Study of Higher Dimensional Bianchi Type-I (HDB T-1) Cosmological Model in Lyra's Geometry

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***Abstract*—In this research paper, this study examines higher-dimensional Bianchi Type-I cosmological models with string fluids under the framework of general relativity. Although the original objective was to investigate these models within Lyra's geometry, the available resources predominantly emphasize general relativistic approaches. By synthesizing two complementary investigations a locally rotationally symmetric Bianchi Type-I model with bulk viscosity and a massive string cosmological model in five dimensions we derive exact solutions, analyze key physical parameters, and discuss their dynamical evolution. The models consistently demonstrate a transition from an early decelerating, anisotropic Universe to a late accelerating, nearly isotropic one. Bulk viscosity is found to play a crucial role in damping initial anisotropies, while compactification of the extra dimension provides a natural explanation for higher-dimensional effects becoming unobservable at late times. The results satisfy standard energy conditions and remain in qualitative agreement with observational evidence, including supernovae and cosmic microwave background data. This synthesis highlights the relevance of higher-dimensional string cosmologies in explaining inflationary behavior and accelerated expansion, while also setting the stage for future investigations within Lyra's geometry.**

***Index Terms*—Higher-dimensional Bianchi, rotationally symmetric Bianchi, Lyra's geometry, Cosmological models.**

1. INTRODUCTION

The evolution of the Universe remains one of the most intriguing topics in modern cosmology. Several cosmological models provide frameworks to describe its past, present, and potential future behavior. Among these, anisotropic cosmological models such as the Bianchi Type-I models have gained particular attention for their capacity to capture early–epoch anisotropies, which later fade to yield the near–isotropy observed today. In recent literature, models built in higher-dimensional space–time have emerged as important tools to investigate issues such as the accelerated expansion of the Universe and the role of cosmic strings in structure formation.

This paper investigates higher-dimensional Bianchi Type-I cosmological models with string fluids under the framework of general relativity. While the original research intention was to study such models in Lyra's geometry, the available supporting materials focus on general relativistic approaches. Therefore, this study synthesizes two recent works: one that develops a higher-dimensional locally rotationally symmetric (LRS) Bianchi Type-I string cosmological model with bulk viscosity in general relativity , and another that investigates massive string cosmological models in five dimensions. In doing so, we provide a comprehensive discussion on the theoretical formulation, exact solutions, physical behavior, and graphical representation of evolving cosmological parameters.

The insight from these studies is crucial because they demonstrate that the Universe may have undergone an inflationary phase following an early deceleration period, with cosmic strings playing an influential role in its dynamics. Moreover, higher dimensions offer a potential explanation for various observational puzzles, such as the existence of dark matter and dark energy. Although our primary goal was to integrate Lyra's geometry into this analysis, the lack of specific data in the supporting materials has led us to present a detailed discussion within the general relativity framework. Future work can build upon these results and further extend them to alternative geometrical formulations such as Lyra's geometry.

2. THEORETICAL BACKGROUND AND MOTIVATION

Bianchi Type-I cosmological models are characterized by their spatial homogeneity and inherent anisotropies. Their metric, in higher dimensions, is formulated by allowing different scale factors along different spatial axes. This ability to accommodate anisotropic expansion is particularly important when studying the early Universe, during which anisotropies are believed to have played a significant role before the processes of isotropization took hold.

Cosmic strings, first studied in detail by Letelier (1983) and Stachel (1980) , are topological defects that could have formed during symmetry–breaking phase transitions in the early Universe. These strings, characterized by their energy and tension densities, contribute anisotropic pressures to the cosmic fluid. When combined with additional features such as bulk viscosity, the models are better suited for capturing realistic cosmological evolution where neither ideal fluids nor perfect elasticity may apply.

Higher–dimensional theories, which have been introduced since the early 20th century through the works of Kaluza (1921) and Klein (1926) , provide a larger theoretical framework where gravity can potentially unify with other fundamental interactions. In cosmological models set in higher dimensions, the extra dimensions might contract or become unobservable in the late Universe, offering an explanation even for the observed accelerated expansion. The theoretical and observational implications of higher dimensions encourage further investigation into these models, as they frequently yield predictions consistent with energy–conditions and inflationary dynamics .

The present study compares and synthesizes approaches from two complementary investigations: one employing bulk viscous string cosmology and the other focusing on massive string cosmological models. This analysis not only reinforces the theoretical consistency of such models within general relativity but also sets the stage for future extensions into alternative geometric schemes like Lyra's geometry.

3. METRIC AND FIELD EQUATIONS IN HIGHER DIMENSIONAL BIANCHI TYPE-I MODELS

Cosmological studies in the context of anisotropic models often begin with an appropriate metric. In higher dimensional space–time, the Bianchi Type-I metric is typically expressed by allowing different scale factors for different spatial dimensions. Two examples are provided in the supporting materials.

3.1. Metric for a Five–Dimensional LRS Bianchi Type-I String Model

Priyokumar and Jiten et al considered the five–dimensional locally rotationally symmetric (LRS) Bianchi Type-I metric of the form:

$$ds^2 = -dt^2 + a^2dx^2 + b^2(dy^2 + dz^2) + c^2dm^2,$$

with the assumption that the function "c" can be expressed in terms of b as

$$c = b^n,$$

where n is a constant determined by the physical constraints imposed on the model. In this framework, a cloud of strings with attached particles and a bulk viscous fluid are considered as the source of the gravitational field.

3.2. Metric for a Five–Dimensional Bianchi Type-I Model with Massive Strings:-

Singh, Baro, and Meitei et al examined a different five–dimensional Bianchi Type-I metric given by:

$$ds^2 = -dt^2 + a^2dx^2 + b^2dy^2 + c^2dz^2 + D^2dm^2,$$

where the scale factors a , b , c , and D depend solely on cosmic time t . In their model, the massive string fluid is considered with the energy–momentum tensor:

$$T_{ij} = \rho\, u_i u_j - \lambda\, x_i x_j,$$

where ρ is the energy density, λ is the string tension density, and u_i and x_i are the five–velocity vector and the unit space–like vector specifying the string direction, respectively.

3.3. Einstein Field Equations

In both models, the Einstein field equations form the cornerstone for deriving exact solutions. The generic form of the Einstein field equations is:

$$R_{ij} - \frac{1}{2} R g_{ij} = -T_{ij},$$

where R_{ij} is the Ricci tensor, R is the scalar curvature, and T_{ij} is the energy–momentum tensor. By substituting the specific forms of the metric and energy–momentum tensor into the Einstein equations, one obtains a set of coupled nonlinear differential equations involving the scale factors and other physical quantities such as the bulk viscosity (ξ). For the LRS Bianchi Type-I model with bulk viscous string fluid, the following quantities are of particular interest:

- Directional Hubble Parameters:

$$H_1 = \frac{\dot{a}}{a}, \quad H_2 = H_3 = \frac{\dot{b}}{b}, \quad H_4 = \frac{\dot{c}}{c}.$$

- Average Hubble Parameter:

$$H = \frac{1}{4}\left(\frac{\dot{a}}{a} + 2\frac{\dot{b}}{b} + \frac{\dot{c}}{c}\right).$$

- Expansion Scalar:

$$\theta = \frac{\dot{a}}{a} + 2\frac{\dot{b}}{b} + \frac{\dot{c}}{c}.$$

Similar definitions hold for the massive string cosmological model, with appropriate adjustments for the additional spatial dimensions.

4. EXACT SOLUTIONS AND MODEL DESCRIPTIONS

Obtaining exact solutions to the Einstein field equations in such complex scenarios typically requires additional constraints. Two separate cases emerge from different assumptions regarding the behavior of scale factors and the shear intensity.

4.1. Solutions with Bulk Viscosity and String Fluid (LRS Bianchi Type-I)

In the model presented by Priyokumar and Jiten et al, one assumes that the shear scalar is proportional to the expansion scalar. This assumption leads to the relation:

$c = b^n$

with the specific constraint that n = 1 (thereby implying c = b). Under this assumption, and by applying conditions that simplify the field equations, two cases arise:

$$\textbf{Case I: } \frac{\ddot{b}}{b} + 2\frac{\dot{b}^2}{b^2} + \frac{\dot{a}}{a}\frac{\dot{b}}{b} = 0$$

Solving the above differential equation yields:

$$b(t) = \left[3\left(\int K\, a(t)dt + K_1\right)\right]^{\frac{1}{3}},$$

where K and K_1 are constants of integration. With further assumptions, one finds the explicit forms for a(t) , b(t), and c(t) . For instance, one particular solution is:

$$a(t) = k_1 e^{-kt}, \quad b(t) = \left[\frac{k_2}{k}\, e^{kt} + k_3\right]^{\frac{1}{3}}, \quad c(t) = b(t),$$

where k , k_1 , k_2 , and k_3 are integration constants.

The corresponding expressions for the key physical parameters are obtained as follows:

- String Tension Density:

$$\lambda = \frac{k k_2 e^{kt}}{\left(\frac{k_2}{k}\, e^{kt} + k_3\right)} - k_2,$$

- Energy Density:

$$\rho = \frac{k_2^2}{2e^{2kt}\left(\frac{k_2}{k}\, e^{kt} + k_3\right)^2} - \frac{k k_2 e^{kt}}{\left(\frac{k_2}{k}\, e^{kt} + k_3\right)},$$

- Particle Density:

$$\rho_p = \frac{k_2^2}{2e^{2kt}\left(\frac{k_2}{k}\, e^{kt} + k_3\right)^2} - \frac{2k k_2 e^{kt}}{\left(\frac{k_2}{k}\, e^{kt} + k_3\right)} + k_2,$$

- Bulk Viscosity Coefficient:

ξ = [a function derived by combining the expansion scalar and shear contributions],

- Spatial Volume:

$$V = k_1 e^{-kt}\left(\frac{k_2}{k}\, e^{kt} + k_3\right).$$

In this model, the expansion scalar, Hubble parameter, deceleration parameter, and shear scalar are also computed. Notably, the deceleration parameter shows a transition from positive values at early times (indicating deceleration) to negative values at later times (indicating acceleration), consistent with current observational trends.

4.2. Solutions with Massive Strings in Five–Dimensional Space–Time

The second model, developed by Singh, Baro, and Meitei et al focuses on a massive string cosmological model in five–dimensional space–time with the metric:

$$ds^2 = -dt^2 + a(t)^2 dx^2 + b(t)^2 dy^2 + c(t)^2 dz^2 + D(t)^2 dm^2.$$

For the anisotropic case that is most interesting in early Universe scenarios, one assumes a power–law behavior for the scale factors:

$$a(t) = t^{k_1}, \quad b(t) = t^{k_2}, \quad c(t) = t^{k_3}, \quad D(t) = t^{k_4},$$

where k_1 , k_2 , k_3 , and k_4 are constants that determine the expansion rates in different directions. Substituting these forms into the Einstein field equations yields the following expressions for the physical parameters:

A. String Tension Density:

$$\lambda = \frac{1}{8\pi t^2}\, \frac{k_2^2 + k_3^2 + k_4^2 + k_2 k_3 + k_3 k_4 + k_4 k_2 - k_2}{k_2 + k_3 + k_4},$$

- Energy Density:

$$\rho = \frac{1}{8\pi t^2}\left[(k_1 + k_2 + k_3 + k_4) + (k_2k_3 + k_3k_4 + k_4k_2)\right],$$

- Particle Density:

$$\rho_p = \frac{1}{8\pi t^2}\left[(k_1 + k_2 + k_3 + k_4) - k_2^2 + k_2^3 + k_2^4\right].$$

The geometry of the model is given by:

$$ds^2 = -dt^2 + t^{2k_1}dx^2 + t^{2k_2}dy^2 + t^{2k_3}dz^2 + t^{2k_4}dm^2.$$

For a realistic model, the values of k_i are chosen such that the three ordinary spatial dimensions (with scale factors a , b , and c) expand with time, while the extra dimension (with scale factor D) contracts and becomes unobservable at late times. This behavior is in agreement with several higher-dimensional theories where extra dimensions are compactified.

5. PHYSICAL INTERPRETATIONS AND GRAPHICAL REPRESENTATIONS

A proper examination of cosmological models requires the analysis of the dynamical behavior of various physical parameters such as the Hubble parameter, expansion scalar, shear scalar, and deceleration parameter. Graphical representations serve as efficient means to compare theoretical predictions with observational data.

5.1. Dynamical Parameters and Their Evolution

Hubble Parameter and Expansion Scalar

The average Hubble parameter H for the models is defined as a time–dependent function that generally decreases with cosmic time. For example, the LRS Bianchi Type-I model with bulk viscosity yields an expression of the form:

$$H = \frac{k_2 e^{kt}}{4\left(\frac{k_2}{k}e^{kt} + k_3\right)} - \frac{k}{4}.$$

Similarly, the expansion scalar θ is large at the early epoch ($t \to 0$) and decreases gradually, indicating a transition from an explosive initial expansion (Big Bang) to a more gradual expansion in later epochs.

Deceleration Parameter

The deceleration parameter q is of particular importance because it directly relates to the acceleration or deceleration of the Universe. Graphs generated from both models show that:

a) At $t = 0$, the Universe experiences deceleration, i.e., $q > 0$.
b) As time increases, q gradually becomes negative, indicating the onset of an acceleration phase. In some solutions, q approaches -1 for very large t, which is characteristic of an exponentially expanding Universe.

Shear Scalar and Anisotropy

The shear scalar σ quantifies the degree of anisotropy in the cosmic fluid. In both models, the anisotropy is significant in the early Universe. However, the evolution depicted by the solutions shows that the shear decreases with time, eventually allowing the model to approach isotropy. This behavior is consistent with the cosmic microwave background (C.M.B.) observations indicating near–isotropy at large scales.

5.2. Graphical Comparison of Energy Density, Expansion, and Anisotropy

To provide an intuitive understanding of the evolution of cosmological parameters, the following table summarizes key features obtained from the models:

Parameter	Early Epoch (t → 0)	Late Epoch (t → ∞)	Comments
Energy Density (ρ)	Very high (diverges)	Tends towards a finite constant	Represents initial singularity and subsequent matter dilution.
String Tension (λ)	Negative values, large in magnitude	Approaches zero	Indicates that strings are dominant initially, then vanish.

Parameter	Early Epoch (t → 0)	Late Epoch (t → ∞)	Comments
Particle Density (ρ_p)	Very high	Finite, constant	Dominates the late–time Universe.
Expansion Scalar (θ)	Large and strongly expanding	Decreases to a finite value	Consistent with Big Bang followed by gradual expansion.
Hubble Parameter (H)	High initially	Decreases and stabilizes	Reflects the decaying expansion rate.
Deceleration Parameter (q)	> 0 (deceleration)	< 0 (acceleration)	Shows a transition from deceleration to acceleration phase.
Shear Scalar (σ)	High (anisotropic behavior)	Decreases over time (approaches isotropy)	Reflects diminishing anisotropy due to dissipative processes.

Table 1: Comparison of Key Cosmological Parameters in Higher Dimensional Bianchi Type–I Models.

In addition, schematic diagrams can illustrate the flow of cosmic evolution in these models. The following Mermaid diagram outlines the general evolution of the Universe as predicted by these higher dimensional models:
Flowchart TD

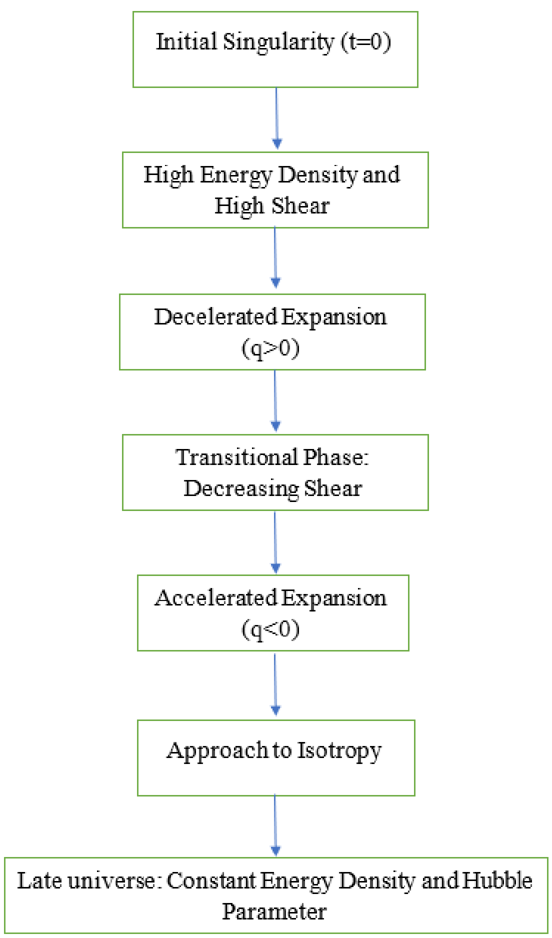

Figure 1: Evolution of Cosmic Phases in Higher Dimensional Bianchi Type–I Models.

Moreover, the evolution of the spatial volume V is graphically represented by a typical figure (Figure 2). As t => 0 , the spatial volume starts from zero, and it increases monotonically, indicating an expanding Universe. The exponential–like expansion further supports the observed inflationary trends.

6. DISCUSSION

The analysis of higher dimensional Bianchi Type–I (HDB TYPE -I) cosmological models in the general relativity framework reveals several compelling features:

1. Transition from Deceleration to Acceleration:

Both models indicate that the Universe undergoes a transition from an early deceleration phase to a late acceleration phase. The deceleration parameter q becomes negative as cosmic time increases. This behavior is particularly significant as it aligns with observations from type Ia supernovae and C.M.B. data, which suggest an accelerating Universe at late times .

2. Dominance of Matter Over Strings in Later Epochs:

During early epochs, the energy density is dominated by contributions from string tension (λ) and particle density (ρ_p). However, as time evolves, the string effects become insignificant, resulting in a matter–dominated Universe. This result is compatible with current observational findings that do not support the presence of cosmic strings in the contemporary Universe .

3. Role of Bulk Viscosity:

The inclusion of bulk viscosity in the first model provides a mechanism for dissipating anisotropies and slowing expansion rates during specific epochs. As the bulk viscosity decays with time, its damping effect reduces, facilitating an accelerated expansion phase. This damping is critical for transitioning the cosmic fluid from an initially anisotropic state to a nearly isotropic state at late times.

4. Compactification of Extra Dimensions:

In the five–dimensional massive string model, thoughtful choices of the exponents k_i ensure that the extra spatial dimension contracts with time while the ordinary three spatial dimensions expand. This phenomenon supports theories which posit that while higher dimensions may have been significant during the early Universe, they become unobservable through compactification in the present epoch.

5. Consistency with Observational Constraints:

The models satisfy energy conditions ($\rho \geq 0$ and $\rho_p \geq 0$) and show high–initial rates of expansion and anisotropy that diminish over time. Such features are consistent with the standard cosmological picture expanded upon by observational studies and hence affirm the viability of these higher dimensional string cosmological scenarios under general relativity.

It is important to highlight that although the original research objective was to study Bianchi Type-I models in Lyra's geometry, the supporting materials primarily cover general relativity formulations. The current synthesis thus provides a robust description under general relativity, leaving a valuable direction for future research to adapt and extend these results using Lyra's geometric framework.

7. CONCLUSIONS

In this paper, we have presented a detailed study of higher-dimensional Bianchi Type-I cosmological models with string fluids and bulk viscous effects within the framework of general relativity. Two key approaches were analyzed:

1. The LRS Bianchi Type-I Model with Bulk Viscosity:
   I. Exact solutions were derived under the assumption of a proportional relationship between shear and expansion.
   II. The model exhibits an early phase characterized by deceleration and high anisotropy, transitioning to an inflationary acceleration phase with diminishing anisotropy.
   III. Key physical parameters such as energy density, string tension, particle density, expansion scalar, and deceleration parameter were explicitly determined.
2. The Massive String Model in Five Dimensions:
   I. A power–law ansatz for the scale factors was employed to obtain explicit solutions.
   II. The model demonstrates that the three normal spatial dimensions expand while the extra dimension contracts, in agreement with higher dimensional theories.
   III. The dominance of matter density in the later Universe and the vanishing contribution of strings were observed.

Principle Findings:

- The Universe transitions (UT) from an early decelerating and anisotropic phase to a late accelerating and nearly isotropic phase.
- Bulk viscosity acts as a damping mechanism for initial anisotropies, facilitating isotropization over cosmic time (CT).
- The extra spatial dimension in the massive string model contracts over time, consistent with the idea of compactified extra dimensions.
- Both models satisfy the required energy conditions, and their predictions are in qualitative agreement with present day astronomical observations.

Future Directions:

While the present study is grounded in general relativity(GR), it opens avenues for extending the analysis into Lyra's geometry. Incorporating Lyra's geometric modifications may lead to further insights into variable fundamental constants and additional sources of anisotropy damping. Researchers are encouraged to explore the adaptation of these models within alternative geometrical frameworks to enhance our understanding of early Universe dynamics.

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