A Parameter-Dependent Second-Derivative General Linear Method with Nearly Almost Runge–Kutta Stability

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***Abstract*—This paper presents already derived parameter-dependent second-derivative general linear method (SD-GLM) for the numerical solution of ordinary differential equations (ODEs) and establishing the method with nearly almost Runge-Kutta stability properties. The proposed method will be known as the parameter dependent second derivative general linear method (PD-SDGLM) with nearly almost Runge-Kutta (NAR-K) stability and it incorporates both the first and second derivatives of the differential function and depends on adjustable parameters that enhance accuracy and stability. This is one of the possible generalizations of the classical second derivative Runge-Kutta method (SD-RKM). Order conditions are derived up to the fourth order through Taylor series expansion, and a general form of the amplification matrix is established for stability analysis using the Dahlquist test equation. By appropriate parameter selection, the method achieves "nearly almost Runge–Kutta stability," characterized by large regions of absolute stability. The resulting framework generalizes multi-derivative general linear methods and provides a flexible foundation for constructing high-order, stiffly-stable time integrators.**

***Index Terms*—Parameter dependent, Second derivative, general linear method, nearly almost Runge-Kutta stability.**

I. INTRODUCTION

According to Hairer and Wanner (1996). The numerical integration of ordinary differential equations (ODEs) remains a central concern in scientific computing among the various classes of numerical integrators. Also, Butcher and Jackiewicz (1998) opined that General Linear Methods (GLMs) form a broad framework that unifies multistep and Runge–Kutta (RK) schemes. When higher-order derivatives of the right-hand side function are incorporated, multi-derivative GLMs arise, offering improved accuracy per computational stage. In the work of Butcher (2016) the special class of general linear methods known as Almost Runge-Kutta methods (ARK) was introduced. The works of Chan and Tsai (2019), Okuonghae, Ikhile and Ogunleye (2013) shows that the ARK method has properties that are very much close to the classical explicit RKM. This paper presents a parameter-dependent second-derivative General Linear Method (SD-GLM), derives its order conditions up to the fourth order through Taylor series expansion and eventually established that the method has the properties of nearly almost Runge-Kutta stability. According to Jackiewicz (2009). The proposed formulation allows tuning of parameters to mimic the stability characteristics of implicit Runge–Kutta schemes, hence achieving nearly almost Runge–Kutta stability.

Consider the initial value problem (IVP)

$$y'(x) = f(x, y(x)), \quad y(x_0) = y_0 \quad \backslash \qquad (1)$$
$$with \;\; f; \Re \times \Re^m \to \Re^m$$

Define the total derivative of f along thje exact solution as

$$f' = \frac{d}{dx} f(x, y(x)) = f_x + f_y f \qquad (2)$$

Incorporating both f and f' allows the construction of higher-order schemes while maintaining compact step structures.

According to Okuonghae and Ikhile (2014) an example of the ARK method which can be explicit in nature is

$$\left(\begin{array}{c|c} A & U \\ \hline B & V \end{array}\right) = \left(\begin{array}{cccc|ccc} 0 & 0 & 0 & 0 & 1 & 1 & \frac{1}{2} \\ \frac{1}{16} & 0 & 0 & 0 & 1 & \frac{7}{16} & \frac{1}{16} \\ -\frac{1}{4} & 2 & 0 & 0 & 1 & -\frac{3}{4} & -\frac{1}{4} \\ 0 & \frac{2}{3} & \frac{1}{6} & 0 & 1 & \frac{1}{6} & 0 \\ \hline 0 & \frac{2}{3} & \frac{1}{6} & 0 & 1 & \frac{1}{6} & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ -\frac{1}{3} & 0 & -\frac{2}{3} & 2 & 0 & -1 & 0 \end{array}\right) \tag{3}$$

The abscissa vector of this method in (3) is $c = [1, \frac{1}{2}, 1, 1]^T$. The method in (3) behaves like the well-known fourth order explicit RKM because the stages are computed in sequence using derivative values computed from the previous stages. Despite the similarity of ARK and ERKM, there is a dissimilarity, the ARK method in (3) requires an input value such as y_{n-1} to compute an approximation to $y(x_{n-1})$, $hy'(x_{n-1})$ and $h^2 y''(x_{n-1})$ respectively, where $y_1^{[n-1]} = y(x_{n-1})$, $y_2^{[n-1]} = hy'(x_{n-1})$, and $y_3^{[n-1]} = h^2 y''(x_{n-1})$ are the input approximations. The differential equation which the numerical solution is to be sought for in this paper is (1). The application of the GLM in (3) to the test scalar problem $y' = \lambda y,\ where\ z = \lambda h$ yields the stability matrix

$$M(z) = V + zB(I - zA)^{-1}U \tag{4}$$

With the scheme in (3)

$$M(z) = \begin{pmatrix} 1 + \frac{5z}{6} + \frac{z^2}{3} + \frac{z^3}{48} & \frac{1}{6} + \frac{z}{6} + \frac{7z^2}{48} + \frac{z^3}{48} & \frac{z^2}{48} + \frac{z^3}{96} \\ z + \frac{5z^2}{6} + \frac{z^3}{3} + \frac{z^4}{48} & \frac{z}{6} + \frac{z^2}{6} + \frac{7z^3}{48} + \frac{z^4}{48} & \frac{z^3}{48} + \frac{z^4}{96} \\ z + \frac{z^2}{2} + \frac{7z^3}{12} + \frac{z^4}{24} & -1 + \frac{z}{2} - \frac{z^2}{12} + \frac{3z^3}{24} + \frac{z^4}{24} & \frac{z^4}{48} \end{pmatrix} \tag{5}$$

Definition 1.1. According to Okuonghae and Ikhile (2014). A GLM has RK-stability if the stability matrix $M(z)$ given by (4) has only a single nonzero eigenvalue.

Definition 1.2; Also, **a**ccording to According to Okuonghae and Ikhile (2014). A GLM is said to be zero-stable if the non-zero eigen-value of the matrix V is one.

These are conditions that must be satisfied before a GLM can be regarded as ARK method.

Let $R(z)$ be the non-zero eigenvalues of $M(z)$ so that

$Det(Iw - M(z)) = w^{s-1}(w - R(z))$

For an explicit ARK-stability GLM then

$$R(z) = 1 + z + \frac{z^2}{2!} + \ldots + \frac{z^\ell}{\ell!}, \quad \ell \geq 1$$

II. GENERAL FORM OF THE PARAMETER-DEPENDENT SD-GLM

Let the step size be h > 0, and denote by $y^{(n)}$ the external stage vector that stores combinations of previously computed solution values. Let Y denote the vector of internal stage approximations. The general parameter-dependent second-derivative GLM is written in matrix form as:

Stage equations: $Y = hA\, f(Y) + h^2 \hat{A}\, \dot{x}f(Y) + U\, y^{(n)}$

Update equations: $y^{(n+1)} = hB\, f(Y) + h^2 \hat{B}\, \dot{x}f(Y) + V\, y^{(n)}$

The entries of these matrices may include free parameters which are adjusted to satisfy order and stability requirements. When $r = 1$ and $V = 1$, the method reduces to a one-step, one-output form.

III. ORDER CONDITIONS

Expanding the exact and numerical solutions in Taylor series and matching coefficients yields the order conditions. For a one-output scheme, the first four order conditions are summarized as:

Order	Condition
1	$\Sigma b_i = 1$
2	$\Sigma b_i c_i + \Sigma \hat{b}_i = ½$
3	$\Sigma b_i c_i^2/2 + \Sigma \hat{b}_i c_i + \Sigma b_i a_{ij} = 1/6$
4	$\Sigma b_i c_i^3/6 + \Sigma \hat{b}_i c_i^2/2 + \Sigma \hat{b}_i a_{ij} + \Sigma b_i a_{ij} a_{jk} = 1/24$

IV. ANALYSIS OF PDSD-GLM WITH NEARLY ALMOST R-K STABILITY

Considering the Stage equations and update equations of the general form parameter dependent second derivative general linear method written in matrix form below

Stage equations: $Y = hA\, f(Y) + h^2 \hat{A}\, \dot{x}f(Y) + U\, y^{(n)}$

Update equations: $y^{(n+1)} = hB\, f(Y) + h^2 \hat{B}\, \dot{x}f(Y) + V\, y^{(n)}$

The stability is analyzed using the Dahlquist test equation

$$y' = \lambda y\ , f(Y) = \lambda Y \text{ and } \dot{x}f(Y) = \lambda^2 Y \tag{8}$$

Substituting (8) into the matrix above gives the amplification matrix

$$M(z) = (zB + z^2 \hat{B})(I - zA - z^2 \hat{A})^{-1}\, U + V \tag{9}$$

where $z = h\lambda$. For scalar methods, this reduces to $y_{n+1} = R(z)y_n$. The method is A-stable if $|R(z)| \leq 1$ *for* $Re(z) \leq 0$, and L-stable if $\lim\limits_{z\to\infty} R(z) = 0$.

V. ACHIEVING THE PD-SDGLM WITH NEARLY ALMOST RUNGE–KUTTA STABILITY

Almost Runge–Kutta stability refers to selecting parameters so that *R(z)* resembles the stability function of an implicit Runge–Kutta method. This ensures wide stability regions and absence of spurious parasitic roots. Key constraints include one-step consistency, stage initialization, and implicitness for stiffness. An example parameterized family is to Consider a two-stage parameter-dependent SD-GLM $(s = 2,\ r = 1)$ with parameters a_{21}, $\hat{a}_{21}$, b_i, and $\hat{b}_i$. Applying the order conditions up to order 3 yields a linear system that can be solved for these coefficients. Parameter tuning was performed numerically to optimize the stability function *R(z)* and almost Runge–Kutta stability which combine the high-order accuracy and robust stability of implicit RK schemes with the flexibility of the GLM framework. The parameter dependent second derivative general linear method with nearly almost Runge-Kutta stability polynomial was obtained as compared with the RK3 stability polynomial below

$$R_{alm}(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{7} + \ldots \tag{10}$$

which is slightly different from RK3 polynomial as the coefficient of z^3 of $R_{alm}(z)$ is slightly different from that of $RK3$ polynomial

$$RK3(z) = 1 + z + \frac{z^2}{2} + \frac{z^3}{6} + \ldots \tag{11}$$

A plot of the Jordan curve of the parameter dependent second derivative general linear method with nearly almost Runge-Kutta stability curve as compared with that of RK3 is as shown below

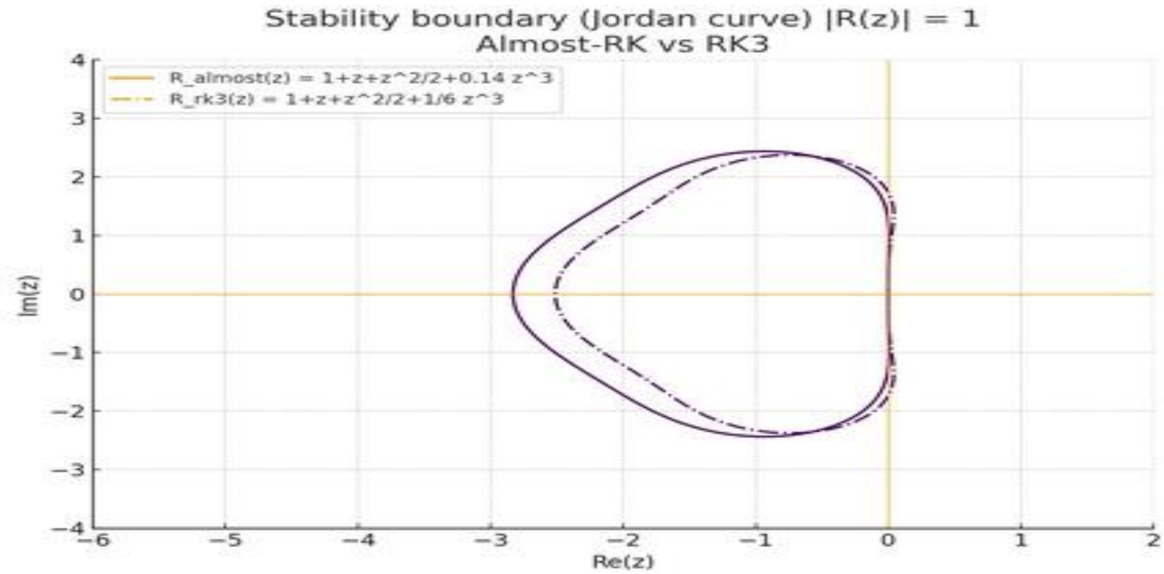

Fig 1: the Jordan curve of the parameter dependent second derivative general linear method with nearly almost Runge-Kutta stability curve as compared with that of RK3.

The plotted curve represent the contour *|R(z)| =1* in the complex plane. The boundary of the method's stability region forms the closed Jordan curve

VII. DISCUSSION AND CONCLUSION

Figure 1 shows the Jordan curve of the parameter dependent second derivative general linear method with nearly almost Runge-Kutta stability curve as compared with that of RK3. The plotted curve represent the contour *|R(z)| =1* in the complex plane. The boundary of the method's stability region forms the closed Jordan curve. It can be ascertained from the Jordan curve of the parameter dependent second derivative general linear method with nearly almost Runge-Kutta stability curve as compared with that of RK3 the little margin between the two and that the new method was almost close to the Runge-Kutta method.

In conclusion, a parameter-dependent second-derivative general linear method (SD-GLM) was developed and analyzed. The method incorporates both first and second derivatives and introduces free parameters that can be optimized for order and stability. By appropriate parameter tuning, the method attains nearly almost Runge–Kutta stability, combining the high-order accuracy and robust stability of implicit RK schemes with the flexibility of the GLM framework.

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