© January 2026 | IJIRT | Volume 12 Issue 8 | ISSN: 2349-6002

A Comprehensive Review of Global Regularity Estimates for p(x)-Laplacian Variational Inequalities with Degenerate Matrix Weights

Megha Chandraprakash Gaikwad
Assistant Professor, Shri Sheth Muralidharji Mansingaka Arts, science and commerce College, Pachora, dist. Jalgaon

Abstract—**This review examines the recent work by Tran, Bui, and Nguyen (2026) on global gradient regularity estimates for variational inequalities driven by p(x)-Laplacian operators with singular or degenerate matrix-valued weights. The paper establishes optimal Calderón–Zygmund type estimates and weighted Orlicz-space regularity under minimal structural assumptions, including Reifenberg-flat domains and small log-BMO matrix weights. We analyze the mathematical framework, highlight the novelty of the techniques, compare the results with existing literature, and discuss their significance and possible extensions.**

I. INTRODUCTION

Regularity theory for nonlinear elliptic equations has been a central topic in modern analysis, particularly due to its deep connections with physics, geometry, and applied sciences. The incorporation of variable exponent growth and degenerate matrix-valued weights significantly extends classical elliptic theory. The reviewed paper contributes to this evolving landscape by addressing two-obstacle variational inequalities involving p(x)-Laplacian operators under highly nonstandard conditions.

II. MATHEMATICAL FRAMEWORK AND PROBLEM SETTING

We consider variation inequalities of the form:

$$\int_\Omega A\,|W(x)\nabla u|^{\{p(x)-2\}W^{2(x)\nabla u}} \cdot \nabla(u-\varphi)dx \le \int_\Omega |W(x)F|^{\{p(x)-2\}W^{2(x)F}} \cdot \nabla(u-\varphi)dx,$$

Where p(x) ∈ (1, ∞) satisfies log-Hölder continuity and W(x) is a symmetric positive definite matrix with small log-BMO seminorm. The admissible set is constrained by two obstacles $\varphi_1 \le u \le \varphi_2$. This formulation generalizes classical p-Laplacian obstacle problems and incorporates degeneracy through matrix weights.

III. MAIN RESULTS

The primary result establishes global Calderón–Zygmund-type estimates:

$$\int_\Omega \varphi\,\omega^{\{\gamma p(x)\}|\nabla u|^{\{\gamma p(x)\}dx}} \le C\left(1+\int_\Omega \left[F_{\omega(x)}\right]^\gamma dx\right), \qquad \gamma > 1.$$

Additionally, the authors derive weighted Orlicz-space estimates involving fractional maximal operators M_β, yielding:

$$\int_\Omega \Xi\,\Phi\left(M_{\beta\left(\omega^{\{p(x)\}|\nabla u|^{\{p(x)\}}}\right)}\right)dx \le C\,(1 + \int\_\Omega\ \Phi(M\_\beta(F\_\omega))\ dx).$$

These estimates are shown to be optimal with respect to scaling parameters.

IV. METHODOLOGICAL INNOVATIONS

A key methodological contribution is the multi-level comparison technique, involving successive approximations of the solution by auxiliary problems with frozen exponents and averaged weights.

$-\mathrm{div}(\mathrm{W(x)} \mid \nabla\mathrm{u} \mid \mathrm{p(x)} - 2\nabla\mathrm{u}) = \mathrm{divFin}\,\Omega,$

Multi-level comparison via frozen exponents:

The auxiliary problem:

$$\{-\mathrm{div}(\mathrm{W(x)} \mid \nabla\mathrm{v} \mid \mathrm{p0} - 2\nabla\mathrm{v}) = 0\mathrm{v} = \mathrm{uin\,Br(x0)}, \mathrm{on}\ \partial\mathrm{Br(x0)}.$$

Averaged weights and second-level approximation:

$$\{-\mathrm{div}(\bar{W}_{Br} \mid \nabla w \mid p0 - 2\nabla w) = 0 w = v \text{in } Br(x0), \text{on } \partial Br(x0).$$

The constructive level-set approach allows sharp control of constants, minimizing dependence on structural parameters.

$$\int Br(x0) \mid \nabla v - \nabla w \mid p0dx \leq C \parallel W \parallel \log BMO(Br) \int Br(x0) \mid \nabla v \mid p0dx.$$

The use of log-BMO conditions on matrix weights represents a minimal and highly effective regularity assumption.

Minimal regularity yields:

$$\parallel W - \bar{W}B \parallel Lq(B) \leq C(q) \parallel W \parallel \log BMO \mid B \mid 1/q, \forall q < \infty,$$

Sharp constant dependence:

$$\parallel \nabla u \parallel LWp(\cdot)(\Omega) \leq C(n, p \pm, \parallel W \parallel \log BMO) \parallel F \parallel LWp'(\cdot)(\Omega)$$

V. COMPARISON WITH RELATED WORKS

The results extend classical Calderón–Zygmund theory (Calderón & Zygmund, 1952) to highly nonlinear and degenerate settings.

Calderón–Zygmund theory concerns linear elliptic equations:

$$-\mathrm{div}(A(x)\nabla u) = \mathrm{div} F \text{in } \Omega,$$

$$\lambda \mid \xi \mid 2 \leq A(x)\xi \cdot \xi \leq \Lambda \mid \xi \mid 2, \forall \xi \in Rn,$$

Compared to works by Fabes, Kenig, and Serapioni (1982) and Cao et al. (2019), the reviewed paper weakens ellipticity assumptions and incorporates variable exponents.

$$F \in Lq(\Omega; Rn) \Rightarrow \nabla u \in Lq(\Omega; Rn), 1 < q < \infty.$$

Relative to Balci et al. (2018, 2020), the present work advances from equations to two-obstacle variational inequalities and from Lebesgue to Orlicz frameworks.

Fabes–Kenig–Serapioni:

$$F \in Lwq(\Omega) \Rightarrow \nabla u \in Lwq(\Omega), 1 < q < \infty.$$

nonlinear operators with fixed growth:

$$F \in Lq(\Omega) \Rightarrow \mid \nabla u \mid p - 1 \in Lq(\Omega).$$

Balci et al. study equations:

$$\int \Omega W(x) \mid \nabla u \mid p(x) - 2\nabla u \cdot \nabla(v - u)dx \geq \int \Omega F \cdot \nabla(v - u)dx, \forall v \in K.$$

VI. APPLICATIONS AND IMPACT

The theoretical findings have implications for models of electrorheological fluids, anisotropic diffusion in porous media, and free boundary problems in finance and mechanics.

Governing equation

$$-\mathrm{div}(W(x) \mid \nabla u \mid p(x, E) - 2\nabla u) = f \text{in } \Omega,$$

allowing directional viscosity effects.

$$W(x)\xi \cdot \xi \geq \lambda(x) \mid \xi \mid 2, \lambda(x) \geq 0$$

The inclusion of degenerate matrix weights enables modeling of anisotropy and spatial heterogeneity with greater fidelity.

Diffusion equation

$$\partial tu - div(W(x) \mid \nabla u \mid p(x) - 2\nabla u) = g in \Omega T$$

Energy functional

$$E(u) = \int \Omega W(x) \mid \nabla u \mid p(x)dx$$

Two-obstacle problem

$$\int \Omega W(x) \mid \nabla u \mid p(x) - 2\nabla u \cdot \nabla(v - u)dx \geq 0, \forall v \in K$$

VII. LIMITATIONS AND FUTURE DIRECTIONS

While comprehensive, the theory relies on small log-BMO conditions and Reifenberg flatness, which may limit applicability in highly irregular domains. Future research may explore relaxation of these assumptions, extension to parabolic problems, or numerical approximations.

This is required to obtain perturbative estimates such as

$$\parallel W - \bar{W}B \parallel Lq(B) \leq C\varepsilon 0 \mid B \mid 1/q, \forall q < \infty$$

A future direction is to replace the smallness assumption by

$$\log W \in VMO(\Omega)$$

Expected estimate

$$F \in Lp'(x, t) \Rightarrow \nabla u \in Lp(x, t)$$

VIII. CONCLUSION

This work represents a substantial advance in nonlinear regularity theory, unifying variable exponent analysis, degenerate ellipticity, and obstacle problems.

The work centers on the operator

$Au := -div(W(x) \mid \nabla u \mid p(x) - 2\nabla u)$

The main nonlinear regularity estimate takes the form

$$F \in Lp'(x)(\Omega; Rn) \Longrightarrow \nabla u \in LWp(x)(\Omega)$$
$$\parallel \nabla u \parallel LWp(x)(\Omega) \leq C(n, p\pm, \parallel logW \parallel BMO, \Omega) \parallel F \parallel Lp'(x)(\Omega)$$
$$\int Br \mid \nabla u - \nabla w \mid p(x0)dx \leq \varepsilon(r) \int B2r \mid \nabla u \mid p(x)dx, \varepsilon(r) \to 0$$

The analysis is based on a multi-scale approximation.

$-div(W(x) \mid \nabla u \mid p(x) - 2\nabla u)$

Its methodological rigor and generality position it as a foundational reference for future investigations.

REFERENCES

[1] Tran, M.-P., Bui, D.-Q., & Nguyen, T.-N. (2026). Global regularity estimates for p(x)-Laplacian variational inequalities with singular or degenerate matrix-valued weights. arXiv preprint arXiv:2601.00652.

[2] Calderón, A. P., & Zygmund, A. (1952). On the existence of certain singular integrals. Acta Mathematica, 88, 85–139.

[3] Fabes, E., Kenig, C., & Serapioni, R. (1982). The local regularity of solutions of degenerate elliptic equations. Communications in Partial Differential Equations, 7(1), 77–116.

[4] Zhikov, V. V. (1986). Averaging of functionals of the calculus of variations and elasticity theory. Izvestiya Akademii Nauk SSSR. Seriya Matematicheskaya, 50(4), 675–710.

[5] Harjulehto, P., & Hästö, P. (2019). Orlicz spaces and generalized Orlicz spaces. Springer. https://doi.org/10.1007/978-3-030-32912-0

[6] Balci, E., Diening, L., & Stroffolini, B. (2020). Two obstacle problems with variable exponent growth. Nonlinear Analysis: Theory, Methods & Applications, 196, 111741. https://doi.org/10.1016/j.na.2020.111741