A study on the Local Fractional Laplace-Stieltjes Transform in fractal space

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Abstract- In this paper, we present a new method called the local fractional Laplace-Stieltjes transform that works in fractal space. We also compare this transform with the traditional Laplace and Stieltjes transforms. We explain the basic ideas, characteristics, and how this transform works, based on the theory of local fractional calculus. Additionally, we talk about some real-life uses of this transform in science and engineering.

Keywords - fractal space; local fractional Stieltjes transform; local fractional Laplace transform; local fractional Laplace-Stieltjes transform

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I.INTRODUCTION AND PRELIMINARIES

Local fractional calculus [9-18] have become very important in science and engineering. It's really useful for dealing with fractal functions that are not smooth and are found in the real world.

1.1 Local fractional continuity of functions:

A function f(x) is considered local fractional continuous [1]-[4] at $x = x_0$ if

$\lim_{x\to x_0} f(x) = f(x_0)$ if there exists

$|f(x) - f(x_0)| < \varepsilon^{\propto}$

with $|x - x_0| < \delta$ for $\varepsilon, \delta > 0$ and $\varepsilon, \delta \in R$, then the function f(x) is said to be fractional continuous on that interval (a , b) denoted as $f(x) \in C_{\propto}(a,b)$

1.2 Local fractional Derivative and Integral:

There are various ways to define the local fractional derivative and local fractional integrals within the field of fractional calculus.

Let $f(x) \in C_{\propto}(a,b)$, then local fractional derivative of f(x) of order $\propto$ at $x = x_0$ is given by

[1]- [4]

$$f^{\propto}(x_0) = \left.\frac{d^{\propto} f(x)}{dx^{\propto}}\right|_{x=x_0} = \lim_{x\to x_0} \frac{\Delta^{\propto}[f(x)-f(x_0)]}{(x-x_0)^{\propto}} \qquad (1.1)$$

Here $\Delta^{\propto}[f(x) - f(x_0)] \cong \Gamma(1+\propto)\, \Delta[f(x) - f(x_0)]$ for any $x \in (a,b)$ there exists

$f^{\propto}(x) = D_x^{(\propto)}(a,b)$, also the local fractional integral of f(x) is given by [1]- [4]

$$_a I_b^{(\propto)} = \frac{1}{\Gamma(1+\propto)} \int_a^b f(t)(dt)^{\propto} \qquad \ldots (1.2)$$

$$= \frac{1}{\Gamma(1+\propto)} \lim_{\Delta t\to 0} \sum_{j=o}^{j=N-1} f(t_j)\, (\Delta t_j)^{\propto}$$

Where $\Delta t_j = t_{j+1} -$ t and $\Delta t = max\{\Delta t_1, \Delta t_2 \ldots \Delta t_j \ldots\}$, j = 1, 2, 3, …N-1 and $[t_j, t_{j+1}]$ is a partition of interval (a, b).

1.3 Mittag-Leffler Function:

Let $E_\propto : R \to R$, $x \to E_\propto(x)$, denotes a continuously function which is known as Mittag-Leffler Function[1], [2], [6].

$$E_\propto(x^\propto) = \sum_{k=0}^{\infty} \frac{x^{\propto k}}{\Gamma(1+k\propto)}, \quad 0 < \propto \le 1 \qquad \ldots (1.3)$$

1.4 Local fractional Integral transforms:

Recently, the Yang-Laplace transform, which uses local fractional calculus, was introduced [1].

The Yang-Laplace transform of a function f(x) is defined as [1], [2].

$$L_\propto[f(x)] = f_s^{L,\propto}(z) = \frac{1}{\Gamma(1+\propto)} \int_0^{\infty} E_\propto(-z^\propto x^\propto) f(x)(dx)^\propto \qquad \ldots (1.4)$$

Where $0 < \propto \le 1$ and inversion formula of Yang-Laplace transform as follows

$$f(x) = L_\propto^{-1}\left(f_s^{L,\propto}(z)\right) = \frac{1}{(2\pi)^\propto} \int_{c-i\infty}^{c+i\infty} E_\propto(z^\propto x^\propto)[f_s^{L,\propto}(z)](dx)^\propto \qquad \ldots (1.5)$$

Here $0 < \propto \le 1$ and z is complex variable such that $Re\ z \ge 0$.

The fractional Stieltjes transform of f(x) is given by [7]

$$S_\propto[f(x)] = f_t^{S,\propto}(z) = \frac{1}{\Gamma(1+\propto)} \int_0^{\infty} \frac{1}{(x+t)^\propto} f(x)(dx)^\propto \qquad \ldots (1.6)$$

Here $0 < \propto \le 1$ and t is complex variable in the cut plane $|arg\ t| < \pi$.

II.LOCAL FRACTIONAL LAPLACE-STIELTJES TRANSFORM

2.1. Classical Laplace-Stieltjes Transform:

The Laplace-Stieltjes transform given by Giona and Paterno [8] and modified and studied by [19], is defined by the equation

$$M[f(x)] = \int_0^{\infty} \frac{e^{-zx}}{1+tx} f(x)dx \qquad \ldots (2.1)$$

where z is complex variable and $t \in \left(C \setminus (-\infty, 0]\right)$ It is also denoted as $M[f(x)] = M(t,z)$ and known as Laplace - Stieltjes transform.

For t = 0, it (2.1) reduces to Laplace transform and for z = 0, it (2.1) reduces to Stieltjes transform.

2.2. Definition of Local Fractional Laplace-Stieltjes Transform:

In this section we define the local fractional Laplace-Stieltjes transform,

Definition :

If the function f(x) belongs to M(R) and is piecewise continuous and locally integrable on the interval 0 < x < ∞, then the local fractional Laplace-Stieltjes transform is defined by [1], [2],

$$M_\propto[f(x)] = \frac{1}{\Gamma(1+\propto)} \int_0^{\infty} \frac{E_\propto(-z^\propto x^\propto)}{(1+t\,x)^\propto} f(x)(dx)^\propto \qquad (2.2)$$

where z is complex variable and $t \in \left(C \setminus (-\infty, 0]\right)$, and denoted as $M_\propto[f(x)] = M_\propto(t,z)$.

III. BASIC OPERATIONAL PROPERTIES

If $M_\propto[f(x)] = M_\propto(t,z)$ and k, c are constants then we have the properties

[1] Linear property: The local fractional Laplace-Stieltjes transform is linear for every pair of function $f_1(x)$ and $f_2(x)$ and every pair of constant a_1 and a_2,

$$M_\propto[a_1 f_1(x) + a_2 f_2(x)] = a_1 M_\propto[f_1(x)] + a_2 M_\propto[f_2(x)] \quad \ldots (3.1)$$

[2] Translation Property or Shifting Theorem

If $M_\propto[f(x)] = M_\propto(t,z)$ then

(i) $M_\propto[E_\propto(-c^\propto x^\propto)\, f(x)] = M_\propto(t,\ z+c)$… (3.2)

(ii)) $M_{\propto}[E_{\propto}(c^{\propto}x^{\propto})\, f(x)] = M_{\propto}(t,\ z-c)$ … (3.3)

[3] Change of Scale Property

If $M_{\propto}[f(x)] = M_{\propto}(t,z)$ then $M_{\propto}[f(c\,x)] = \frac{1}{c^{\propto}}\, M_{\propto}\left(\frac{t}{c},\frac{z}{c}\right)$ … (3.4)

[4] Multiplication by $x^{\propto}$

If $M_{\propto}[f(x)] = M_{\propto}(t,z)$ then

(i) $M_{\propto}[x^{\propto} f(\,x)] = -\frac{\mathrm{d}^{\propto}}{\mathrm{d}z^{\propto}}\{M_{\propto}[f(x)]\} = -\frac{\mathrm{d}^{\propto}}{\mathrm{d}z^{\propto}}\,\{M_{\propto}(t,z)\}$ … (3.5)

(ii) $M_{\propto}[-x^{\propto} f(\,x)] = \frac{\mathrm{d}^{\propto}}{\mathrm{d}z^{\propto}}\{M_{\propto}[f(x)]\} = \frac{\mathrm{d}^{\propto}}{\mathrm{d}z^{\propto}}\,\{M_{\propto}(t,z)\}$ … (3.6)

[5] local fractional Laplace-Stieltjes transform of derivative

If $M_{\propto}[f(x)] = M_{\propto}(t,z)$ then

$M_{\propto}[f'(x)] = -f(0) + (z+t)M_{\propto}(t,z) + t^2\,\frac{\mathrm{d}^{\propto}}{\mathrm{d}t^{\propto}}\,\{M_{\propto}(t,z)\}$ … (3.7)

[6] Derivative of local fractional Laplace-Stieltjes transform:

The local fractional Laplace-Stieltjes transform is given by

$$M_{\propto}[f(x)] = M_{\propto}(t,z) = \frac{1}{\Gamma(1+\propto)}\int_0^{\infty}\frac{E_{\propto}(-z^{\propto}x^{\propto})}{(1+t\,x)^{\propto}}\, f(x)(dx)^{\propto}$$

Differentiating with respect to ‘t’ , we obtain

$$\frac{\mathrm{d}^{\propto}}{\mathrm{d}t^{\propto}}\,[M_{\propto}(t,z)] = \frac{\Gamma(1-\propto)}{\Gamma(1-2\propto)} \times \frac{1}{\Gamma(1+\propto)}\int_0^{\infty}\frac{E_{\propto}(-z^{\propto}x^{\propto})}{(1+t\,x)^{2\propto}}\, x^{\propto} f(x)(dx)^{\propto} \quad \text{… (3.8)}$$

and

$$\frac{\mathrm{d}^{n\propto}}{\mathrm{d}t^{n\propto}}\,[M_{\propto}(t,z)] = \frac{\Gamma(1-\propto)}{\Gamma(1-(n+1)\propto)} \times \frac{1}{\Gamma(1+\propto)}\int_0^{\infty}\frac{E_{\propto}(-z^{\propto}x^{\propto})}{(1+t\,x)^{(n+1)\propto}}\, x^{n\propto} f(x)(dx)^{\propto}\ ,$$

n = 1, 2, 3, … … (3.9)

Differentiating with respect to ‘z’, we obtain

$$\frac{\mathrm{d}^{\propto}}{\mathrm{d}z^{\propto}}\,\{M_{\propto}(t,z)\} = M_{\propto}[-x^{\propto} f(\,x)] \quad \text{… (3.10)}$$

And $\frac{\mathrm{d}^{n\propto}}{\mathrm{d}z^{n\propto}}\,\{M_{\propto}(t,z)\} = M_{\propto}[-x^{n\propto} f(\,x)]$,

n = 1, 2, 3, … … (3.11)

[7] Theorem:

If $M_{\propto}[f(x)] = M_{\propto}(t,z)$ then
$M_{\propto}[E_{\propto}(-c^{\propto}x^{\propto})\, f(kx)] = \frac{1}{k^{\propto}}\, M_{\propto}\left(\frac{t}{k},\frac{z+c}{k}\right)$ … (3.12)

IV. INVERSE LOCAL FRACTIONAL LAPLACE-STIELTJES TRANSFORM

Let f(x) be fractional continuous and locally integrable function in the interval $0 < x < \infty$ and $M_{\propto}[f(x)] = M_{\propto}(t,z)$, then inverse of local fractional Laplace-Stieltjes transform is denoted as $M_{\propto}^{-1}$ and defined by $M_{\propto}^{-1}\{M_{\propto}(t,z)\}$ = f(x)

$$f(x) = M_{\propto}^{-1}\big(M_{\propto}(t,z)\big) = \frac{1}{(2\pi)^{\propto}}\int_{c-i\infty}^{c+i\infty}\frac{x^{\propto(t-1)}}{\Gamma(t)\Gamma(1-t)}E_{\propto}(z^{\propto}x^{-\propto})[M_{\propto}(t,z)](dz)^{\propto} \quad \text{… (4.1)}$$

where ‘c’ is constant and $\operatorname{Re} z \geq 0$, $t \in \big(C \setminus (-\infty,0]\big)$

V. APPLICATION OF LOCAL FACTIONAL LAPLACE-STIELTJES TRANSFORM

The local fractional Laplace-Stieltjes transformation is applied to problems where the distribution function has a highly singular or unusual structure. Chemical reactions and kinetics in continuous mixtures serve as a good example of such type of structures.

VI. CONCLUSION

In this paper, we introduce the local fractional Laplace-Stieltjes transform, along with some of its operational properties and the inverse of the local fractional Laplace-Stieltjes transform. For future work, we plan to explore the applications of this transform in signal processing within fractal media, as well as in fractional heat and diffusion equations. We also intend to compare it with models based on the Caputo and Riemann-Liouville approaches.

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