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Game Theory in Topological Spaces: A Study of Strategic Equilibria and Fixed-Point Principles

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***Abstract*—Game theory traditionally analyses strategic interactions in finite or convex spaces. However, many real-world systems—such as distributed networks, behavioural economics, and evolutionary biology—exhibit complex topological structures. This paper explores the application of game theory within the framework of topological spaces, highlighting the role of fixed point theorems, continuity, and compactness in the existence and stability of Nash equilibria. We provide formal definitions, illustrate theoretical models, present key results, and discuss implications for applied game-theoretic problems across economics, control systems, and information theory and a real-life example of an infinite game along with a solution approach using game theory and topology.**

***Index Terms*—Game Theory, Equilibria, Strategies, Nash Equilibrium.**

I. INTRODUCTION

Game theory models rational decision-making in multi-agent systems. Typically, strategies are chosen from convex, Euclidean spaces under assumptions of continuity and completeness. However, in advanced scenarios—such as infinite strategy spaces, network games, or function-based strategy sets—the underlying structure is better captured by topological spaces.

Topological tools such as Brouwer's, Kakutani's, and Fan's Fixed Point Theorems provide foundational mechanisms for equilibrium analysis in such spaces. The fusion of topology with game theory enables the study of infinite games, topological preference spaces, and strategic interactions on manifolds and graphs.

II. PRELIMINARIES AND DEFINITIONS

Definition 2.1 Topological Space - A topological space (X, τ) is a set X equipped with a topology τ, i.e., a collection of open sets satisfying:

i. $\emptyset, X \in \tau$
ii. Any union of sets in τ belongs to τ
iii. Any finite intersection of sets in τ belongs to τ

Definition 2.2 Strategic Game- A strategic game in topological terms is a tuple
$G = (N, \{S_i\}_{i\in N}, \{u_i\}_{i\in N})$
where:

i. N is the finite set of players
ii. S_i is a non-empty compact topological space of strategies for player i
iii. $u_i\colon \prod_{j\in N} S_j \to \mathbb{R}$ is a continuous utility function

Definition 2.2 Nash Equilibrium in Topological Spaces- A strategy profile $s^* \in \prod_{i\in N} S_i$ is a Nash equilibrium if:

$$u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*) \forall s_i \in S_i, i \in N$$

III. EXISTENCE THEOREMS IN TOPOLOGICAL GAMES

Theorem 3.1 Brouwer's Fixed Point Theorem- If $S \subset \mathbb{R}^n$ is convex, compact, and non-empty, and $f\colon S \to S$ is continuous, then there exists $x \in S$ such that $f(x) = x$.
Used in proving existence of Nash equilibria when strategy spaces are finite-dimensional.
Theorem 3.2 Kakutani's Fixed Point Theorem- Let S be a compact convex subset of a topological vector space, and $F\colon S \to S$ be an upper semi-continuous correspondence with non-empty, convex values. Then F has a fixed point.

This generalizes Brouwer's theorem and is instrumental in games with mixed strategies and discontinuous preferences.

Theorem 3.3 Fan–Glicksberg Fixed Point Theorem- Let S be a compact convex subset of a locally convex topological vector space and $f: S \to S$ be a continuous map. Then f has a fixed point. Extends Nash equilibrium existence to infinite-dimensional strategy spaces.

IV. TOPOLOGICAL CONCEPTS IN GAME MODELS

4.1 Infinite Games

i. Players choose strategies from $[0,1]^{\mathbb{N}}$
ii. Requires Tychonoff topology (product topology)
iii. Fixed-point theorems still apply if compactness is ensured

4.2 Games on Networks (Graph Topologies)

i. Strategy sets are nodes
ii. Topology derived from adjacency relations
iii. Distance games or congestion games use these topologies for payoffs

4.3 Manifold Games

i. Strategy space is a smooth manifold (e.g., sphere, torus)
ii. Applications: quantum games, evolutionary dynamics
iii. Nash equilibria characterized by topological degree theory

V. REAL-LIFE EXAMPLE: PRICE COMPETITION IN ONLINE MARKETPLACES

Here's a real-life example of an infinite game along with a solution approach using game theory and topology:

Scenario:

Two competing online retailers Amazon and Flipkart are selling the same product (say, a smartphone). Each retailer can choose any price from a continuous interval, say [5000, 50000] INR. Prices can be chosen up to infinite precision (e.g., ₹9999.99, ₹9999.9999, etc.). This results in an infinite strategy space.

Players:

Player 1: Amazon

Player 2: Flipkart

Strategy Space:

Each player chooses a price $p_i \in [5000,50000]$, a compact interval on the real line ⇒ an infinite set of strategies.

Payoff Function: Let customers choose the retailer with the lower price.

If both offer the same price, they split the market.

Let Market size = ₹10,00,000

Cost per phone = ₹8000

Then the utility function for each retailer is:

$$u_i(p_i, p_j) = \begin{cases} (q.(p_i - 8000)) & if p_i < p_j \\ \left(\frac{q}{2}.(p_i - 8000)\right) & if p_i = p_j \\ 0 & if p_i > p_j \end{cases}$$

where q is the number of units sold.

Game Theoretic Nature:

This is a Bertrand competition with continuous strategy spaces.

Players can undercut each other infinitely finely.

This is a non-cooperative, zero-sum game with infinite strategies.

Infinite Game Characteristics:

- Strategy space: $[5000,50000] \subset\Rightarrow \mathbb{R}$ uncountably infinite
- Payoff function: Piecewise, depends on price comparison
- Not guaranteed to have a pure strategy Nash equilibrium
- Requires mixed strategies or equilibrium in the limit

VI. SOLUTION USING FAN–GLICKSBERG FIXED POINT THEOREM

Step 1: Define the Game: We model a 2-player Bertrand price competition game as follows:

Players: Player 1: Amazon

Player 2: Flipkart

Strategy Spaces:

Each player chooses a price $p_i \in S = [p_{min}, p_{max}] = [8000,50000]$

So strategy space $S_1 = S_2 = [8000,50000] \subset \mathbb{R}$

These are:

i. Compact: Closed and bounded interval
ii. Convex: Every convex combination of prices is also a valid price

iii. Nonempty

Step 2: Define Payoff Functions
Let each player sell a fixed number of units (say, 100 units).
Each chooses a price $p_i \in S$, where cost = ₹8000.

Let payoff for player 1 (Amazon) be:

$$u_1(p_1, p_2) = \begin{cases} (100.(p_1 - 8000)) & if p_1 < p_2 \\ (50.(p_1 - 8000)) & if p_1 = p_2 \\ 0 & if p_2 > p_1 \end{cases}$$

Similarly,

$$u_1(p_1, p_2) = \begin{cases} (100.(p_2 - 8000)) & if p_1 < p_2 \\ (50.(p_2 - 8000)) & if p_1 = p_2 \\ 0 & if p_2 > p_1 \end{cases}$$

Step 3: Convert to Mixed Strategy Game
Each player can now randomize over the interval S using probability measures (i.e., choose a distribution over prices).
Let: Δ(S) : set of Borel probability measures on $S = [8000,50000]$
So, Player 1's mixed strategy $\mu_1 \in \Delta(S)$
Player 2's mixed strategy $\mu_2 \in \Delta(S)$
Payoff becomes:

$$U_1(\mu_1, \mu_2) = \int_{S \times S} u_1(p_1, p_2)\, d\mu_1(p_1) d\mu_2(p_2)$$

$$U_2(\mu_1, \mu_2) = \int_{S \times S} u_2(p_1, p_2)\, d\mu_1(p_1) d\mu_2(p_2)$$

Step 4: Verify Conditions of Fan–Glicksberg Theorem
Strategy spaces $\Delta(S_1), \Delta(S_1)$ are:

i. Non-empty: There exist probability measure on nonempty compact interval S

So $\Delta(S) \neq \emptyset$.

ii. Compact: S is compact metric; by Prokhorov's theorem (specialized to compact metric spaces) the set Δ(S) of Borel probability measures is tight and hence sequentially compact for the weak (narrow) topology; in fact Δ(S) is compact and metrizable
iii. Convex: if $\mu, \nu \in \Delta(S)$ and $\lambda \in [0,1]$ then $\lambda\mu + (1-\lambda)\nu$ is a probability measure on S . So $\Delta(S)$ convex.
iv. Locally convex: $\Delta(S)$ lies in the dual of $C(S)$, which is locally convex

Payoffs U_1, U_2 are upper semi-continuous in mixed strategies,
Best response correspondence $BR_i(\mu_{-i})$ is:

i. Nonempty (due to Weierstrass)
ii. Convex-valued (since maximizing over linear functionals)
iii. Upper semi-continuous

All conditions of Fan–Glicksberg fixed point theorem are satisfied.

There exists at least one Nash equilibrium $(\boldsymbol{\mu_1^*}, \boldsymbol{\mu_2^*}) \in \boldsymbol{\Delta(S)} \times \boldsymbol{\Delta(S)}$
i.e., Amazon and Flipkart have equilibrium mixed strategies, choosing prices from distributions over [8000,50000], such that no player benefits by unilaterally deviating.

VII. SIMPLIFIED PRICE COMPETITION GAME

Let's construct a complete numerical solution for the infinite game of price competition between Amazon and Flipkart
Setup:
Two players (Amazon and Flipkart) choose prices from:
Price Set $P = \{9000, 9500, 10000, 10500, 11000\}$

Each player's cost per unit: ₹8000
Market size: 100 units
Customers always buy from the cheaper seller. If prices are equal, each seller gets 50 units.

7.1 Payoff Rules

If player i's price p_i is:
Lower than p_j: sells 100 units
Equal to p_j: sells 50 units
Higher than p_j: sells 0 units

$$Profit = (unitssold) \times (p_i - 8000)$$

7.2 Payoff Matrix Construction

Let rows = Amazon prices, columns = Flipkart prices
Each cell = (Amazon profit, Flipkart profit)

Price A / Price F	9000	9500	10000	10500	11000
9000	(50×1000, 50×1000) = (50k, 50k)	(100×1000, 0) = (100k, 0)	(100k, 0)	(100k, 0)	(100k, 0)
9500	(0, 100×1000) = (0, 100k)	(50×1500, 50×1500) = (75k, 75k)	(100×1500, 0) = (150k, 0)	(150k, 0)	(150k, 0)
10000	(0, 100k)	(0, 150k)	(50×2000, 50×2000) = (100k, 100k)	(200k, 0)	(200k, 0)
10500	(0, 100k)	(0, 150k)	(0, 200k)	(50×2500, 50×2500) = (125k, 125k)	(250k, 0)
11000	(0, 100k)	(0, 150k)	(0, 200k)	(0, 250k)	(50×3000, 50×3000) = (150k, 150k)

7.3 Best responses (pure strategy analysis)

We analyze Amazon's best response to each Flipkart pure price (i.e., for each column, which row gives Amazon the largest payoff):

If Flipkart = 9000 (col 1): Amazon's payoffs by row = [50k, 0, 0, 0, 0]

best = 50k at 9000 (tie is best here).

If Flipkart = 9500 (col 2): Amazon's payoffs = [100k, 75k, 0, 0, 0]

best = 100k at 9000 (undercut to 9000).

If Flipkart = 10000 (col 3): Amazon's payoffs = [100k,150k,100k,0,0]

best = 150k at 9500.

If Flipkart = 10500 (col 4): Amazon's payoffs = [100k,150k,200k,125k,0]

best = 200k at 10000.

If Flipkart = 11000 (col 5): Amazon's payoffs = [100k,150k,200k,250k,150k]

best = 250k at 10500.

So Amazon's pure best responses vector : [9000, 9000, 9500, 10000, 10500].

Symmetrically, Flipkart's best responses to Amazon's pure prices

If Amazon = 9000 (row1): Flipkart payoffs by col = [50k, 0, 0, 0, 0]

best = 50k at 9000 (tie).

If Amazon = 9500: Flipkart's payoffs = [100k,75k,0,0,0]

best = 100k at 9000 (undercut).

If Amazon = 10000: payoffs = [100k,150k,100k,0,0]

best = 150k at 9500.

If Amazon = 10500: payoffs = [100k,150k,200k,125k,0]

best = 200k at 10000.

If Amazon = 11000: payoffs = [100k,150k,200k,250k,150k]

best = 250k at 10500.

So Flipkart's pure best responses: [9000,9000,9500,10000,10500] symmetric pattern.

7.4 Existence of pure-strategy Nash equilibrium

A pure-strategy Nash equilibrium (p_A^*, p_F^*) must satisfy that p_A^* is a best response to p_F^* and p_F^* is a best response to p_A^*.

Check diagonal candidates (ties)

first: (9000,9000):

Amazon best response to 9000 is 9000 (yes);

Flipkart best response to 9000 is 9000 (yes).

So (9000,9000) is a pure-strategy Nash equilibrium on this discrete grid.

Second: (9500,9500): Is 9500 best response to 9500? Amazon's best response to 9500 is 9000 (not 9500) → not Nash equilibrium.

Third: (10000,10000): best response to 10000 is 9500 → not Nash equilibrium.

Fourth: (10500,10500): best response to 10500 is 10000 → not Nash equilibrium
Fifth: (11000,11000): best response to 11000 is 10500 → not Nash equilibrium
On this discrete grid, (9000,9000) is the only pure-strategy NE. That is because 9000 is the lowest available price in the grid — it blocks further undercutting. If grid included 8000 (cost), (8000,8000) might appear; if grid allowed arbitrarily fine prices down to cost, there would be no pure-strategy NE above cost.

VIII. MIXED STRATEGY NASH EQUILIBRIUM (NUMERICAL APPROXIMATION)

Let's assume both players randomize (i.e., mixed strategy) over {9000,9500,10000}
Let Amazon plays $p_1 = 9000$ with probability x_1, 9000 with x_1, and 10000 with x_3
Flipkart does the same with y_1, y_2, y_3
Let's assume symmetric strategy: $x_1 = y_1 = 0.5, x_2 = y_2 = 0.3, x_3 = y_3 = 0.2$
Now compute Amazon's expected profit:

$$E(u_A) = \sum_{i,j} x_i y_j \,. Amazon'sprofitat(p_i, p_j)$$

A Price	F Price	Prob.	A Profit
9000	9000	0.25	50k → 12.5k
9000	9500	0.15	100k → 15k
9000	10000	0.10	100k → 10k
9500	9000	0.15	0 → 0
9500	9500	0.09	75k → 6.75k
9500	10000	0.06	150k → 9k
10000	9000	0.10	0 → 0
10000	9500	0.06	0 → 0
10000	10000	0.04	100k → 4k

Total Expected Profit for Amazon**:**

$$E(u_A) = 12.5 + 15 + 10 + 0 + 6.75 + 9 + 0 + 4 = 57.25 thousand$$

Same expected profit for Flipkart due to symmetry.

8.1 Final Interpretation

i. Firms randomize pricing strategies to avoid being undercut
ii. No pure-strategy equilibrium exists
iii. Mixed strategy ensures both players earn stable, expected profits
iv. As the price space becomes more granular (e.g., ₹1 increments), the game approaches a true infinite game, solvable using Fan–Glicksberg fixed point theorem

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