Zero Forcing Problem-A Survey

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***Abstract*—The zero forcing problem is a graph-theoretic process that arose from linear algebraic motivations, namely in limiting the maximum nullity of matrices connected with graphs. It has now grown into a substantial research area, with applications including quantum controllability, network monitoring, and combinatorial optimisation. This paper provides an overview of the zero forcing problem, including definitions, key results with proofs, computational complexity, and applications, as well as a discussion of current research objectives.**

***Index Terms*—Zero Forcing Set, Zero Forcing Number.**

I. INTRODUCTION

Graph theory has emerged as a major research area in mathematics and computer science, with applications ranging from network theory to physics, electrical engineering, and social science. One of the most important challenges in this subject is the Zero Forcing Problem, which offers a combinatorial technique to understanding the spread of influence or knowledge across networks. Beyond its obvious interpretation, the zero forcing problem has important implications for linear algebra, combinatorial optimisation, and theoretical computer science.

The American Institute of Mathematics (AIM) established the notion of zero forcing in 2006 as a method for limiting the maximum nullity of graph-related matrices [7]. In linear algebra, the family of real symmetric matrices with nonzero off-diagonal components corresponds to the adjacency of a graph G G. The nullity of such a matrix relates to the size of its kernel. It was discovered that the zero forcing number offers an upper bound for the maximal nullity in this family, linking graph theory and linear algebra.

Suppose we have a simple graph G with some vertices colored blue. The zero forcing rule is as follows: If a blue vertex v has a neighborhood with a unique uncolored vertex w, w can be colored blue. A zero forcing set of G, is a set of vertices S that, when colored blue, can eventually color all vertices of G using the zero forcing rule. We sometimes simply state that the set S is forcing in G. When a vertex v colors another vertex w, this is referred to as v forcing w.

The study of zero forcing sets in graphs has sparked interest due to its role as a coloring process and its relevance to various fields, including linear algebra [1,2,3], power grid dominance strategies [4,5,6], theoretical computer science [11], modelling physical phenomena [8], quantum system control [9], and rumour spreading models [10]. The zero forcing technique, first described by Burgarth and Giovannetti, has received substantial scientific attention [12,13,14].

The AIM research group established the idea of zero forcing number, which limits the maximum nullity of a graph and is related to other graph characteristics. Research on zero forcing has been conducted in both randomised and reconfigurable settings [15, 16,17]. A few more variations of zero forcing have also been thoroughly investigated.

II. DEFINITION

Let G= (V, E) be a finite, simple, undirected graph.

Zero Forcing Rule: If a black vertex u has exactly one white neighbor v, then v is forced to become black. This is called the color-change rule.

Zero Forcing Set: A set $S \subseteq V(G)$ such that, when the vertices in S are initially colored black and all others white, repeated application of the color-change rule results in all vertices becoming black.

Zero Forcing Number: The minimum cardinality of a zero forcing set in G, denoted Z(G)

Theorem 1.

For a path graph P_n, with $n \geq 2$, Z(Pn)=1.

Proof.

Let Pn=v1−v2−⋯−vn.. Choose S={v1}as the initial black vertex. Then v1 has exactly one white neighbor, v2, so v1 forces v2. Now v2 is black and has exactly one white neighbor, v3 so v2 forces v3

Repeating this argument, each vertex forces the next one along the path. Thus, all vertices are eventually black, so S is a zero forcing set of size 1. Clearly, no smaller set exists, hence Z(Pn)=1

Theorem 2.
For a cycle graph, Cn,with n≥3, Z(Cn)=2.
Proof.
First, observe that a single black vertex cannot force, since every vertex in a cycle has degree 2. Therefore, Z(Cn)≥2. Now, choose two adjacent vertices, say v1, v2, to be black. Then v1 has a unique white neighbor vn (since v2 is black). Thus, v1 forces vn. Similarly, v2 forces v3. This creates two propagation chains in opposite directions, which eventually color the entire cycle black. Thus, two vertices suffice, so Z(Cn)=2.

Theorem 3.
For a complete graph, Kn , with n≥2, Z(Kn)=n−1.
Proof.
Suppose S S is a zero forcing set with ∣S∣≤n−2. Then at least two vertices remain white initially. Since every vertex in Kn is adjacent to all others, each black vertex has at least two white neighbors, so no forcing is possible. Hence Z(Kn)≥n−1.
Conversely, if n−1 vertices are black, then only one vertex remains white. Each black vertex has exactly one white neighbor, so the forcing rule applies immediately, turning the last vertex black. Thus,
Z(Kn)=n−1

Theorem 4.
For a star graph, K1, n−1, with n≥3, Z (K1, n−1) =n−2.
Proof.
Let the central vertex be c and the leaves be ℓ1, ℓ2, ℓn−1. Suppose fewer than n−2 leaves are initially black. Then at least two white leaves exist. The center c, regardless of color, cannot force because it is adjacent to multiple white vertices. Similarly, no leaf can force since each is only adjacent to c. Hence, no forcing occurs.
If n−2 leaves are black, then only the center c and one leaf remain white. Each black leaf sees exactly one white neighbor (the center). Thus, any black leaf forces c. Once c is black, it has exactly one white neighbor (the last leaf), so it forces that leaf. Therefore, Z (K1, n−1) = n−2.

III. ZERO FORCING AND LINEAR ALGEBRA

The study of zero forcing is deeply connected to linear algebra, particularly through the concept of maximum nullity of matrices associated with a graph. Given a graph G, one considers the family of real symmetric matrices whose off-diagonal nonzero entries correspond to the adjacency of G. The maximum nullity M(G) is the largest possible nullity among these matrices, while the minimum rank is
∣V(G)∣−M(G). It was shown that the zero forcing number Z(G) provides an upper bound for the maximum nullity, i.e., M(G) ≤ Z(G). This bound links a purely combinatorial process—the propagation of colors under the zero forcing rule—to an algebraic invariant of matrices, making zero forcing a powerful tool for estimating rank-related parameters and for bridging graph theory with spectral and matrix theory [1,2.3.9]. Given a graph G, consider the family of real symmetric matrices A whose off-diagonal entries correspond to the adjacency of G. The maximum nullity of G, denoted M(G), is defined as:
M(G)=max{nullity(A): A∈S(G)}, where S(G) is the set of such matrices. It is known that: M(G)≤Z(G). Thus, the zero forcing number provides an upper bound for maximum nullity, connecting combinatorial and algebraic graph theory

IV. COMPUTATIONAL COMPLEXITY

Determining Z(G) for an arbitrary graph is NP-hard [8]. However, polynomial-time algorithms exist for restricted classes such as trees and graphs of bounded pathwidth. Z(G) of a general graph is NP-hard (Aazami, 2008), which indicates that no polynomial-time solution exists to solve it across all graph families. This difficulty stems from the combinatorial explosion of potential beginning sets that must be examined to determine whether they are zero forcing sets. Despite this general difficulty, efficient algorithms exist for specific types of graphs. For example, zero force numbers can be computed in polynomial time for pathways, cycles, trees, and graphs with bounded treewidth. Approximation algorithms and heuristic techniques have also been investigated, although the gap between exact computing and efficient approximation remains a topic of interest in theoretical computer science [18,19]

V. APPLICATIONS

One of the most notable uses of zero forcing is in the study of quantum systems, specifically the controllability of quantum spin networks. In these networks, vertices represent quantum spins, while edges reflect their interactions. The capacity to steer the entire system to a desired quantum state with only a small number of control inputs is critical for quantum computation and information processing. Zero forcing is a combinatorial method for testing whether such controllability is possible: if the initially controlled vertices form a zero forcing set in the underlying interaction graph, the entire system can be controlled in principle. This connection, first highlighted by Burgarth and Giovannetti [10,13] demonstrates how a purely graph-theoretic process can capture deep physical behavior, linking structural properties of graphs to the dynamics of quantum systems.

Zero forcing also finds important applications in the monitoring and control of networks, particularly in the context of electrical power grids and sensor placement problems. In such systems, operators aim to observe or control the entire network using measurements from only a small subset of nodes. The zero forcing process models how information, once available at certain key vertices, can propagate through the network under well-defined rules. In power systems, this concept is closely related to the power domination problem, where the placement of phase measurement units (PMUs) is optimized to guarantee complete observability of the grid. By identifying minimum zero forcing sets, one can determine efficient strategies for network monitoring that minimize cost while ensuring reliability. This has broader applications in communication networks, epidemic tracking, and infrastructure monitoring, where the spread of observability mimics the spread of color in the zero forcing process. Network Monitoring: Closely related to the power domination problem, zero forcing model's sensor placement for full observability of electric power grids [4,5,6,10].

Zero forcing provides a mathematical framework to study controlled information propagation, helping in optimizing influencer selection, predicting network reach, and designing efficient communication strategies. It serves as a model for rumor spreading, fault detection, and infection processes in networks [20 - 27]

VII. CONCLUSION

The zero forcing problem brings graph theory, linear algebra, and network research together in a unique way. While essential conclusions for standard graphs are known, there are still issues in determining accurate values for larger classes and understanding the complexity of connected parameters. The field continues to draw attention because to its elegant rules, computational problems, and broad applicability.

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