The Minimum Reduced Sombor Index of Unitary Cayley

Dr.Syeda Asma kauser[1], Syed Sajjad Hussain[2], Azra Sultana[3] and Syed Shahbaaz Hussain[4]
[1]*Assistant professor, Department of mathematics, Global Institute of engineering and technology, Moinabad, Hyderabad*
[2]*Assistant professor, Department of physics, Anwarul Uloom College, Hyderabad*
[3]*Assistant professor, Department of mathematics, Anwarul Uloom College, Hyderabad*
[4]*Assistant professor, Department of mathematics, Anwarul Uloom College, Hyderabad*

Abstract—**Aims: The paper investigates the Reduced Sombor Index (RSO) for Unitary Cayley graphs. It aims to determine and characterize the Unitary Cayley graphs that attain the minimum RSO index among all Unitary Cayley graphs of a given order.**
Study design: This is a theoretical mathematical study based on graph theory and topological indices. The study involves in defining and analyzing the Reduced Sombor Index by comparing RSO values across different Unitary Cayley graphs. To obtain the minimum RSO index of Unitary Cayley graphs different lemmas and theorems are proved.
Methodology: Unitary cayley graphs of different orders are analyzed. The study proves multiple lemmas that give different RSO values by demonstrating whether a specific structural modification increases or decreases the RSO value.
Results: The minimum RSO value in unitary cayley graphs is achieved. Several lemmas and theorem are established for unitary cayley graph of various orders.

Index Terms—**Reduced Sombor index; topological index; graph invariant; unitary cayley graph.**

I. INTRODUCTION

Throughout this article, by a graph G, we mean an ordered pair (V_G, E_G), and the members of the sets V_G and E_G respectively are the vertices and edges of the graph. The set of vertices that are adjacent to a vertex u in G is denoted as $N_G(u)$ and it is called as "the open neighbourhood" of u in G. The term "closed neighbourhood" is $N_G[v] = N_G(v) \cup \{v\}$ .We mean a path is a (u, v)-Path if $uu_1u_2 \dots v$ is a sequence of distinct members of the set V_G and the vertices u,v are usually known as *the origin* and *the terminal* of the path P respectively. The concept of distance between any two vertices $x, y \in V_G$ is usually defined as the length of the smallest (x, y)-path that exists in G. If $d_G(v) = 1$, then v is a pendant vertex and it is adjacent to a unique vertex in G, say u which is called a support vertex. For more on graphs and related works, the reader is referred to [1, 2, 3].

A graph variant also called as the topological indices, plays a major role in the chemical graph theory as they help to analyze the behaviour of the molecule structures and their inter-relationships. There are numerous topological indices available in the literature; a few of them are the Sombor index, Zagreb index, and so on. The topological indices were defined with minor and major modifications in the past and several classes of topological indices are available for the Sombor index and Zagreb index. Given a graph G, the Sombor (SO) index is defined (by Gutman [4]) as

$$SO(G) = \sum_{uv \in E(G)} \sqrt{d_G(u)^2 + d_G(v)^2}.$$

The Sombor index, in recent years, received huge attentions from academicians and researchers [4, 5, 6, 7, 8]. For some recent surveys in Sombor index, one can refer to the articles [9,10]. Chemical applications have been carried out in the articles [11, 12]. For various results and versions of Sombor index, one can refer [13, 14, 15]. The reduced Sombor index is defined as

$$RSO(G) = \sum_{uv \in E(G)} \sqrt{(d_G(u) - 1)^2 + (d_G(v) - 1)^2}.$$

A recently introduced term is the reduced Sombor index and some of its works are found in [2, 16].

The reduced Sombor index of unitary cayley graphs of various order has been studied in this article.

II. METHODOLOGY

The main results on the minimum Reduced Sombor index for the class of unitary cayley graphs is studied in this section. If the order n of G is one, there is no edge in the graph; hence by a graph G, throughout this article, we mean a graph with minimum two vertices. Also, by a graph G, we mean only a connected graph, unless explicitly stated. Cayley graphs with more number of vertices for a given diameter and for a given number of edges per vertex were studied previously. An immediate example of Cayley graph is a unitary cayley graph. More information about Cayley graphs can be found in the books on algebraic graph theory by Biggs [17] and by Godsil and Royle [18]. Unitary Cayley graphs are highly symmetric. They have some remarkable properties connecting graph theory and number theory.

All graphs discussed in this paper are simple graphs. The unitary Cayley graph $X_n$has vertex set $Z_n = \{0,1,2,..n-1\}$ where the vertices a, b are adjacent, if $gcd(a-b,n) = 1$.

If we represent the elements of Z_n by the integers $0,1,\ldots,n-1$, then it is well known that X_n has vertex set $V(X_n) = Z_n = \{0,1,\ldots..n-1\}$and edge set

$$E(X_n) = \{\{a,b\}: a,b \in Z_n, \gcd(a-b,n) = 1\}.$$

The multiplicative indices are based on the distances of vertices in a graph. Some new multiplicative indices of unitary cayley graphs has been studied in [19].Status sombor indices of unitary cayley graph are studied in [20]

A characterization of graphs with the minimum values of $RSO(G)$ on the class of Unitary cayley graphs is provided in this section.

Lemma 1: Let $X_n$is a unitary cayley graph where $n = p$ is a prime number, then $X_n = K_p$ is a complete graph on p vertices. In a complete graph we have $\frac{n(n-1)}{2}$ edges and the degree of every vertex $d_{X_n}(u) = n-1$.

Lemma 2: Let $X_n$is a unitary cayley graph where $n = 6$ then X_n is a cycle. In a cycle we have n edges and the degree of every vertex $d_{X_n}(u) = 2$.

Lemma 3[19]: Let $X_n$is a unitary Cayley graphs for n as any even number such that $n = 2p$ where p is a prime and $p \neq 2$, 3.It has $2n$ edges, every vertex has $d_{X_n}(u) = n-1$.

Lemma 4: Let $X_n$is a unitary Cayley graphs for n as any even number such that $n = p\alpha$ where p is a prime, $p = 2$ *and* $\alpha = 3, 4$...

From [19] this type of graphs has $2n$ edges, every vertex has $d_{X_n}(u) = \frac{n}{2}$.

III. RESULTS AND DISCUSSION

The following theorem is the consequence of the lemma 1 from the previous section.

Theorem 5: The minimum reduced Sombor index of a unitary Cayley graph $X_n$where $n = p$ is a prime number is $\sqrt{2}(n-2)^2$.

Proof: Since from Lemma 1, if $X_n$is a unitary cayley graph where $n = p$ is a prime number, then $X_n = K_p$ is a complete graph on p vertices. We know that in a complete graph we have $\frac{n(n-1)}{2}$ edges and the degree of every vertex is $d_{X_n}(u) = n-1$.

Hence $RSO(X_n) = \sum_{uv \in E(G)} \sqrt{(d_G(u)-1)^2 + (d_G(v)-1)^2}$ $=\sqrt{(n-1-1)^2 + (n-1-1)^2}$

$=\sqrt{2(n-2)^2} = \sqrt{2}(n-2)^2$.

Theorem 6: The minimum reduced Sombor index of a unitary Cayley graph $X_n where\, n = 6$ is an even number is $\sqrt{2}$.

Proof: Since from Lemma 2, if $X_n$is a unitary cayley graph where $n = 6$ then X_n is a cycle. In a cycle we have n edges and the degree of every vertex $d_{X_n}(u) = 2$.

Hence $RSO(X_n) = \sum_{uv \in E(G)} \sqrt{(d_G(u)-1)^2 + (d_G(v)-1)^2}$ $=\sqrt{(2-1)^2 + (2-1)^2} = \sqrt{2}$.

Theorem 7: The minimum reduced Sombor index of a unitary Cayley graph X_n for n as any even number such that $n = 2p$ where p is a prime and $p \neq 2, 3$ is $\sqrt{2}(n-2)^2$.

Proof: Since from Lemma 3, if $X_n$is a unitary cayley graph for n as any even number such that $n = 2p$ where p is a prime and $p \neq 2, 3$.It has $2n$ edges, every vertex has $d_{X_n}(u) = n - 1$.

Hence $RSO(X_n)$=$\sqrt{2}(n-2)^2$.

Theorem 8: The minimum reduced Sombor index of a unitary Cayley graph X_n for n as any even number such that $n = p\alpha$ where p is a prime, $p = 2$ and $\alpha = 3$, 4...is $\frac{\sqrt{2}}{2}(n-2)$

Proof: Since from Lemma 4, if $X_n$is a unitary cayley graph for n as any even number such that $n = p\alpha$ where p is a prime, $p = 2$ and $\alpha = 3$, 4...

From [19] this type of graphs has $2n$ edges, every vertex has $d_{X_n}(u) = \frac{n}{2}$.

Hence $RSO(X_n) = \sum_{uv \in E(G)} \sqrt{(d_G(u)-1)^2 + (d_G(v)-1)^2} = \sqrt{\left(\frac{n}{2}-1\right)^2 + \left(\frac{n}{2}-1\right)^2}$

$$=\frac{\sqrt{2}}{2}(n-2).$$

IV. CONCLUSION

The study successfully characterizes unitary cayley graphs with the minimum Reduced Sombor Index (RSO).The findings contribute to chemical graph theory by refining how topological indices behave in molecular graph models.

REFERENCE

[1] Bondy J. A., Murty U. S. R., Graph Theory, Springer, 2008 doi:10.1007/978-1-84628-970-5.

[2] *Fangxia Wang, Baoyindureng Wu, The Proof of a Conjecture on the Reduced SomborIndex, MATCH Commun. Math. Comput. Chem. 88 (2022) 583-591.* doi: 10.46793/match.88-3.583W

[3] Haynes, T., Hedetniemi, S., Slater, P.:Fundamentals of Domination in Graphs. Marcel Dekker, New York (1998). https://doi.org/10.1201/9781482246582

[4] . Das K. C., Gutman I., On Sombor index of trees, Appl. Math. Comput. 412 (2022) #126575. https://doi.org/10.1016/j.amc.2021.126575

[5] *Gutman I., Kulli V. R., Redˇzepoviˊc I., Sombor index of Kragujevac trees, Sci. Publ. Univ. Novi Pazar Ser. A 13 (2021) 61–70.* DOI:10.5937/SPSUNP2102061G

[6] *Gutman I., Redzepovic I., Furtula B., On the product of Sombor and modified Sombor index, Open Journal of Applied Discrete Mathematics, 6(2) (2023) 1-6.* DOI: https://www.doi.org/10.30538/psrp-odam2023.0083

[7] *Gutman I., Sombor index – one year later, Bull. Acad. Serb. Sci. Arts 153 (2020)43–55.http://elib.mi.sanu.ac.rs/files/journals/bltn/45/bltnn45p43-55.pdf*

[8] *Redˇzepoviˊc I., Chemical applicability of Sombor indices, J. Serb. Chem. Soc. 86(2021) 445–457.* https://doi.org/10.2298/JSC201215006R

[9] Chen H., Li W., Wang J., Extremal values on the Sombor index of trees, MATCH Commun. Math. Comput. Chem. 87 (2022) 23–49. doi: 10.46793/match.87-1.023C

[10] *Gutman I., Kulli V. R., Redˇzepoviˊc I., Sombor index of Kragujevac trees, Sci. Publ. Univ. Novi Pazar Ser. A 13 (2021) 61–70.* DOI:10.5937/SPSUNP2102061G

[11] Deng H., Tang Z., Wu R., Molecular trees with extremal values of Sombor indices, Int. J. Quantum Chem. 121 (2021) #e26622. DOI:10.1002/qua.26622

[12] *Li S., Wang Z., Zhang M., On the extremal Sombor index of trees with a given diameter, Appl. Math. Comput. 416 (2022) #126731.* https://doi.org/10.1016/j.amc.2021.126731

[13] *Gutman I., Geometric approach to degree-based topological indices: Sombor indices, MATCH Commun. Math. Comput. Chem. 86 (2021) 11–16. https://match.pmf.kg.ac.rs/electronic_versions/Match86/n1/match86n1_11-16.pdf*

[14] *Liu H., Gutman I., You L., Huang Y., Sombor index: review of extremal results and bounds, J. Math. Chem. 60 (2022) 771–798.* DOI:10.1007/s10910-022-01333-y

[15] *Sun X., Du J., On Sombor index of trees with fixed domination number, Appl. Math. Comput. 421 (2022) #126946.* https://doi.org/10.1016/j.amc.2022.126946

[16] *Hechao Liu, Lihua You, Zikai Tang, Jia–Bao Liu, On the Reduced Sombor Index and Its Applications, MATCH Commun. Math. Comput. Chem. 86 (2021) 729-753. https://match.pmf.kg.ac.rs/electronic_versions/Match86/n3/match86n3_729-753.pdf*

[17] Biggs, N. Algebraic graph theory. Second Edition. Cambridge Mathematical Library. Cambridge University Press, 1993. https://superoles.wordpress.com/wp-content/uploads/2015/09/n-biggs-algebraic-graph-theory-1993.pdf.

[18] Godsil, C., and Royle, G. Algebraic graph theory. Graduate Texts in Mathematics. Vol 207.Springer,2001,https://books.google.co.in/books/about/Algebraic_Graph_Theory.html?id=pYfJe-ZVUyAC&redir_esc=y.

[19] Syeda Asma Kauser. Some Multiplicative Indices of Unitary Cayley Graphs. Global Journal of Pure and Applied Mathematics.2023 ,19(2),319-328. DOI:10.37622/GJPAM/19.2.2023.319-328.

[20] Syeda Asma Kauser,Mashaer Alsaeedi.Status Sombor indices of unitary cayley graphs.Indian journal of science and technology. 2024,17(46),4861-4866.DOI. https://doi.org/10.17485/IJST/v17i46.3569.