lfloor \frac{n+2}{3} \right\rfloor (= \left\lfloor \frac{n}{3} \right\rfloor)$. Therefore, $\partial_R(C_n) = \partial_R(G'')$. Again, we subdivided the edges v_1v_2, v_2v_3, v_3v_4 in C_n, the Max-Radial number of the resulting graph $G''' \cong C_{n+3}$ as 0, $\partial_R(C_n) > \partial_R(G''')$. Next, we subdivide the edges v_iv_{i+1}, $i = 1,2,3,4$ in C_n, we get the Max-Radial number of the resulting graph $G^{iv} \cong C_{n+2}$ as $\left\lfloor\frac{n+4}{3}\right\rfloor (> \left\lfloor\frac{n}{3}\right\rfloor)$. Therefore, $\partial_R(C_n) < \partial_R(G^{iv})$. Hence $sd_{\partial_R}(G) = 4$.

Proposition 3.5 For any complete bipartite graph $K_{m,n}(m,n \geq 2)$, $sd_{\partial_R}(K_{m,n}) = 1$.

Proof Let $V(K_{m,n}) = \{u_1, u_2 \dots u_m, v_1, v_2 \dots v_n\}$. In [9], Fact 2.5, $\partial_R(K_{m,n}) = m + n - 4$. Now, we subdivide an edge u_1v_1 in G, the resulting graph as G' and $V(G') = \{u_1, u_2 \dots u_m, v_1, v_2 \dots v_n, w\}$. Let $X = \{w\}$, $B_R(X) = \{u_2, u_3 \dots u_m, v_2, v_3 \dots v_n\}$.Then $\partial_R(X) = |B_R(X)| - |X| = m + n - 3$. For any other subset S in $V(G')$, $\partial_R(S) \leq m + n - 3$. Therefore, $\partial_R(G') = m + n - 3$, we have $\partial_R(G) < \partial_R(G')$. Therefore, we subdivide an edge u_1v_1 in G, the Max-Radial number to be increase. Thus, $sd_{\partial_R}(G) = 1$.

From the definition of Max-Radial Subdivision number in graphs, the following propositions can be easily verified.

Proposition 3.6 For any connected graph G of order $n \geq 2$, $1 \leq sd_{\partial_R}(G) \leq 4$.

By the definition of Max-Radial subdivision number, the subdivision is atleast one. Therefore, $sd_{\partial_R}(G) \geq 1$. If $G \not\equiv C_n(n \equiv 0(mod\ 3))$ and $|V(G)|$ is high, the Max-Radial subdivision number is at most 3. In the odd cycle, the subdivision of edges is 2 or 4. Hence the Max-Radial subdivision number is at the most 4. Therefore, $sd_{\partial_R}(G) \leq 4$.

Bound is sharp. (i) A graph $G \cong P_n$ $(n\ is\ even)$, $sd_{\partial_R}(G) = 1$. (ii) A graph $G \cong C_n$ $(n \equiv 0(mod3)$, $sd_{\partial_R}(G) = 4$.

Proposition 3.7 For any graph G with radius 1and $|V(G)| \geq 4$ if and only if $sd_{\partial_R}(G) = 3$.

Proof

Let $v_1, v_2 \dots v_n$ be the vertices of a graph G with radius 1. Then by Theorem 2.3, we have $\partial_R(G) = n - 2$. Let v_1 be a full vertex in G. Now we subdivide an edge v_1v_2 in G, the new vertex as u_1 and existing graph as G' with $r(G') = 2$. Let $X = \{u_1\} \subseteq V(G), B_R(X) = \{v_3, v_4 \dots v_n\}$.Then $\partial_R(X) = |B_R(X)| - |X| = n - 3$. For any subset $X \subseteq V(G')$, $\partial_R(X) \leq n - 3$. Therefore, $\partial_R(G') = n - 3$. Hence $\partial_R(G) > \partial_R(G')$. Next we subdivide the edges v_1v_2, v_1v_3 in G, the new vertices are u_1, u_2 respectively, then resulting graph as G'' with $r(G'') = 2$. Let $X' = \{u_1, v_1\} \subseteq V(G'')$, $B_R(X') = \{u_2, v_2, v_3, \dots v_n\}$. Then $\partial_R(X') = |B_R(X')| - |X'| = n - 2$. Therefore, $\partial_R(G'') = n - 2$. $\partial_R(G) > \partial_R(G'')$. Next we subdivide the edges v_1v_2, v_1v_3, v_1v_4 in G, the new vertices are u_1, u_2, u_3 respectively, then existing graph as G''' with $r(G''') = 2$. Let $X' = \{u_1, v_1\} \subseteq V(G''')$, $B_R(X') = \{u_2, u_3, v_2, v_3, \dots v_n\}$. Then $\partial_R(X') = |B_R(X')| - |X'| = (n + 1) - 2 = n - 1$. Therefore, $\partial_R(G''') = n - 1$, $\partial_R(G) < \partial_R(G''')$. Therefore three edges can be subdivide in such that increase the Max-Radial number. Thus, $sd_{\partial_R}(G) = 3$. Conversely we assume $sd_{\partial_R}(G) = 3$. We claim that $r(G) = 1$. Suppose $r(G) \neq 1$.Then G has no full vertex. Therefore, $\partial_R(G) < n - 2 = \partial_R(G''') - 1$. Implies $\partial_R(G) < \partial_R(G''') - 1$ which is contradiction to $\partial_R(G) < \partial_R(G''')$. Thus, G has a full vertex. Hence $r(G) = 1$.

Proposition 3.8 If G_1 and G_2 be any two connected graphs, then $sd_{\partial_R}(G_1 + G_2) \leq sd_{\partial_R}(G_1) + sd_{\partial_R}(G_2)$.

Proof Given G_1 and G_2 are any two graphs. By Proposition 3.6, $sd_{\partial_R}(G_1 + G_2) \leq 4$. Also $sd_{\partial_R}(G_1) \leq 4$ and $sd_{\partial_R}(G_2) \leq 4$. Hence the inequality holds. Bound is sharp. $sd_{\partial_R}(G_1 + G_2) = sd_{\partial_R}(G_1) + sd_{\partial_R}(G_2)$ if $G_1 \cong K_5$ and $G_2 \cong C_4$.

Corollary 3.9 For any graph G and $v \in V(G)$, $sd_{\partial_R}(G + v) \leq sd_{\partial_R}(G)$.

Proof Let $v_1, v_2, \dots, v_n$ be the vertices of a graph G.

Case(i) G contains a full vertex v_1.

Then by Proposition 3.7, we have $sd_{\partial_R}(G) = 3$.

Case(ii) G contains no full vertex.

Then $\partial_R(G) < n - 2$.

Subcase(i) Subdivide one edge in G.

Suppose we subdivide an edge v_1v_2 in G, the resulting graph G' with $\partial_R(G) < \partial_R(G')$. Therefore, $sd_{\partial_R}(G) = 1$.

Subcase(ii) Subdivide two edges in G.

Suppose we subdivide two edges in G, the resulting graph G'' with $\partial_R(G) < \partial_R(G'')$. Therefore, $sd_{\partial_R}(G) = 2$.

Subcase(iii) Subdivide three edges in G.

Suppose we subdivide three edges in G, the resulting graph G''' with $\partial_R(G) < \partial_R(G''')$. Therefore, $sd_{\partial_R}(G) = 3$.

Subcase(iv) Subdivide four edges in G.

Suppose $G \cong C_n(n \equiv 0(mod\ 3)$, Now, we subdivide four edges in G, the resulting graph G'^v with $\partial_R(G) < \partial_R(G^{iv})$. Therefore, $sd_{\partial_R}(G) = 4$.

Subcase(v) There is no edges in G.

Now, we subdivide the edges in G, the resulting graph G^* with $\partial_R(G) \geq \partial_R(G^*)$. Therefore, $sd_{\partial_R}(G) = \infty$. Thus $sd_{\partial_R}(G) \leq 4$.

We construct a graph $G' \cong G + v$ with $V(G') = V(G) \cup \{v\}$ and $rad(G') = 1$. By Proposition 3.7, we have $sd_{\partial_R}(G') = 3$. Thus, $sd_{\partial_R}(G) \geq sd_{\partial_R}(G + v)$.

Proposition 3.10 Any graph G with radius 1, $sd_{\partial_R}(G) = sd_{\partial_R}(R(G))$.

For consider a graph G with radius 1, $G \cong R(G)$. Therefore, $sd_{\partial_R}(G) = sd_{\partial_R}(R(G))$.

Proposition 3.11 For any connected k-regular graph G $(k \leq 2)$, $1 \leq sd_{\partial_R}(G) \leq k + 2$.

Proof By proposition 3.4.8 , Since G is connected, $sd_{\partial_R}(G) \geq 1$. By proposition 3.4.6, $G \cong C_n(n \equiv 0(mod\ 3)$, $sd_{\partial_R}(G) = 4$. Since G is 2-regular graph. Hence $sd_{\partial_R}(G) \leq k + 2$.

Proposition 3.12

For any graph $G \cong K_p(p > 4)(or)K_{m,n}(m, n > 2)$ having a minimum $\partial_R(G)$-set S where the subgraph G' induced by $V - S$, $sd_{\partial_R}(G) = sd_{\partial_R}(G')$.

Proof

Case(i) $G \cong K_n$. Let $V(G) = \{v_1, v_2 \dots v_n\}$. By Proposition 3.7, $sd_{\partial_R}(G) = 3$. Let S be a minimum ∂_R-set of G. Construct a subgraph G' induced by $V - S$, which is a complete graph. Therefore, $sd_{\partial_R}(G') = 3$. Hence $sd_{\partial_R}(G) = sd_{\partial_R}(G')$.

Case(ii) $G \cong K_{m,n}$. Let $V(G) = \{u_1, u_2 \dots u_m, v_1, v_2 \dots v_n\}$. By Proposition 3.5, $sd_{\partial_R}(G) = 1$. Construct a subgraph G' induced by $V - S$, which is a bipartite complete graph. Therefore, $sd_{\partial_R}(G') = 1$. Hence $sd_{\partial_R}(G) = sd_{\partial_R}(G')$.

We note that the converse of the above proposition is not true. For example, consider the graph G as shown in Figure 3.11.

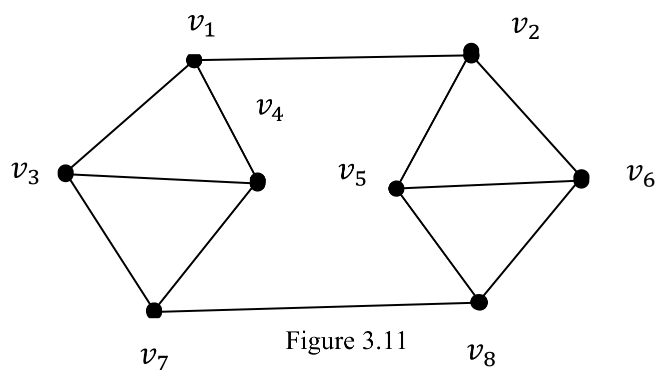


Figure 3.11

Here $S = \{v_3\}$ be the minimum ∂_R-set of G, $\partial_R(G) = 1$. Now we subdivide an edge v_1v_3 in G, the Max-Radial number of resulting graph G^* as 2. Therefore, $sd_{\partial_R}(G) = 1$.

Now we construct a subgraph G' induced by $V - S$.

$G' = G[V - S]$:

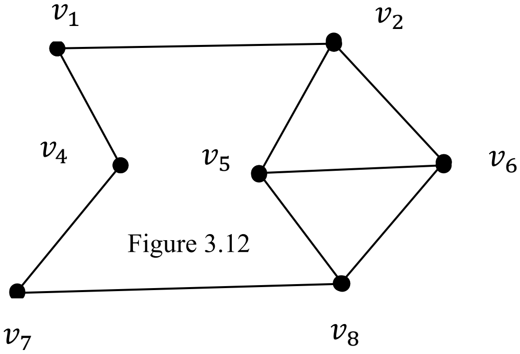


Figure 3.12

Here $S = \{v_4\}$ be the minimum ∂_R-set of G, $\partial_R(G') = 1$. Now we subdivide an edge v_1v_4 in G', we get the Max-Radial number of resulting graph G^{**} as 2. Therefore, $sd_{\partial_R}(G') = 1$. Therefore, $sd_{\partial_R}(G) = sd_{\partial_R}(G')$. But $G \ncong K_n (or) K_{m,n}$.

Proposition 3.13 If $G \cong P_n(n \geq 3), S_n, W_n(n \geq 4), K_{m,n}(m \geq 2, n \geq 3)$, then $sd_{\partial_R}(DS(G))$=1 where $DS(G)$is the degree splitting graph of G.

Proof Let $G \cong P_n(n \geq 3), S_n, W_n(n \geq 4), K_{m,n}(m \geq 2, n \geq 3)$. Then, we subdivide exactly one edge in $DS(G)$, the Max-Radial number of the resulting graph G' is increased. Therefore, $\partial_R(DS(G)) < \partial_R(G')$. Hence $sd_{\partial_R}(DS(G))$=1.

Proposition 3.14 For any k-regular graph G, $sd_{\partial_R}(DS(G))$=3.

Proof Let G be a k-regular graph. Then $DS(G)$ contains a full vertex v. Therefore, by proposition 3.7, we have $sd_{\partial_R}(DS(G))$=3.

Theorem 3.15

For any given natural number n, there exists no graph G such that $\partial_R(G) = n - 3$.

Proof Suppose a graph G exists to contrary of the statement.

Let X be a ∂_R-set of G.

Case (i): X contains exactly one vertex, v

Subcase(i): v is a full vertex.

Then $\partial_R(G) = n - 2$ which is a contradiction.

Subcase(ii): v is not a full vertex.

Then v is adjacent to at most $n - 2$ vertices in G, Therefore, $|B_R(X)| \leq n - 3$

$\Rightarrow |B_R(X)| - |X| \leq n - 4$. Implies $\partial_R(X) < \partial_R(G)$ which is contradiction to $\partial_R(G) = \partial_R(X)$.

Case (ii): X contains atleast two vertices.

That is, $|X| \geq 2$.

Now $\partial_R(G) = \partial_R(X)$

$\Rightarrow \partial_R(G) = |B_R(X)| - |X|$

$\Rightarrow \partial_R(G) + |X| = |B_R(X)|$

$\Rightarrow |B_R(X)| \geq (n - 3) + 2 = n - 1$

$\Rightarrow |B_R(X)| \geq n - 1$

But $|V(G)| \geq |X \cup B_R(X)|$

$\Rightarrow n \geq |X| + |B_R(X)|$

$\Rightarrow n \geq n + 1$ which is a contradiction. Our assumption is wrong. Therefore there exists no graph G with order n such that $\partial_R(G) = n - 3$.

Theorem 3.16 For any given natural number $n \geq 5$, there exists a graph G such that $\partial_R(G) = n - 4$ and $sd_{\partial_R}(G) = 1$.

Proof Given natural number $n \geq 5$. We construct a graph G with $V(G) = \{v_1, v_2, v_3, v_4, w_1, w_2, w_3, \ldots, w_{n-4}\}$ and $E(G) = \{v_1v_2, v_2v_3, v_3v_4, v_3w_i, v_4w_i : 1 \leq i \leq n - 4\}$. We claim that $\partial_R(G) = n - 4$. Let $X = \{v_2\} \subseteq V(G)$, then $B_R(X) = \{w_1, w_2, \ldots, w_{n-4}, v_4\}$. Therefore, $\partial_R(X) = |B_R(X)| - |X| = n - 4$. Thus, $\partial_R(G) \geq n - 4$. It is enough to prove that $\partial_R(X) \leq n - 4$. Suppose X contains at least two vertices, $B_R(X)$ contains at most $n - 2$ vertices. Then $\partial_R(X) = |B_R(X)| - |X| \leq n - 4$. Thus, $\partial_R(G) < n - 4$. Therefore, $X = \{v_2\}$ is only ∂_R-set of G. Hence $\partial_R(G) = n - 4$. Next we claim that $sd_{\partial_R}(G) = 1$. Now, we subdivide an edge of v_1v_2 in G, the new vertex as u and existing graph as G' with radius $r(G') = 2$. Let $X = \{v_2\}$, then $B_R(X) = \{v_1, v_4, w_1, w_2 \ldots, w_{n-4}\}$. Therefore, $\partial_R(X) = |B_R(X)| - |X| = n - 3$ which is minimum, $\partial_R(G') = n - 3$. Therefore, $\partial_R(G) \leq \partial_R(G')$. Thus $sd_{\partial_R}(G) = 1$. Example 3.17 When $n = 9$, $m = 5$ in Theorem 3.16, the constructed graph G is shown in Figure 3.13

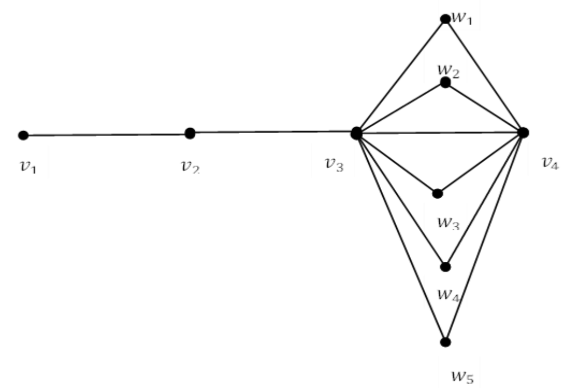


Figure 3.13

Here $S = \{v_2\}$ be the minimum ∂_R-set of G, $\partial_R(G) = 5$. Now we subdivide an edge v_1v_2, the Max-Radial number of resultant graph as 6. Hence $sd_{\partial_R}(G) = 1$.

Note that the other construction of a graph G with $\partial_R(G) = m - 4$ and $sd_{\partial_R}(G) = 1$ as given below.

Given a natural number $m \geq 6$. Construct a graph G with $V(G) = \{v_1, v_2, v_3, v_4, v_5, w_1, w_2, \ldots, w_k, u_1, u_2, \ldots, u_{m-(k+5)} : 1 \leq k \leq m - 5$ and $E(G) = \{v_1v_2, v_2v_3, v_3v_4, v_4v_5, v_5v_1, v_3w_i, v_4u_j : 1 \leq i \leq$

k and $1 \leq j \leq m-(k+5)\}$ with radius 2. Now, we claim that $\partial_R(G) = n-4$. Let $X = \{v_3, v_4\}$, then $B_R(X) = \{v_1, v_2, v_5, w_1, w_2, \dots, w_k, u_1, u_2, \dots, u_{m-(k+5)}\}$. Therefore, $\partial_R(X) = |B_R(X)| - |X| = m-4$. Thus $\partial_R(G) \geq m-4$. It is enough to prove that $\partial_R(G) \leq m-4$. Let $S \subseteq V(G)$ containing at least two vertices, Then $B_R(S)$ contains at most $m-2$ vertices. We have $\partial_R(S) \leq m-4$. Therefore, $\partial_R(G) = n-4$. Next we claim that $sd_{\partial_R}(G) = 2$. Now we subdivide an edge in the cycle $C: v_1v_2v_3v_4v_5v_1$, the new vertex as x_1 and existing graph as $G^{'}$ with $r(G^{'}) = 3$. Let $X \subseteq V(G)$ be ∂_R-set with cardinality 2. Then $B_R(X)$ contains $m-3$ vertices. Therefore, $\partial_R(X) \leq m-5$, $\partial_R(G^{'}) = m-5$. Hence $\partial_R(G) > \partial_R(G^{''})$. Next we subdivide an edge u_jv_4 or w_kv_3 in G, the new vertex x_2 and resulting graph as $G^{'}$ with $r(G^{''}) = 2$. Let $X^{'} = \{v_3, v_4\} \subseteq V(G^{'})$. Then $B_R(X^{'}) = \{v_1, v_2, v_5, x_1, w_1, w_2, \dots, w_k, u_1, u_2, \dots, u_{m-(k+5)}\}$. Therefore, $\partial_R(X^{'}) = |B_R(X^{'})| - |X^{'}| = m-3$. For any subset $X \subseteq V(G^{'})$, we have $\partial_R(X) \leq m-3$. Therefore, $\partial_R(G^{''}) = m-3$. Hence $\partial_R(G) < \partial_R(G^{''})$. Thus $sd_{\partial_R}(G) = 1$.

Example 3.18 When $m = 10$ in Theorem 3.16, the constructed graph is shown in Figure 3.14.

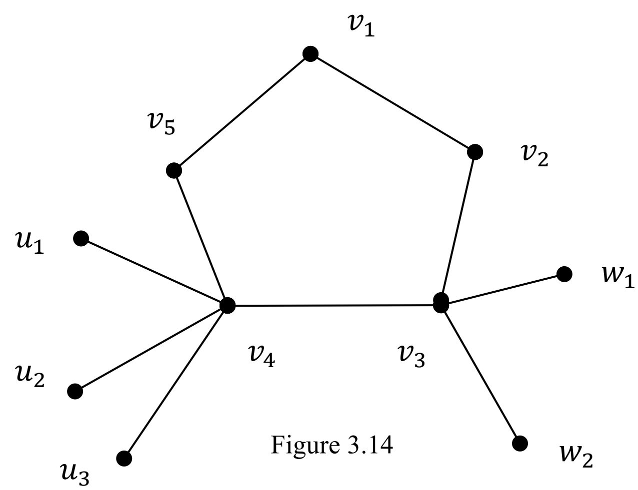


Figure 3.14

Here $S = \{v_3, v_4\}$ be the minimum ∂_R-set of G, $\partial_R(G) = 6$. Now we subdivide an edge u_1v_4 or w_1v_3, the Max-Radial number of the resultant graph as 7. Hence $sd_{\partial_R}(G) = 1$.

Theorem 3.19 For given any natural number $m \geq 5$, there exists a graph G such that $\partial_R(G) = m-5$ and $sd_{\partial_R}(G) = 1$.

Proof Given natural number $m \geq 5$. We construct a graph G with $V(G) = \{v_1, v_2, v_3, v_4; w_1, w_2, \dots, w_k;\ u_1, u_2, \dots, u_{m-(k+5)+1}\}$where$1 \leq k \leq m-5$ and $E(G) = \{v_1v_2, v_1v_3, v_1v_4, v_2v_3, v_3v_4, v_2w_i, v_4u_j, w_iw_{i+1}, u_ju_{j+1} /\ 1 \leq i \leq k$ and $1 \leq j \leq m-(k+5)\}$ with radius 2. Now, we claim that $\partial_R(G) = m-5$. Let $X = \{v_1\}$ or $\{v_3\}$, then $B_R(X) = \{w_1, w_2, \dots, w_k, u_1, u_2, \dots, u_{m-(k+5)+1}\}$. Therefore, $\partial_R(X) = |B_R(X)| - |X| = (m-4) - 1 = m-5$. Thus $\partial_R(G) \geq m-5$. It is enough we prove that $\partial_R(G) \leq m-5$. Suppose a set S contains at least two vertices in G, then $B_R(S)$ contains at most $m-3$ vertices. Therefore, $\partial_R(S) < m-5$. For any subset $X^{'} \subseteq V(G)$, $\partial_R(X^{'}) \leq m-5$. Therefore, $X = \{v_1\}$ or $\{v_3\}$ is only ∂_R-set of G. Thus $\partial_R(G) = m-5$. Next we claim that $sd_{\partial_R}(G) = 1$. Now, we subdivide an edge of v_1v_3 in G, the new vertex as x_1 and existing graph as $G^{'}$. Then radius $r(G^{'}) = 2$. Let $X = \{v_3\} \subseteq V(G')$, then $B_R(X) = \{v_1, w_1, w_2, \dots, w_k, u_1, u_2, \dots, u_{m-(k+4)}\}$. Therefore, $\partial_R(X) = |B_R(X)| - |X| = (m-3) - 1 = m-4$. Also, for any subset $X^{'} \subseteq V(G^{'})$, we have $\partial_R(X^{'}) \leq m-4$. Therefore, $X = \{v_3\}$$or$ $\{v_1\}$ is a ∂_R-set of G, $\partial_R(G^{'}) = m-4$. Hence $\partial_R(G) < \partial_R(G^{'})$. Therefore, we subdivide only one edge in G, the Max-Radial number is increase. Thus $sd_{\partial_R}(G) = 1$.

Example 3.20 When $m = 13$ in Theorem 3.19, the constructed graph is shown in Figure 3.15.

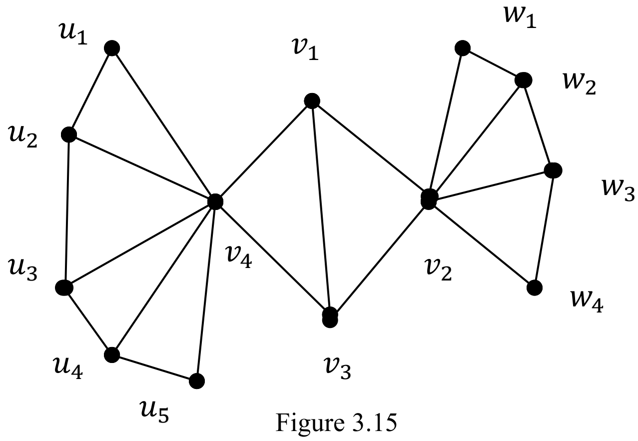


Figure 3.15

Here $S = \{v_1\}$ *or* $\{v_3\}$ be the minimum ∂_R-set of G, $\partial_R(G) = 8$. Now we subdivide an edge v_1v_3, we get Max-Radial number of resultant graph as 9. Hence $sd_{\partial_R}(G) = 1$.

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