A Study of Divided Square Difference Cordial Labeling in Graphs

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Abstract— **Let G = (V (G),E(G)) be a simple graph and f : V (G) → {1, 2, · · · , |V (G)|} be a bijection function. For each edge uv, assign the label 1 if 2|(f(u)+f(v)) and the label 0 otherwise. The function f is called a sum divisor cordial (SDC) labeling if $|e_f (1) - e_f (0)| \leq 1$, where ef (s) denotes the number of edges labeled with s (s = 0, 1). A graph which admits a sum divisor cordial labeling is called a sum divisor cordial (SDC) graph. In this paper I have proved nine new results admitting Divided Square Difference Cordial Labeling.**

Keywords: Divided square difference cordial labeling, star graph, Path graph, Fan.
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I. INTRODUCTION

Let G = (V, E) be a simple, finite, undirected and non-trivial graph with the vertex set V. The number of elements of V, denoted as |V(G)| is called the order of G while the number of elements of E, denoted as |E(G)| is called the size of G. More detail of graph labeling results and its applications can be found in Gallian [2]. We provide brief summary of definitions and other related information which are useful for the further investigations.

The concept of divided square difference cordial labeling was given by A. Alfred Leo, R. Vikramaprasad and R. Dhavaseelan[4]. They proved that path, cycle, Star graph, K2,n are divided square difference cordial graphs.

II. MAIN RESULTS

Theorem 1. Let g be a divided square difference cordial labeling of a graph G of order p and size q. Let $z \in V(G)$ be a vertex such that g(z) = 2. The graph obtained by identifying a vertex z in G and a vertex of degree m in $K_{1,m}$ admits a divided square difference cordial labeling if one of the following conditions holds:

(1) m is even,

(2) m is odd and q is even,

(3) m is odd, q is odd, p is even and $e_g(0) = \frac{q+1}{2}$,

(4) m is odd, q is odd, p is odd and $e_g(0) = \frac{q-1}{2}$.

Proof. Let g be a divided square difference cordial labeling of a graph G of order p and size q. That is the vertices of G are labeled with numbers $\{1,2,\dots,p\}$ and $\left|e_g(0) - e_g(1)\right| \leq 1$.

Let $z \in V(G)$ be such that g(z) = 2. Let us denote by H the graph obtained by identifying a vertex z in G and a vertex of degree m in $K_{1,m}$.

We define a vertex labeling h of H such that

$$h(v) = g(v), v \in V(G);$$

$$h(x_j) = p + j, \qquad j = 1,2,\dots,m.$$

Thus for the induced edge labeling we get

$$h(uv) = g(vu), v \in V(G);$$

for $j = 1,2,\dots,m$.

$h(z\,x_j) = 0$ if (p is odd and j is odd) or (p is even and j is even).

$H(z\,x_j) = 1$ if (p is odd and j is even) or (p is even and j is odd).

Let us denote by $e^*{}_h(k)$ number of edges $z\,x_j$ labelled with k, where k = 0, 1.

Then $|e_h(0) - e_h(1)| = \left|\left(e_g(1) - e^*{}_h(1)\right) - (e_g(0) - e^*{}_h(0))\right|$

$= \left|e_g(0) - e_g(1) + e^*{}_h(0) - e^*{}_h(1)\right|$

$$= \left|\frac{q}{2} - \frac{q}{2} + \frac{m}{2} - \frac{m}{2}\right|$$
$$= 1, \text{if (m is even and q is even)};$$

$$= \left|\frac{q+1}{2} - \frac{q-1}{2} + \frac{m}{2} - \frac{m}{2}\right|$$
$$= 0 \text{ if } \left(\text{m is even, q is odd and } e_g(0) = \frac{q+1}{2}\right);$$

$$= \left|\frac{q-1}{2} - \frac{q+1}{2} + \frac{m}{2} - \frac{m}{2}\right|$$
$$= 0 \text{ if } \left(\text{m is even, q is odd and } e_g(0) = \frac{q-1}{2}\right);$$

$$= \left|\frac{q}{2} - \frac{q}{2} + \frac{m-1}{2} - \frac{m+1}{2}\right|$$
$$= 0, \text{if (m is odd, p and q are even)};$$

$$= \left|\frac{q}{2} - \frac{q}{2} + \frac{m+1}{2} - \frac{m-1}{2}\right|$$
$$= 0, \text{if (m and p are odd and q is even)};$$

$$= \left|\frac{q+1}{2} - \frac{q-1}{2} + \frac{m-1}{2} - \frac{m+1}{2}\right|$$
$$= 1 \text{ if } \left(\begin{matrix}\text{m is odd, p is even q is odd} \\ \text{and } e_g(0) = \dfrac{q+1}{2}\end{matrix}\right)$$

$$= \left|\frac{q-1}{2} - \frac{q+1}{2} + \frac{m+1}{2} - \frac{m-1}{2}\right|$$
$$= 1, \text{if } \left(\begin{matrix}\text{m is odd, p is odd q is odd} \\ \text{and } e_g(0) = \dfrac{q-1}{2}\end{matrix}\right)$$

In the above two case, we observe that $\left|e_g(0) - e_g(1)\right| \leq 1$.

Hence Proved.

Theorem 2. Let g be a divided square difference cordial labeling of a graph G of order p and size q. Let $z \in V(G)$ be a vertex such that g(z) = 1. The graph obtained by identifying a vertex z in G and a vertex of degree m in $K_{1,m}$ admits a divided square difference cordial labeling if one of the following conditions holds:

(1) m is even,

(2) m is odd and q is even,

(3) m is odd, q is odd, p is odd and $e_g(0) = \frac{q+1}{2}$,

(4) m is odd, q is odd, p is even and $e_g(0) = \frac{q-1}{2}$.

Proof. The labelling pattern is same as Theorem 1.

Theorem 3. Let g be a divided square difference cordial labeling of a graph G of order p and size q. Let $z \in V(G)$ be a vertex such that $g(z) = 2^r$, where $r = 0,1,2, \cdots$. The graph obtained by identifying a vertex z in G and a vertex of degree m in $K_{1,m}$ admits a divided square difference cordial labeling.

Proof. The labelling pattern is same as Theorem 1.

Theorem 4. Let g be a divided square difference cordial labeling of a graph G of order p and size q. Let $z \in V(G)$ be a vertex such that g(z) = 2. The graph obtained by identifying a vertex z in G and a vertex of degree 1 in P_m admits a divided square difference cordial labeling if one of the following conditions holds:

(1) m is odd,

(2) m is even and q is even,

(3) m is even, q is odd, p is even and $e_g(0) = \frac{q+1}{2}$,

(4) m is even, q is odd, p is odd and $e_g(0) = \frac{q-1}{2}$.

Proof. Let g be a divided square difference cordial labeling of a graph G of order p and size q. That is the vertices of G are labeled with numbers $\{1,2,\ldots,p\}$ and $\left|e_g(0) - e_g(1)\right| \leq 1$.

Let $z \in V(G)$ be such that g(z) = 2. Let us denote by H the graph obtained by identifying a vertex z in G and a vertex of degree 1 in P_m.

We define a vertex labeling h of H such that

$$h(v) = g(v), v \in V(G);$$

If m is odd and p is odd, then define $h(y_j)$ as

$$h(y_j) = \begin{cases} p+j-1; & \text{if } j \equiv 2,3 (\text{mod } 4) \text{ and } 2 \le j \le m, \\ p+j; & \text{if } j \equiv 0 (\text{mod } 4) \text{ and } 2 \le j \le m, \\ p+j-2; & \text{if } j \equiv 1 (\text{mod } 4) \text{ and } 2 \le j \le m. \end{cases}$$

If m is odd and p is even, then define $h(y_j)$ as

$$h(y_j) = \begin{cases} p+j-1; & \text{if } j \equiv 0,1 (\text{mod } 4) \text{ and } 2 \le j \le m, \\ p+j; & \text{if } j \equiv 2 (\text{mod } 4) \text{ and } 2 \le j \le m, \\ p+j-2; & \text{if } j \equiv 3 (\text{mod } 4) \text{ and } 2 \le j \le m. \end{cases}$$

If m is even, then define $h(y_j)$ as

$$h(y_j) = \begin{cases} p+j-1; & \text{if } j \equiv 1,2 (\text{mod } 4) \text{ and } 2 \le j \le m, \\ p+j; & \text{if } j \equiv 3 (\text{mod } 4) \text{ and } 2 \le j \le m \\ p+j-2; & \text{if } j \equiv 0 (\text{mod } 4) \text{ and } 2 \le j \le m \end{cases}$$

Thus for the induced edge labeling we get

$$h(uv) = g(vu), v \in V(G);$$

$$h(z\, y_2) = 1 \text{ if (p and m are even)};$$

$$h(z\, y_2) = 0 \text{ if (m is odd) or (p is odd and m is even)};$$

$$h(y_{2j-1}\, y_{2j}) = 0 \text{ if m is odd and } 2 \le j \le \frac{m-1}{2};$$

$$h(y_{2j-1}\, y_{2j}) = 1 \text{ if m is even and } 2 \le j \le \frac{m}{2};$$

$$h(y_{2j}\, y_{2j+1}) = 1 \text{ if m is odd and } 1 \le j \le \frac{m-1}{2}$$

$$h(y_{2j}\, y_{2j+1}) = 0 \text{ if m is even and } 1 \le j \le \frac{m-2}{2}.$$

Let us denote by ${e^*}_h(k)$ number of edges $z\, y_2, y_j\, y_{j+1}$ labelled with k, where k = 0, 1.

Then $|e_h(0) - e_h(1)| = \left|\left(e_g(0) - {e^*}_h(0)\right) - (e_g(1) - {e^*}_h(1))\right|$

$$= \left|e_g(0) - e_g(1) + {e^*}_h(0) - {e^*}_h(1)\right|$$

$$= \left|\frac{q}{2} - \frac{q}{2} + \frac{m-1}{2} - \frac{m-1}{2}\right| = 1, \text{if (m is odd and q is even)};$$

$$= \left|\frac{q+1}{2} - \frac{q-1}{2} + \frac{m-1}{2} - \frac{m-1}{2}\right|$$
$$= 0 \text{ if } \left(\text{m and q are odd and } e_g(0) = \frac{q+1}{2}\right);$$

$$= \left|\frac{q-1}{2} - \frac{q+1}{2} + \frac{m-1}{2} - \frac{m-1}{2}\right|$$
$$= 0 \text{ if } \left(\text{m and q are odd and } e_g(0) = \frac{q-1}{2}\right);$$

$$= \left|\frac{q}{2} - \frac{q}{2} + \frac{m}{2} - \frac{m-2}{2}\right|$$
$$= 0, \text{if (m and q are even, p is odd)};$$

$$= \left|\frac{q}{2} - \frac{q}{2} + \frac{m-2}{2} - \frac{m}{2}\right| = 0, \text{if (m , p and q are even)};$$

$$= \left|\frac{q+1}{2} - \frac{q-1}{2} + \frac{m-2}{2} - \frac{m}{2}\right|$$
$$= 1 \text{ if } \left(\begin{matrix}\text{m and p are even, q is odd} \\ \text{and } e_g(0) = \dfrac{q+1}{2}\end{matrix}\right);$$

$$= \left|\frac{q-1}{2} - \frac{q+1}{2} + \frac{m}{2} - \frac{m-2}{2}\right|$$
$$= 1, \text{if } \left(\begin{matrix}\text{m is even, p and q are odd} \\ \text{and } e_g(0) = \dfrac{q-1}{2}\end{matrix}\right).$$

In the above two case, we observe that $\left|e_g(0) - e_g(1)\right| \le 1$.

Hence Proved.

Theorem 5. Let g be a divided square difference cordial labeling of a graph G of order p and size q. Let $z \in V(G)$ be a vertex such that g(z) = 1. The graph obtained by identifying a vertex z in G and a vertex of degree 1 in P_m admits a divided square difference cordial labeling if one of the following conditions holds:

(1) m is odd,

(2) m and q are even,

(3) m is even, q is odd, p is even and $e_g(0) = \frac{q-1}{2}$,

(4) m is even, q is odd, p is odd and $e_g(0) = \frac{q+1}{2}$.

Proof. The labeling pattern is same as Theorem 4.

Theorem 6. Let g be a divided square difference cordial labeling of a graph G of order p and size q. Let $z \in V(G)$ be a vertex such that $g(z) = 2^r$, where $r = 0,1,2, \cdots$. The graph obtained by identifying a vertex z in G and a vertex of degree m in P_m admits a divided square difference cordial labeling.

Proof. The labelling pattern is same as Theorem 4.

Theorem 7. Let g be a divided square difference cordial labeling of a graph G of order p and size q. Let $z \in V(G)$ be a vertex such that g(z) = 2. The graph obtained by identifying a vertex z in G and a vertex of degree m in F_m admits a divided square difference cordial labeling if one of the following conditions holds:

(1) m and q are even,

(2) m is even and q is odd and $e_g(0) = \frac{q+1}{2}$,

(3) m is odd and q is even,

(4) m and q are odd, p is even and $e_g(0) = \frac{q+1}{2}$,

(5) m, p and q are odd and $e_g(0) = \frac{q+1}{2}$.

Proof. Let g be a divided square difference cordial labeling of a graph G of order p and size q. That is the vertices of G are labeled with numbers $\{1,2, \ldots, p\}$ and $|e_g(0) - e_g(1)| \leq 1$.

Let $z \in V(G)$ be such that g(z) = 2. Let us denote by H the graph obtained by identifying a vertex z in G and a vertex of degree m in F_m.

We define a vertex labeling h of H such that

$$h(v) = g(v), v \in V(G);$$

If m is even, then define $h(x_j)$ as

$$h(y_j) = \begin{cases} p+j+1; & \text{if } j \equiv 3 (\text{mod } 4) \text{ and } 1 \leq j \leq m, \\ p+j-1; & \text{if } j \equiv 0 (\text{mod } 4) \text{ and } 1 \leq j \leq m, \\ p+j; & \text{if } j \equiv 1, 2 (\text{mod } 4) \text{ and } 1 \leq j \leq m. \end{cases}$$

If m is odd, then define $h(x_j)$ as

$$h(x_j) = \begin{cases} p+j+1; & \text{if } j \equiv 2 (\text{mod } 4) \text{ and } 1 \leq j \leq m, \\ p+j-1; & \text{if } j \equiv 3 (\text{mod } 4) \text{ and } 1 \leq j \leq m, \\ p+j; & \text{if } j \equiv 0, 1 (\text{mod } 4) \text{ and } 1 \leq j \leq m. \end{cases}$$

Thus for the induced edge labeling we get

$$h(uv) = g(vu), v \in V(G); \text{for } 1 \leq j \leq m,$$

$h(z\, x_j) = 0$, if (p is odd, m is even and j ≡ 0, 1 (mod 4)) or

(p and m are even and j ≡ 2, 3 (mod 4)) or

(p and m are odd and j ≡ 1, 2 (mod 4)) or

(p is even, m is odd and j ≡ 3, 0 (mod 4));

$h(z\, x_j) = 1$, if (p is odd, m is even and j ≡ 2,3 (mod 4)) or

(p and m are even and j ≡ 0, 1 (mod 4)) or

(p and m are odd and j ≡ 0, 3 (mod 4)) or

(p is even, m is odd and j ≡ 1, 2 (mod 4));

For $1 \leq j \leq m-1$,

$h(x_j\, x_{j+1}) = 1$ if (m is even and j is odd) or (m is odd and j is even);

$h(x_j\, x_{j+1}) = 0$ if (m and j are odd) or (m and j are even);

Let us denote by $e^*_h(k)$ number of edges $z\, x_j, x_j\, x_{j+1}$ labelled with k, where k = 0, 1.

Then $|e_h(0) - e_h(1)| = \left| \left(e_g(0) - e^*_h(0)\right) - (e_g(1) - e^*_h(1)) \right|$

$= \left|e_g(0) - e_g(1) + e^*{}_h(0) - e^*{}_h(1)\right|$

$= \left|\frac{q}{2} - \frac{q}{2} + (m+1) - m\right|$

$= 0,$ if (m and q are even);

$= \left|\frac{q+1}{2} - \frac{q-1}{2} + (m+1) - m\right|$

$= 1$ if $\left(\text{m is even and q is odd and } e_g(0) = \frac{q+1}{2}\right);$

$= \left|\frac{q}{2} - \frac{q}{2} + m - (m-1)\right|$

$= 0$ if (m and p are odd and q is even);

$= \left|\frac{q}{2} - \frac{q}{2} + (m-1) - m\right|$

$= 0,$ if (m is odd and p and q are even);

$= \left|\frac{q+1}{2} - \frac{q-1}{2} + (m-1) - m\right|$

$= 1,$ if $\left(\text{m and q are odd, p is even and } e_g(0) = \frac{q+1}{2}\right);$

$= \left|\frac{q+1}{2} - \frac{q-1}{2} + \frac{m-2}{2} - \frac{m}{2}\right|$

$= 1$ if $\left(\begin{array}{c}\text{m and p are even, q is odd} \\ \text{and } e_g(0) = \dfrac{q+1}{2}\end{array}\right);$

$= \left|\frac{q-1}{2} - \frac{q+1}{2} + m - (m-1)\right|$

$= 1,$ if $\left(\begin{array}{c}\text{m , p and q are odd} \\ \text{and } e_g(0) = \dfrac{q-1}{2}\end{array}\right).$

In the above two case, we observe that $\left|e_g(0) - e_g(1)\right| \leq 1$.

Hence Proved.

Theorem 8. Let g be a divided square difference cordial labeling of a graph G of order p and size q. Let $z \in V(G)$ be a vertex such that g(z) = 1. The graph obtained by identifying a vertex z in G and a vertex of degree m in F_m admits a divided square difference cordial labeling if one of the following conditions holds:

(1) m and q are even,

(2) m is even, q is odd and $e_g(0) = \frac{q+1}{2}$,

(3) m is odd, q is even,

(4) m, p and q are odd and $e_g(0) = \frac{q+1}{2}$,

(5) m, q are odd, p is even and $e_g(0) = \frac{q-1}{2}$.

Proof. Let g be a divided square difference cordial labeling of a graph G of order p and size q. That is the vertices of G are labeled with numbers $\{1,2,\dots,p\}$ and $\left|e_g(0) - e_g(1)\right| \leq 1$.

Let $z \in V(G)$ be such that g(z) = 1. Let us denote by H the graph obtained by identifying a vertex z in G and a vertex of degree m in F_m.

We define a vertex labeling h of H such that

$$h(v) = g(v), v \in V(G);$$

If m is even, then define $h(x_j)$ as

$$h(y_j) = \begin{cases} p+j+1; & \text{if } j \equiv 3 (\text{mod } 4) \text{ and } 1 \leq j \leq m, \\ p+j-1; & \text{if } j \equiv 0 (\text{mod } 4) \text{ and } 1 \leq j \leq m, \\ p+j; & \text{if } j \equiv 1,2 (\text{mod } 4) \text{ and } 1 \leq j \leq m. \end{cases}$$

If m is odd, then define $h(x_j)$ as

$$h(y_j) = \begin{cases} p+j+1; & \text{if } j = 2, \\ p+j-1; & \text{if } j = 1, \\ p+j; & \text{if } j = 1, \\ p+j+1 & \text{if } j \equiv 3 (\text{mod } 4) \text{ and } 4 \leq j \leq m, \\ p+j-1; & \text{if } j \equiv 0 (\text{mod } 4) \text{ and } 4 \leq j \leq m, \\ p+j; & \text{if } j \equiv 1,2 (\text{mod } 4) \text{ and } 4 \leq j \leq m. \end{cases}$$

Using the above labeling and similar method of theorem 8, one can easily verify that the graph obtained by identifying a vertex z in G and a vertex of degree m in F_m admits a divided square difference labeling.

Theorem 9. Let g be a divided square difference cordial labeling of a graph G of order p and size q. Let $z \in V(G)$ be a vertex such that $g(z) = 2^r$, where $r = 0,1,2, \cdots$. The graph obtained by identifying a vertex z in G and a vertex of degree m in F_m admits a divided square difference cordial labeling.

Proof. The labelling pattern is same as Theorem 8.

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