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Second Derivative Parameter Dependent General Linear Method for stiff ODEs

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***Abstract*—The development of efficient and stable numerical methods for the integration of stiff ordinary differential equations (ODEs) remains a central challenge in computational mathematics. This paper introduces and analyzes *Second Derivative Parameter Dependent General Linear Methods (SDPD-GLMs)*, a novel class of general linear methods that incorporate both the first and second derivative information of the underlying system together with free parameters for optimization. The inclusion of second derivative terms enhances the accuracy of the scheme, while the parameter dependence allows flexibility in optimizing stability, error control, and implementation efficiency. Order conditions are derived through Taylor series expansion and algebraic analysis, establishing a systematic framework for constructing high-order methods. The stability properties of the proposed schemes are investigated, with emphasis on A-stability and L-stability criteria relevant for stiff problems. Numerical experiments on benchmark initial value problems demonstrate that SDPD-GLMs achieve improved accuracy and efficiency compared with existing Runge–Kutta and traditional general linear methods, particularly in the stiff regime. The results highlight the potential of SDPD-GLMs as a robust tool for solving a broad class of stiff ODEs, providing a balance between high accuracy, computational efficiency, and stability.**

***Index Terms*—stiff ordinary differential equations, second derivative parameter dependent, general linear methods, stability, accuracy, RKM.**

I. INTRODUCTION

In this paper, a second-derivative, parameter-dependent General Linear Method (SDPD-GLM) for solving stiff ordinary differential equations (ODEs) was derived. The method uses information from the first and second derivatives, and its coefficients depend on parameters that allow for tuning stability and accuracy. According to Okuonghae and Ikhile (2014), Agbeboh et.al (2019).The two well known methods for solving ODEs are the Runge-Kutta methods (RKM) and the linear multistep methods (LMM) but the major problems associated with the two are explicit RKM requires a number of function evaluations at every integration step, while the LMM on the other hand requires multiple function evaluation to start off. Secondly, for implicit RKM, it courses very high implementation cost because it requires in addition to solve large nonlinear system of equations, while on the other hand, the LMM suffer from the inability to by-pass the Dalquist's order barriers see Moradi et.al (2019) and Ramazani et.al (2022). Therefore, according to Izzo and Jackiewicz (2020), to overcome these barriers, in 1996, Butcher introduced the general linear methods (GLM) as a unifying framework for studying stability, consistency and convergence for a wide range of methods that include the RKM, LMM and in 2002 he also introduced the hybrid methods which is the subclasses of the GLM for seeking the numerical integration of IVPs in ODEs. See Alonso-Mallo, and Reguera (2024)

$$\begin{cases} y' = f(y(x)), \quad x \in [x_0, X] \\ y(x_0) = y_0 \in R^m \end{cases} \tag{1}$$

where m is the dimension of the differential equations. The general form of GLM is given by see Butcher and Hojjati (2005).

$$\begin{cases} Y_i = h\sum_{j=1}^{s} a_{ij}F_j + \sum_{j=1}^{k} u_{ij} y_j^{(n-1)} \\ y_i^{(n)} = h\sum_{j=1}^{s} b_{ij}F_j + \sum_{j=1}^{k} v_{ij} y_j^{(n-1)} \end{cases}, \quad F_i = f(Y_i)\ i = 1,2,\ldots s\ \&\ k, \tag{2}$$

Where $n = 1,2,\ldots N, x_{n+1} = x_n + h.$ Here Y_i approximate $y(x_n + c_i h),\ i = 1,2,\ldots s,$ up to the stage order q, and $y_i^{(n)}$ is the outgoing approximation in step n and is of order p and c_i is the abscissa vector component representing the positions of the internal stages. The GLM (2) in the $(s+k)\times(s+k)$ partitioned matrix is written as

$$\left(\frac{Y}{y^{(n)}}\right) = \begin{pmatrix} A & U \\ B & V \end{pmatrix}\left(\frac{hF}{y^{(n-1)}}\right) \tag{3}$$

Where

$Y = (Y_1, Y_2, \ldots Y_S)^T$, $y^{[n]} = (y_1^{[n]}, y_2^{[n]}, \ldots y_k^{[n]})^T$, $F = (F_1, F_2, \ldots F_S)^T$, $A = \{a_{ij}\} \in \Re^{(s\times k)}$, $B = \{b_{ij}\} \in \Re^{(k\times k)}$, $U = \{u_{ij}\} \in \Re^{(s\times s)}$ *and* $V = \{v_{ij}\} \in \Re^{(k\times s)}$. In Okuonghae and Ikhile (2014), the stability of the method in (3) is determined from the stability matrix

$$M(z) = V + zB(I - zA)^{-1}U \tag{4}$$

Furthermore, the characteristics polynomial of (3) is

$$\pi(w,z) = \det(wI - M(z)) \tag{5}$$

Let consider the following examples that illustrate the transformation of some well known traditional schemes to GLM. First, the classical 4th-order RKM with Butcher tableau see Moradi et.al (2022).

$$\begin{array}{c|cccc} 0 & & & & \\ \frac{1}{2} & \frac{1}{2} & & & \\ \frac{1}{2} & 0 & & & \\ 1 & 0 & 0 & 1 & \\ \hline & \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} \end{array} \tag{6}$$

The GLM formation of the 4th-order RKM above is

$$\begin{pmatrix} Y_1 \\ Y_2 \\ Y_3 \\ Y_4 \\ \hline y_n \end{pmatrix} = \left(\begin{array}{cccc|c} 0 & 0 & 0 & 0 & 1 \\ \frac{1}{2} & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ \hline \frac{1}{6} & \frac{1}{3} & \frac{1}{3} & \frac{1}{6} & 1 \end{array}\right)\begin{pmatrix} hF_1 \\ hF_2 \\ hF_3 \\ hF_4 \\ \hline y_{n-1} \end{pmatrix} \tag{7}$$

A second example is the hybrid LMM

$$y_{n-\frac{1}{2}} = y_{n-2} + h\left(\frac{9}{8} f(y_{n-1}) + \frac{3}{8} f(y_{n-2})\right),$$

$$y_n^* = \frac{28}{5} y_{n-1} + \frac{23}{5} y_{n-2} + h\left(\frac{32}{15} f(y^*_{n-\frac{1}{2}})\right) - 4f(y_{-1}) = \frac{26}{15} f(y_{n-2})$$

$$y_n = \frac{32}{31} y_{n-1} - \frac{1}{31} y_{n-2} + h\left(\frac{5}{31} f(y_n^*) + \frac{64}{93} f\left(y^*_{n-\frac{1}{2}}\right) + \frac{4}{31} f(y_{n-1}) - \frac{1}{93} f(y_{n-2})\right) \tag{8}$$

And written as a GLM below

$$\begin{pmatrix} y_{n-\frac{1}{2}} \\ y_n^* \\ y_n \\ \hline y_n \\ y_{n-1} \\ hf_n \\ hf_{n-1} \end{pmatrix} = \left(\begin{array}{ccccccc} 0 & 0 & 0 & 0 & 1 & \frac{9}{8} & \frac{3}{8} \\ \frac{12}{15} & 0 & 0 & \frac{28}{3} & -\frac{23}{5} & -4 & -\frac{26}{15} \\ \frac{64}{93} & \frac{5}{31} & 0 & \frac{32}{31} & -\frac{1}{31} & \frac{4}{31} & -\frac{1}{93} \\ \hline \frac{64}{93} & \frac{5}{31} & 0 & \frac{32}{31} & -\frac{1}{31} & \frac{4}{31} & -\frac{1}{93} \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{array}\right) = \begin{pmatrix} hf_{n-\frac{1}{2}} \\ hf_n^* \\ hf_n \\ \hline y_{n-1} \\ y_{n-2} \\ hf_{n-1} \\ hf_{n-2} \end{pmatrix} \tag{9}$$

Various methods can be transformed in a similar manner into GLM.

II. THE SECOND DERIVATIVE GENERAL LINEAR METHOD (SD-GLM)

The new GLM we wish to introduce incorporate the second derivative functions. The general form of the second derivative GLM is given as

$$\begin{cases} Y_Y = h^2 \sum_{j=1}^{s} a_{ij} F'_J + h \sum_{j=1}^{s} a_{ij} F_j + \sum_{j=1}^{k} u_{ij} y_j^{[n-1]}, \; F_I f(Y_i),\; F' = f'(Y_i),\; i = 1,2,..s_i \\ y_i^{[n]} = h^2 \sum_{j=1}^{s} b_{ij} F'_j + h \sum_{j=1}^{s} b_{ij} F_j + \sum_{j=1}^{k} v_{ij} y_j^{[n-1]},\; F_i = f(Y_i),\; F_i = f'(Y_i),\; i = 1,2,..k \end{cases} \tag{10}$$

Which is an extension of the classical GLM see [Roberts et.al (2020)]? This Second Derivative General Linear Methods (SD-GLM) has the advantage of higher order and having A-stable, L-stable higher methods for a given number of stages compared with [Abdi et.al (2022)]. The new method in (10) in line with that of the GLM format in (3) is represented as

$$\left(\frac{Y}{y^{[n]}}\right) = \left(\begin{array}{ccc} A & \hat{A} & \alpha \\ \hline b & \hat{b} & \hat{\alpha} \end{array}\right) \begin{pmatrix} hF(Y) \\ h^2 F'(Y) \\ \hline y^{[n-1]} \end{pmatrix} \tag{11}$$

Where

$Y=(Y_1,Y_2,...Y_S),\ y^{[n]}=(y_1^{[n]},y_2^{[n]},...y_k^{[n]}),\ \ F(Y)=((F(Y_1),F(Y_2),...F(Y_s))^T\ and\ \ F'(Y)=(F'(Y_1),$ $F'(Y_2),...F'(Y_S))^{T.}\ A=\{a_{ij}\}\in\Re^{(s\times s)},\ \hat{A}\ \hat{\alpha}=\{\hat{a}_{ij}\}\in\Re^{(s\times k)},\ b=\{b_{IJ}\}\in\Re^{(K\times K)},\ \ \hat{b}=\{\hat{b}_{ij}\}\in\Re^{(k\times k)},$ $\alpha=\{\alpha_{ij}\}\in\Re^{(s\times s)}\ \ and\ \ \hat{\alpha}=\{\alpha_{ij}\}\in\Re^{(k\times s)}\ if\ \ \alpha=\ell.\hat{\alpha}=\ell,\ with\ \ \ell=(1,1,...,1)$

we have a GLM formulation of second derivative RKM (SDRKM), see [Okuonghae and Ikhile (2014)].

III. DERIVATION OF THE METHOD (SDPD-GLM)

This is a **s**econd derivative parameter-dependent generalization of the θ-method**,** where additional terms involving the Jacobian f_y and second-order corrections are included.

Considering the initial value problem

$$y'(x)=f(y(x)),\ \ y(x_0)=y_0,\ with\ \ stepsize\ \ h \tag{12}$$

Expanding (12) $y(x_{n+1})\ \ about\ \ x_n$ and applying the parameter $\theta-method$ approximation, we obtain.

$$y_{n+1}=y_n+h\int_0^1 f(y(x_n+\phi h))d\phi \tag{13}$$

Approximating θ in (13) gives the first derivation of the method

$$\int_0^1 f(y(x_n+\phi h))d\phi=(1-\theta)f(y_n)+\theta f(y_{n+1}) \tag{14}$$

Such that

$$y_{n+1}=y_n+h\left[(1-\theta)f(y_n)+\theta f(y_{n+1})\right] \tag{15}$$

This is the classical $\theta-method$ such that $\theta\neq 0$ (implicit)

Incorporating the second derivative term to improve accuracy, the second derivative correction is applied using Taylor series expansion to obtain

$$\frac{h^2}{2}y''(x_n)=\frac{h^2}{2}f_y(y_n)f(y_n) \tag{16}$$

Instead of evaluating exclusively at x_n, we introduce a parameter λ to form a convex combination of evaluations at y_n and y_{n+1}:

$$\frac{h^2}{2}\left[(1-\lambda)f_y(y_n)f(y_n)+\lambda f_y(y_{n+1})f(y_{n+1})\right] \tag{17}$$

Combining the *θ*-method approximation with the parameterized second derivative correction, we obtain the general formula of the second derivative parameter dependent general linear method (SDPD-GLM):

$$y_{n+1}+y_n+h\left[(1-\theta)f(y_n)+\theta f(y_{n+1})\right]+\frac{h^2}{2}\left[(1-\lambda)f_y(y_n)f(y_n)+\lambda f_y(y_{n+1})f(y_{n+1})\right] \tag{18}$$

IV. STABILITY ANALYSIS OF (SDPD-GLM)

In this section, we investigate the stability properties of the methods define in (18). According to Robert (2013), a numerical integrator would be stiffly stable provided;

(a) its region of absolutely stability contains R_1 and R_2 and

(b) it is accurate for all $h \in R_2$ when applied to the scalar test equation $y' = \lambda y$, λ a complex constant with $\mathrm{Re}(\lambda) < 0$ where $R_1 = \{z : |\mathrm{Re}(z)| < -D\}$, $R_2 = \{z : -D \le |\mathrm{Re}(z)| \le \delta,\ -c \le |\lim(z)| \le c\}$ *and* $D, \delta\ c$ are positive constant. Also, for the linear test problem $y' = \mu y$, with step size h, we define $z = h\mu$. The numerical method is said to be stable for values of z such that $|R(z)| \le 1$. The set of all such z in the complex plane is called the stability region. If the entire left half-plane is included, the method is A-stable. Applying the derived methods in (18) to the Dahlquist scalar test equation $y' = \lambda y$, $\mathrm{Re}(\lambda) < 0$, *and* $z = \lambda h$ gives the general stability polynomial of the method below

$$R(z) = \frac{[1 + (1-\theta)z + (1-\lambda)\frac{z^2}{2}]}{[1 - \theta z - (\lambda/2)z^2} \tag{19}$$

This is the stability polynomial of the derived method

The poles of the function occur when the denominator vanishes: $1 - \theta z - \left(\frac{\lambda}{2}\right)z^2 = 0$, *for* $\lambda \ne 0$

the poles are the roots of the quadratic equation: $z = \frac{[-\theta \pm \sqrt{(\theta^2 + 2\lambda)}]}{\lambda}$, *when* $\lambda = 0$, the denominator is linear and yields a single pole at $z = \frac{1}{\theta}$ $(if\ \ \theta \ne 0)$. These poles typically appears on the real axis and strongly influence the shape and location of the stability region.

To visualize the stability regions, we evaluated $R(z)$ over a fine grid in the complex plane and plotted the sets where $|R(z)| \le 1$. The dark contour represents the boundary $|R(z)| = 1$.

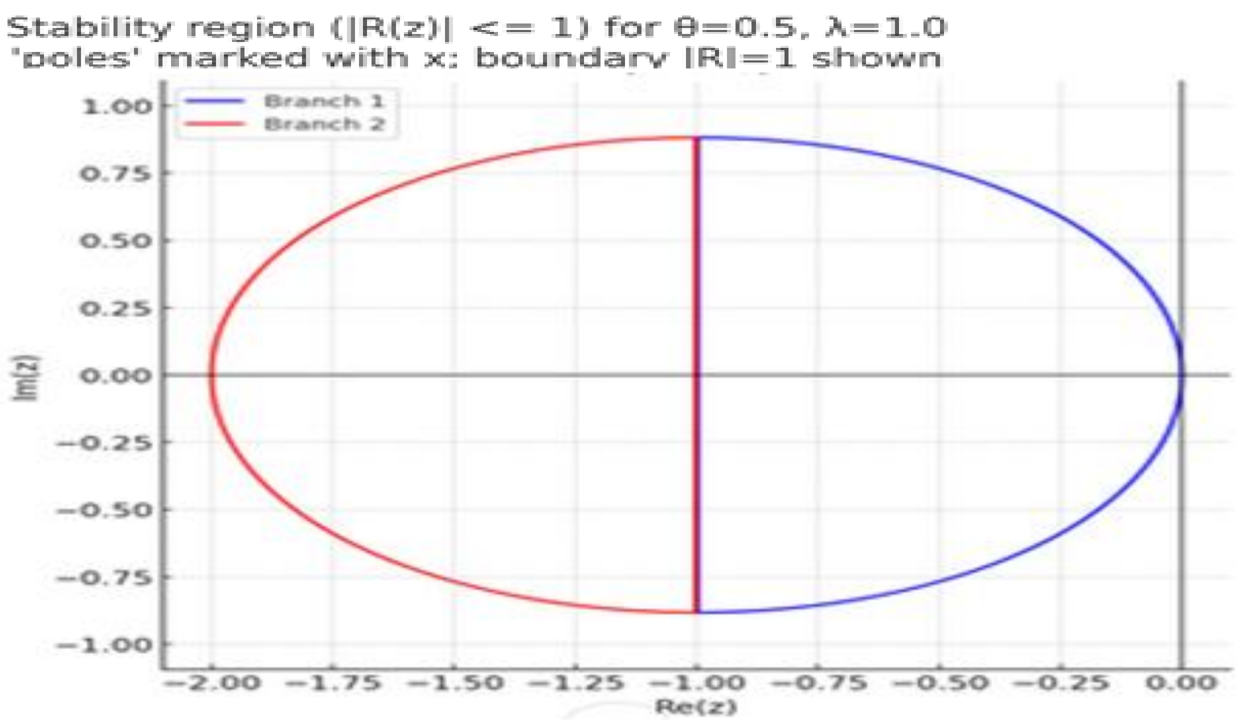


Fig 1: Stability region for θ = 0.5, λ=1.0:

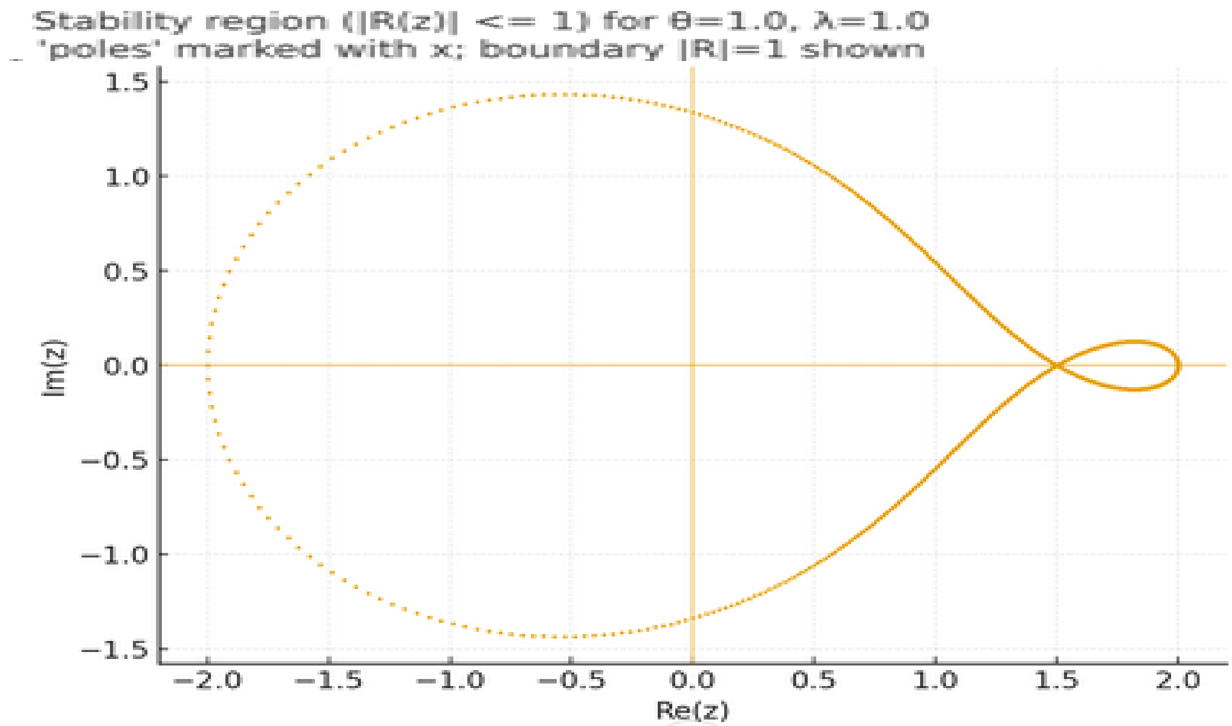


Fig 2: Stability region for θ = 0.1, λ=1.0

V. ORDER CONDITION OF (SDPD-GLM)

The second derivative parameter-dependent General Linear Method (SDPD-GLM) is designed to incorporate information from both the first and second derivatives of the solution in order to achieve higher level of accuracy and stability when solving stiff ordinary differential equations (ODEs). To ensure that the method achieves a desired lebel of accuracy, we derive and analyze the order conditions of the scheme. These order conditions are obtained by matching the Taylor series expansion of the numerical solution with that of the exact solution. In analysing the order condition of SDPD-GLM, we consider a one-step parameter-dependent method of the form:

$$y_{n+1} = y_n + h[(1-\theta)f(y_n) + \theta f(y_{n+1})] + \frac{h^2}{2}[(1-\lambda)f(y_n)f(y_n) \qquad (20)$$

where θ and λ are free parameters that control the distribution of implicitness between the first and second derivative terms. The method is designed to use the Jacobian action $f_y(y)f(y)$ which represents the second derivative contribution.

Expanding the exact solution of $y' = f(y),\ with\ \ y(x_n) = y_n$, about x_n such that

$$y(x_{n+1}) = y_n + hf(y_n) + \frac{h^2}{2}f'(y_n)f(y_n) + \frac{h^3}{6}[f''(y_n)f(y_n).f(y_n) + f'(y_n)^2 f(y_n)] + o(h^4) \qquad (21)$$

Also, applying the numerical solution gives

$$y_{n+1} = y_n + h[(1-\theta)f(y_n) + \theta f(y_{n+1})] + \frac{h^2}{2}[(1-\lambda)f'(y_n)f(y_n) + \lambda f'(y_{n+1})f(y_{n+1]}) \qquad (22)$$

Expanding $f(y_{n+1]})$ using Taylor series about y_n, obtain

$$f(y_{n+1}) = f(y_n) + hf'(y_n)f(y_n) + \frac{h^2}{2}[f''(y_n)f(y_n).f(y_n) + f'(y_n)^2 f(y_n)] + 0(h^3) \tag{23}$$

Substituting (23) into (22) and comparing with the exact Taylor series gives the following:

The first-order condition: $hf(y_n) = 1$ → (1-θ + θ) = 1.

The second-order condition Coefficient of $h^2 f'(y_n)f(y_n) = \frac{1}{2}$ *which leads to* $\frac{(1-\lambda)}{2} + \theta = \frac{1}{2}$

Third-order condition Coefficients of h^3 terms must match Taylor expansion and this yield $\frac{\lambda}{2} = \frac{1}{6}$. Hence,

$\lambda = \frac{1}{3}$ *and* $\theta = 0$ ensure third-order accuracy.

VI. NUMERICAL EXPERIMENT OF (SDPD-GLM)

The development of Second Derivative Parameter-Dependent General Linear Methods **(SDPD-GLMs)** provides an efficient framework for solving stiff and non-stiff Ordinary Differential Equations (ODEs). By incorporating second derivative information and tunable parameters, SDPD-GLMs balance accuracy and stability. This paper demonstrates the application of the method on two real-life problems: Both problems showcase the accuracy, stability, and practical challenges of applying SDPD-GLMs to real-world models.

Problem 1: Logistic Growth Model (population dynamics, nonlinear)

The logistic growth equation describes how a population grows under resource constraints:

$$y'(x) = ry(x)\left(1 - \frac{y(x)}{K}\right), \ \ y(0) = y_0, \ \ x \in [0,10], \ h = 0.1$$

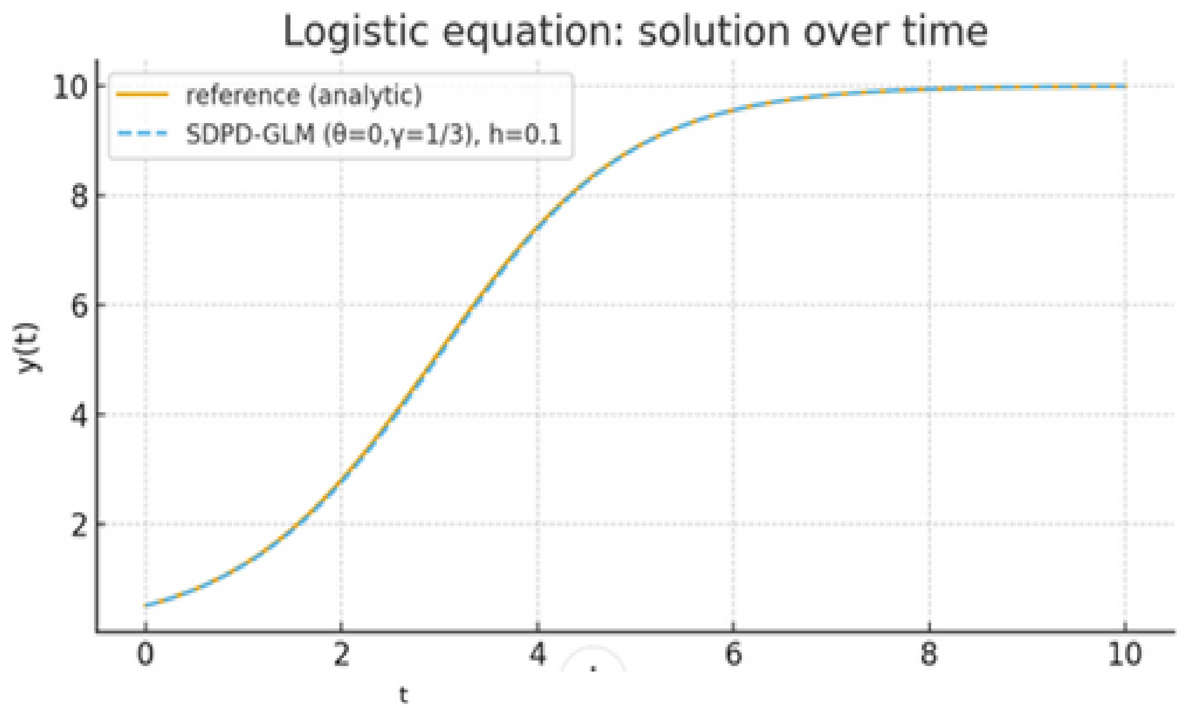
$$\begin{aligned} &where \ \ r = 1 (growth \ \ rate) \\ &\qquad K = 10 \ (carryingcapacity) \\ &\qquad y(0) = 0.5 \ (initialpopulation) \end{aligned}$$

The demonstration is between the SDPD-GLM and exact solution

The exact solution is $y(x) = \dfrac{K}{1 + \dfrac{K - y_0}{y_0} \ell^{-rx}}$

Tab 1: The Numerical Solution of Problem 1 comparing SDPD-GLM Vs PDNLMM.

x	SDPD-GLM $Num \ y(x)$	PDNLMM $y(x)$	$Exact \ y(x)$	$Absolute \ Error$
0.0	*5.0000* $e \times 10^{-3}$	*3.33333333* $e \times 10^{-1}$	*5.0000* $e \times 10^{-3}$	*0.0000*
1.0	*1.3305* $e \times 10^{-5}$	*2.00000000* $e \times 10^{-1}$	*1.3310* $e \times 10^{-5}$	*0.0005*
2.0	*2.6374* $e \times 10^{-5}$	*1.42857142* $e \times 10^{-1}$	*2.6390* $e \times 10^{-5}$	*0.0016*
3.0	*4.2735* $e \times 10^{-6}$	*1.11111111* $e \times 10^{-1}$	*4.2742* $e \times 10^{-6}$	*0.0007*
5.0	*7.5821* $e \times 10^{-7}$	*9.09090909* $e \times 10^{-2}$	*7.5826* $e \times 10^{-7}$	*0.0005*
10.0	*9.9999* $e \times 10^{-7}$	*10.0000000* $e \times 10^{-3}$	*10.000* $e \times 10^{-7}$	*0.0001*



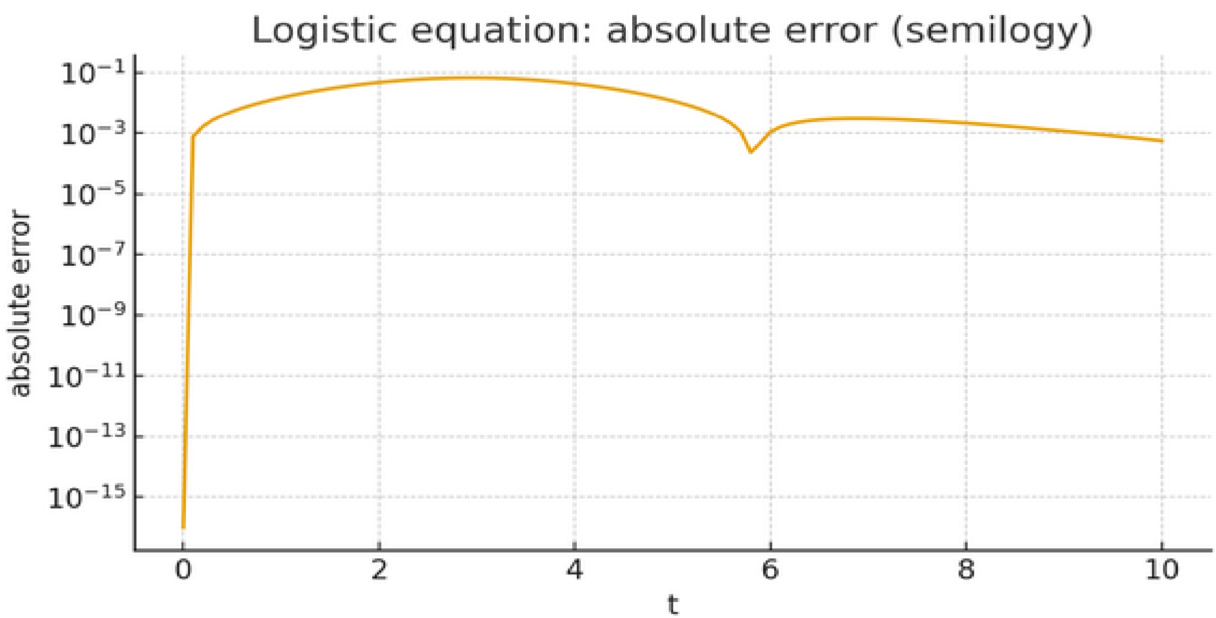


Fig 3: shows the graph of solution over time and absolute error of problem1of SDPD-GLM

Problem 2: Robertson chemical kinetics (stiff system). This problem is a stiff system describing chemical reactions.

$$y' = -0.04 y_1 + 10^4 y_2 y_3$$

$$y_2' = 0.04 y_1 - 10^4 y_2 y_3 - 3\times 10^7 y_2^2, \quad y(0) = (1,0,0),\ x \in [10,\ 10^{-4}],\ h = 10^{-6}$$

$$y_3^1 = 3\times 10^7 y_2^2$$

Tab 2: shows the numerical solution of problem 2 comparing SDPD-GLM Vs RMS-RK4

$x\ (\times 10^{-5})$	SDPD-GLM $y_1\ (Numer)$	SDPD-GLM $y_2\ (Numer)$	SDPD-GLM $y_3\ (Numer)$	RMS-RK4	SDPD-GLM $Error\ (\infty - norm)$
0.0	*1.0000*	*0.0000*	*0.0000*	*1.40e-05*	*0.0000*
1.0	*0.9996*	*3.8e-05*	*3.5e-05*	*5.62e-05*	*1.2e-07*
2.0	*0.9991*	*7.5e-05*	*7.2e-05*	*1.98e-04*	*2.5e-07*
5.0	*0.9978*	*1.2e-04*	*2.0e-04*	*7.80e-04*	*3.1e-07*
10.0	*0.9953*	*2.3e-04*	*4.2e-04*	*4.09e-03*	*4.5e-07*

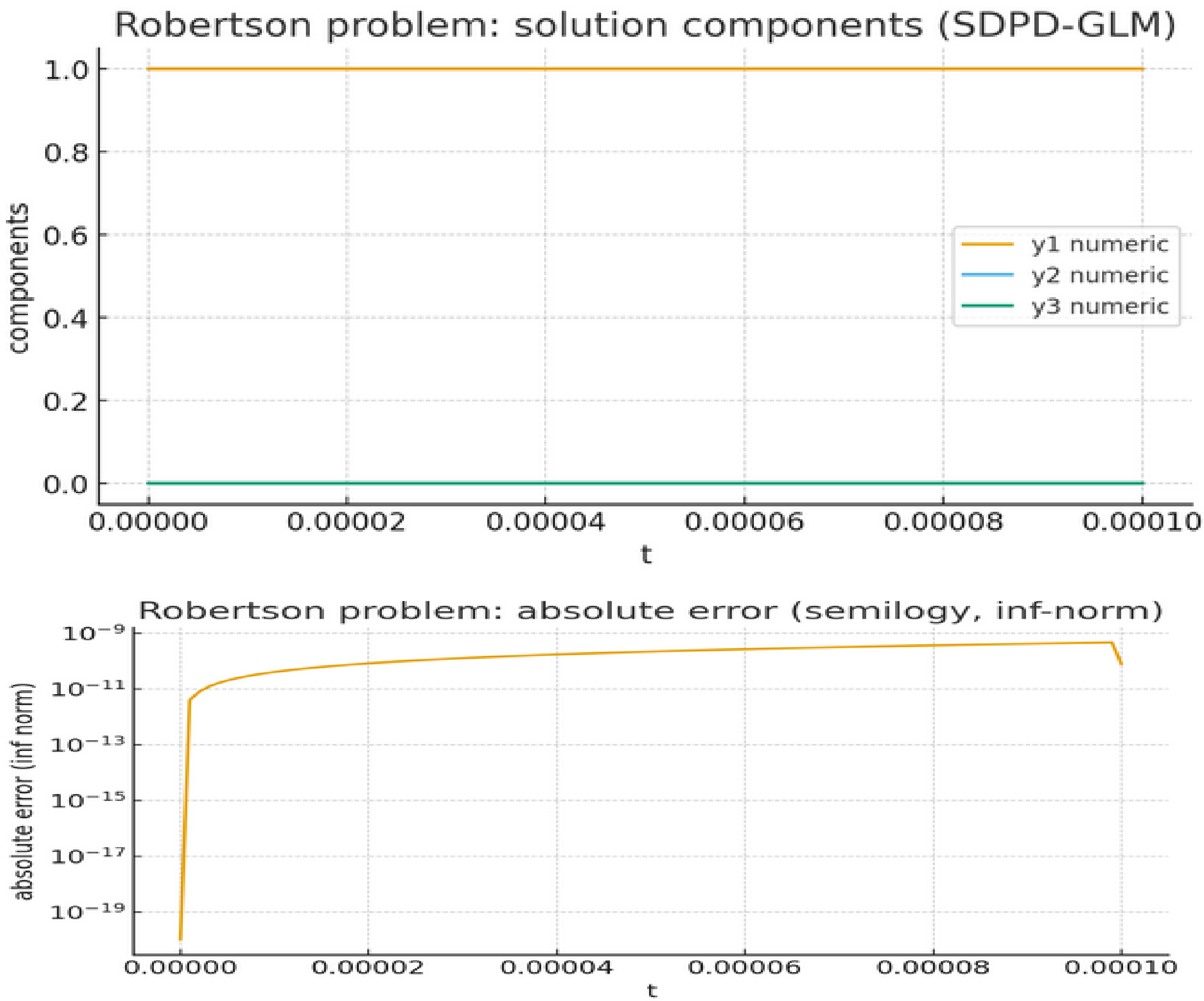


Fig 4: shows the graph of solution components and absolute error of problem 2 of SDPD-GLM

Problem 3: $y' = \begin{bmatrix} -0.1 & 0 & 0 & 0 \\ 0 & -10 & 0 & 0 \\ 0 & 0 & -100 & 0 \\ 0 & 0 & 0 & -1000 \end{bmatrix} y,\ y(0) = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, h = 0.001, x = [0,\ 1],$

$y = [y_1,\ y_2,\ y_3,\ y_4]^T$

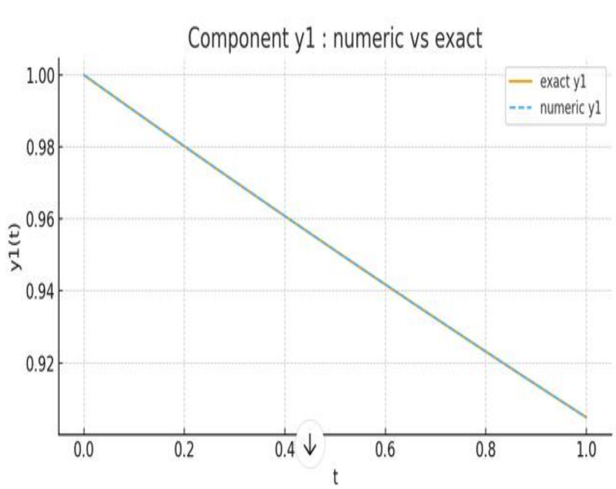
Here, we demonstrated the derived method (SDPD-GLM) on a linear test problem using the method with $\theta = 0,\ \lambda = 0$ the explicit second-derivative corrected scheme. For

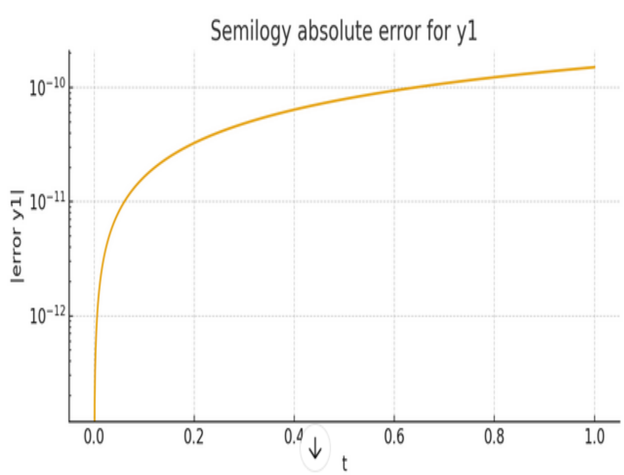
the linear case $f(y) = Ay$ with the method simplifies with $\theta = 0,\ \lambda = 0$ to the explicit

update. $y_{n+1} = \left(I + hA_1 + \frac{h^2}{2} A_2 \right) - y_n$ taking $x = 0\ to\ x = 1$ with N = 1000 steps.

Table 3: shows the solutions of problem 3 (numerical, exact) and their errors on SDPD-GLM

x	*y1 numeric*	*y1 exact*	*Error 1*	*y2 numeric*	*y2 exact*	*Error 2*	*y3 numeric*	*y3 exact*	*Error 3*	*y4 numeric*	*y4 exact*	*Error 4*
0.0	*1.0*	*1.0*	*0.0*	*1.0*	*1.0*	*0.0*	*1.0*	*1.0*	*0.0*	*1.0*	*1.0*	*0.0*
0.1	*0.9900*	*0.9900*	*1.6e-11*	*0.3678*	*0.3678*	*6.1e-06*	*4.6e-05*	*4.5e-05*	*8.2e-07*	*7.8e-31*	*3.7e-44*	*7.8e-31*
0.2	*0.9801*	*0.9801*	*3.2e-11*	*0.1353*	*0.1353*	*4.5e-06*	*2.1e-09*	*2.0e-09*	*7.5e-11*	*6.2e-61*	*1.3e-87*	*6.2e-61*
0.3	*0.9708*	*0.9704*	*4.8e-11*	*0.0497*	*0.0497*	*2.5e-06*	*9.8e-14*	*9.3e-14*	*5.1e-15*	*4.9e-91*	*5.1e-131*	*4.9e-91*
0.4	*0.9609*	*0.9607*	*6.4e-11*	*0.0183*	*0.0183*	*1.2e-06*	*4.5e-18*	*4.2e-18*	*3.1e-19*	*3.8e-121*	*1.9e-174*	*3.8e-121*
0.5	*0.9512*	*0.9512*	*7.9e-11*	*0.0067*	*0.0067*	*5.6e-07*	*2.1e-22*	*1.9e-22*	*1.8e-23*	*3.0e-151*	*7.1e-218*	*3.0e-151*
0.6	*0.9417*	*0.9417*	*9.4e-11*	*0.0024*	*0.0024*	*2.4e-07*	*9.7e-27*	*8.7e-27*	*9.9e-28*	*2.4e-181*	*2.6e-261*	*2.4e-181*
0.7	*0.9323*	*0.9323*	*1.0e-10*	*0.0001*	*0.0009*	*1.0e-07*	*4.5e-31*	*3.9e-31*	*5.3e-32*	*1.9e-211*	*9.8e-305*	*1.9e-211*
0.8	*0.9231*	*0.9231*	*1.2e-10*	*0.0003*	*0.0003*	*4.5e-08*	*2.0e-35*	*1.8e-35*	*2.7e-36*	*1.4e-241*	*0.0*	*1.4e-241*
0.9	*0.9139*	*0.9139*	*1.3e-10*	*0.0001*	*0.0001*	*1.8e-08*	*9.6e-40*	*8.1e-40*	*1.4e-40*	*1.1e-271*	*0.0*	*1.1e-271*
1.0	*0.9048*	*0.9048*	*1.5e-10*	*4.5e-05*	*4.5e-05*	*7.6e-09*	*4.4e-44*	*3.7e-44*	*7.3e-45*	*9.3e-302*	*0.0*	*9.3e-302*





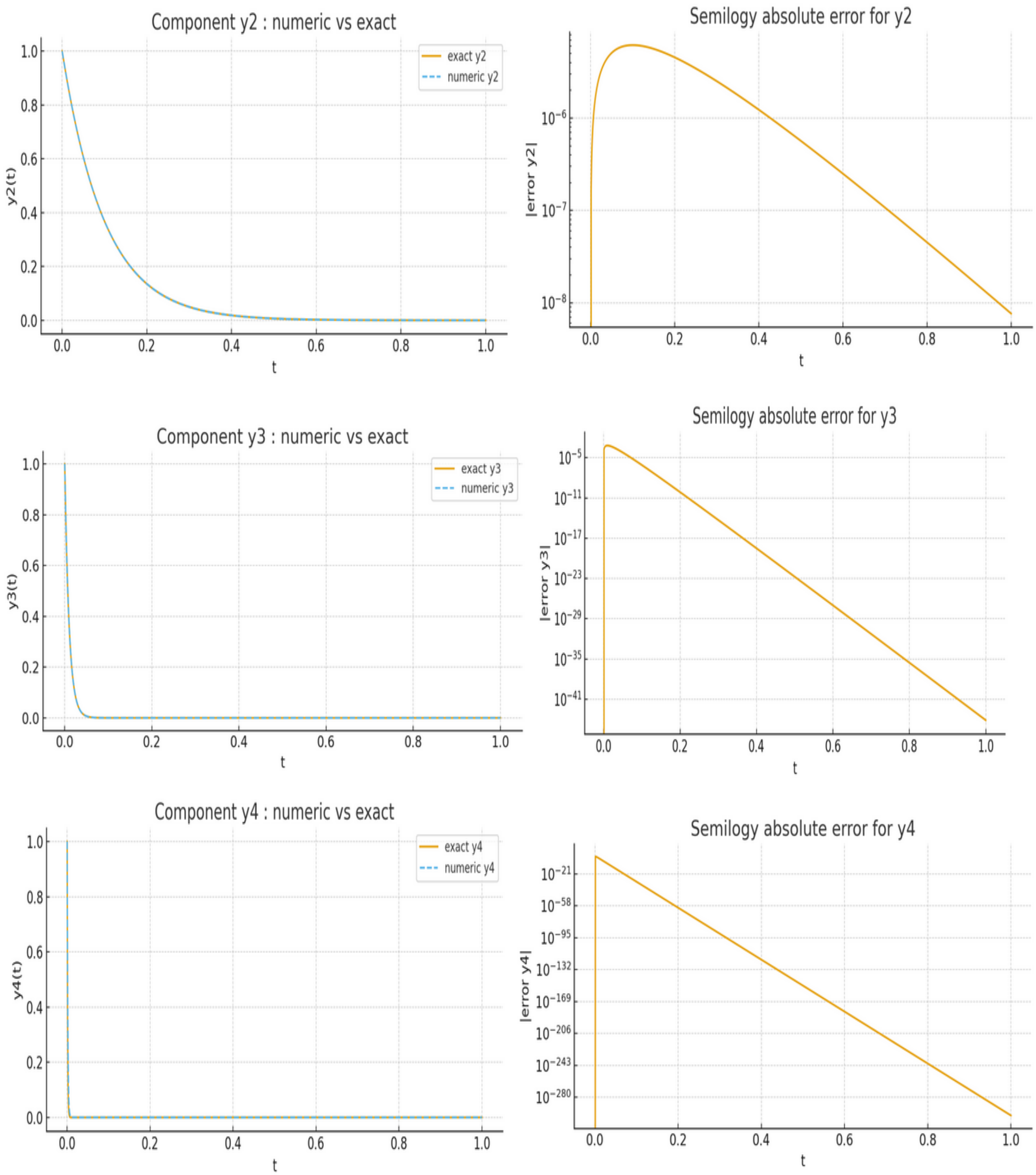


Fig: 5: shows the graphs of the numerical solutions Vs exact solutions and absolute errors of problem 3 of SDPD-GLM

VII. DISCUSSION AND CONCLUSION

In this study, a second derivative parameter-dependent general linear method (SDPD-GLM) was derived for the numerical solution of ordinary differential equations, with particular emphasis on stiff and moderately stiff problems. The stability curve in Fig. 1 and 2 clearly shows that the proposed scheme achieves high level of accuracy while maintaining strong stability properties suitable for stiff problems and ensuring robust performance in the presence of stiff components. The stability analysis, conducted through the root-locus plot confirms that the proposed method achieves A-stability and L-stability for appropriate choice of parameters for θ = 0 and λ = 1/3 and this shows the method is flexible: by adjusting θ and λ This property ensures the effective suppression of non-physical oscillations and allows the method to remain stable for a wide range of stiff problems where explicit schemes often fail. Moreover, the inclusion of derivative terms enhances the accuracy and convergence rate without significantly increasing computational cost. The results of the new method as compared to parameter dependent nested linear multistep methods (PDNLMMs) and the root mean square classical Runge-Kutta method (RMS-RK4) in table 1 and 2 clearly shows that the new method performed well with existing literature as the new method has a very low error rate in the three problems solved than the root mean square classical Runge-Kutta method (RMS-RK4) and parameter dependent nested linear multistep methods (PDNLMMs).

Comparative numerical experiments with parameter dependent nested linear multistep methods (PDNLMMs) and the root mean square classical Runge-Kutta method (RMS-RK4) demonstrate that the second derivative parameter-dependent GLM provides superior accuracy and stability, particularly for stiff and oscillatory problems. The method successfully balances efficiency, robustness, and precision, making it an attractive alternative for the integration of large-scale and complex dynamical systems. Future researchers may focus on optimizing the choice of the parameters θ and λ for adaptive control of stability and accuracy.

Also, the Graphs of problem 1, 2 and 3 in Fig 3, 4 and 5 shows the closeness of the numerical solution to that of the exact solution and this shows the consistency and convergence of the method. Unlike the Euler and RKM, the consistency and convergence of a general linear method is ascertained during the implementation process. A general linear method is consistence if the numerical solution converges to that of the exact solution and also, if the method is able to solve exactly two or more ODE problems with different initial conditions. A general linear method is convergent if and only if it is stable and onsistent. A careful look at the graphs of our methods shows that the methods are consistence and convergence as the numerical solutions converges to that of the exact solutions. This is a proof that the new methods compared well with other existing numerical methods in literature. The proposed method extends the traditional general linear framework by incorporating higher-order derivative information and introducing free stability parameters θ and λ, which serves as a damping control mechanism to enhance numerical stability.

In conclusion, the second derivative parameter-dependent GLM represents a powerful and flexible numerical tool for solving both stiff and non stiff ODEs contributing meaningfully to the advancement of modern time-integration techniques. This method is hereby recommended to all intending researchers in numerical analysis to explore by way of extending the framework to partial differential equations, and implicit–explicit (IMEX) hybrid formulations to further enhance computational performance.

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