A Note on Coretractable Modules

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***Abstract*—We discuss the structures of coretractable and mono-coretractable modules. We provide the characterizations of coretractable modules in term of Kasch rings, CS modules, mono-coretractable modules, projective modules, epi- retractable modules and retractable modules. We derive some equivalent condi- tions related with coretractable and mono-coretractable modules. Among others, some results of coretractable modules in term of symmetric modules are obtained.**

***Index Terms*—retractable module; coretractable module; mono-coretactable mod- ule; Kasch ring; Kasch module. 2010 Mathematics Subject Classification: 16D50; 16D70; 16D80.**

I. INTRODUCTION

Throughout this paper, we assume all rings are associative and all modules are unitary right modules. We follow the terminology of [1, 2]. As in [3], a right R-module M is called *compressible* if for each non-zero submodule N of M, there exists a monomorphism $f : M \to N$. Zelmanowitz has studied the compressible modules in detail in [3, 4]. An R-module M is called *epi-retractable* if every sub- module of M is a homomorphic image of M. McConnel and Robson [2] have considered an epi-retractable module as the dual of a compressible module. An R- module M is said to be *retractable* if $Hom\ (M, N) \vDash 0$ for any non-zero submodule N of M. Retractable modules have been discussed by many authors in a series of papers [5–10]. B. Amini et al. [11] have investigated the dual of retractable mod- ules and proved that completely coretractable rings are two sided perfect. They also provide the condition under which free modules are coretractable. In [6], P.

F. Smith introduced the concept of essentially compressible modules as a gener- alization of compressible modules. Following [8], an R-module M is said to be *essentially retractable* if $Hom\ (M, N) \vDash 0$, for any non-zero essential submodule of M which is a natural generalization of retractable modules. In [9], A. K. Singh has obtained results related with essentially slightly compressible modules and rings.

If N is a submodule of M, we write $N \leq M$ and if N is an essential submodule of M, then we write N Œ M. A partial endomorphism of a module M is a homo- morphism from a submodule of M into M. We say that N is a *dense submodule* of M (written as $N \subseteq_d M$) if, for any $y \epsilon M$ and $x \in M/ \{0\}$, $xy^{-1}N \vDash 0$ (i.e., there exists $r \in R$ such that $xr \vDash 0$, and $yr \in N$). If $N \subseteq_d M$, we also say that M is a *rational extension* of N. We called I to be an *injective hull* of a module M denoted as $E\ (M)$, if I is injective, and is essential over M. Let $I = E\ (M)$, and let $H = End_R(I)$, operating on the left of I. We define *Rational Hull* of a module $\hat{E}\ (M)$ as $\hat{E}\ (M) = \{i \in I{:}\ \forall h \in H, h\,(M) = 0, h(i) = 0\}$. Clearly, this is an R-submodule of I containing M.

A module is called *mono-coretractable* if each of its factors can be embedded in it. That is, a module M is mono-coretractable if for each submodule N of M there exists a monomorphism $g : M/N \to M$. It is easy to see that a mono- coretractable module is coretractable. Every simple and semi simple module is mono-coretractable. $Z_p\infty$ is mono-coretractable and Z(Z)-module is not mono-coretractable. Some characterizations of mono-coretractable and coretractable modules were discussed in [12].

This paper is organized as follow. In section 2 we study some properties of core- tractable modules. We develop some relations between epi-retractable modules, coretractable modules, semi-primitive modules, symmetric modules and Kasch ring. In section 3, we discuss the characterizations of mono-retractable modules in terms of non-singular uniform modules and Kasch ring.

II. PRELIMINARIES

A module M is called coretractable (see [11]) if there exists a non-zero map f: $M/K \to M$ for any proper submodule K of M. A module M is said to be *torsion free module* if $\forall r \in R$ and $m \in M$, $rm = 0 \Rightarrow r = 0$ or $m = 0$. Before we prove some general module theoretic properties of coretractability, we investigate roles of free, dense and rational submodules in coretactable modules. In our first result, we discuss a coretractable module M in terms of torsion free module.

Proposition 2.1. *Let M be coretractable module over a commutative ring R, then M is torsion free.*
Proof. Since M is coretractable thus we have a homomorphism $f : M \to M \neq 0$ with $f(K) = 0$ for a proper submodule K of module M. If possible, suppose M is not torsion free then there exist non-zero element $r \in R$ and $m \in M$ such that $rm = 0$. Thus we have $f(r(M/K)) = f(r(m + K)) = f(rm) + f(rK) = 0$, since $f(K) = 0$. A contradiction to the fact that M is a coretractable module. Hence, M is torsion free.
An R-module M_R is said to be *rationally complete* if it has no proper rational extensions, or equivalently $\hat{E}(M) = M$.

Theorem 2.2. *For any module M_R, the following are equivalent:*
1. *M is rationally complete.*
2. *For any right R-modules $A \le B$ such that $Hom_R(B/A, E(M)) = 0$, any R-homomorphism $f : A \to M$ can be extended to B.*

Clearly, if M is rationally complete then M is not coretractable. We know that an R-module M is called monoform if every submodule is dense in M. Now we are ready to prove the following proposition.

Proposition 2.3. *Let $I \neq R$ be an ideal in a commutative ring R and let $M_R = R/I$, then following are equivalent in M:*
1. *Every submodule in M is dense.*
2. *M is monoform.*
3. *M is not coretractable.*
4. *Every ideal in R/I is prime.*

Proof. (1) ⇒ (2) and (2) ⇒ (3) are clear.
(1) ⇔ (4). Let M_R is a module and $I \neq R$ is an ideal of R. Let us suppose that any submodule N of M is dense in M. This gives a non-zero mapping $f : N/I \to M \Leftrightarrow f(I) = 0 \Leftrightarrow f(ab) = 0 \Leftrightarrow ab = 0 \Leftrightarrow$ *either* $a = 0$ *or* $b = 0 \Leftrightarrow I$ is prime ideal where $a, b \in R$.
(3)⇒ (1) Clear by [11, Proposition 2.14]

Our next result gives relation between an epi-retractable module and a core- tractable module.
Proposition 2.4. *The following statements are equivalent for a module M:*
1. *M is epi-retractable.*
2. *There exist surjective homomorphisms $M \to N$ and $N \to M$ for some epi-retractable module N.*
3. *There exists a surjective homomorphism $M/K \to M$ for some epi-retractable factor module M/K.*
4. *M is a coretractable module.*

Proof. (1) ⇒ (2) and (2) ⇒ (3) is clear from [10, Lemma 2.7].
(3) ⇒ (4) is obvious.
(4)⇒ (1) Let $K \le N$ are proper submodule of M. Since M is coretractable, we have $f : M/K \to M \neq 0$. So we have restriction map $f_{|N/K} : N/K \to N$. Consider a projection map $\pi : M/K \to N/K$ and a canonical epimorphism $g : M \to M/K$. The composition $f_{|N/K}\, \pi g$: $M \to N$ gives a non-zero homomorphism from M/K to M. This implies that M is an epi-retractable module.

Proposition 2.5. *Let M_R be an epi-retractable module, then M_R is coretractable if and only if M is semi-primitive.*
Proof. Let M is a coretractable module then M is semi-simple [11, See 2.3]. So M is semi-primitive. For the sufficient part let M is semi-primitive. Now since M is an epi-retractable module, so we have a non-zero surjective mapping f: $M \to N$ and g: $N \to M$ [10, Lemma 2.7]. Clearly, we have a non-zero homomorphism h $(M/N, N)$. Moreover, the composition $gh \neq 0$ implies that M is a coretractable module.
A projective R-module P is called *projective cover* of module M if there exist a small

epimorphism π: $P \to M$. A ring R is said to be *perfect* if every R-module has a protective cover.

Proposition 2.6. *Let M be a projective R-module, then every epi-retractable module is coretractable.*
Proof. Let M be a projective R-module and N is a submodule of M. Consider a non-zero map f: $M/N \to N$. R is a right perfect ring so we have a surjective map θ: $P \to N$; P is projective cover of N. As M is projective so $f = \theta h \models 0$ where h: $M/N \to P$ is a non-zero map. Since M is epi-retractable module so we have a surjective mapping from π: $M \to N$ and g: $N \to M$ [10, Proposition 2.1]. Clearly the map gf: $M/N \to M \models 0$, proving the proposition.
An R-module M is satisfies *(**)-property* if every $f \epsilon End(M_R) \models 0$, f is an epimorphism from M to M. We say that R is a *right Kasch ring* if every simple right R- module M can be embedded in R_R." *Left Kasch ring*" is defined similarly. A ring R is called a *Kasch ring* if it is both right and left Kasch. A module M is called *Kasch module* if it contains a copy of every simple module in σ (M).

Proposition 2.7. *Let M_R be an epi-retractable module, then:*
1. *M_R is right Kasch.*
2. *M_R satisfies the (**) property.*
3. *M_R is coretractable.*

Proof. (1) ⇒ (2) Let $f \in End_R(M)$ and $f : M \to K$ be a non-zero homomorphism where K is a proper submodule of M. Now M is right Kasch so it contains copy of simple module. Thus, we have $f(M) = K = M$ because M is simple module. So, f is an epimorphism, i.e., M satisfies the (**) property. $i{=}1$ i
⇒ (3) Let M satisfies the (**) property. Also, M is an epi-retractable module
(2) we have a non-zero homomorphism from f: $M/K \to N/K$, where $K \leq N \leq M$. Now we have a natural epimorphism h: $N/K \to N$ and let $g \in End_R(M) \models 0$ where g is an onto map from N to M. So we have a non-zero onto composition ghf : $M/K \to M \models 0$.
(3) ⇒ (1) Follows from [11, Proposition 2.14].
The following two propositions show that coretractable modules M satisfies the (**) property and M is also a direct summand of a free epi-retractable R-modules.

Proposition 2.8. *A coretractable module M satisfies the (**) property.*
Proof. Let M be a coretractable module so we have a non-zero homomorphism f: $M/K \to M$ for some proper submodule K. Consider a non-zero $g \in End_R(M)$. We have $gf(M/K, M) \models 0$ and also the mapping gf is epimorphism. So, M satisfies the (**) property.

Proposition 2.9. *Let R be a ring and α be an infinite ordinal $\geq |R|$. Suppose that $M = K\ N$ where K is a free epi-retractable R-module with a basic set of cardinalities α and N is a β-generated R-module with $\alpha \geq \beta$. Then, M_R is a coretractable module.*
Proof. Sine K is an epi-retractable module, N is a homomorphic image of K. Since K ▾ K K, then there exist surjective homomorphisms f: $M \to K$ and g $K \to M$. Let $L = kerf$, then there exists natural homomorphism h: $M/L \to M$. Hence, M is a coretractable module.
A module M is said to be *fully retractable* if for every non-zero submodule N of M and every non-zero element $g \in Hom_R(N, M)$, we have $Hom_R(M, N)g \models 0$. This definition motivates us to give our next result, whose proof is clear from Proposition 2.4.

Proposition 2.10. *Every fully retractable module is coretractable if $Hom_R(M, N)g$ is an epimorphism.*

Proposition 2.11. *Let R be a ring. If M_R is a cyclic CS module, then M is a coretractable.*
Proof. Since M is a cyclic CS module then M is a direct sum of uniform modules i.e., $M = {}^{\mathrm{L}}K$ $N i$ where each $K's$ is essential in M. Let K be any submodule of M then K be essential in direct summand of M. We know that that every quotient module of a cyclic module is cyclic, so we get a homomorphism $Hom_R(M/K, M) \models 0$, proving that M is coretractable.

Proposition 2.12. *Let R be a commutative artinian ring, then the following are equivalent:*
1. *R is a Kasch ring.*
2. *R_R is coretractable.*

3. If p is any prime ideal of R, then p is maximal.
4. $Hom_R(R/p, R) \models 0$.
5. R_R *is semi-simple.*

Proof. (1) ⇐⇒ (2) These are equivalent conditions because every proper ideal of R is contained in a maximal ideal.

(1) ⇒ (3) Let p be any prime ideal of R, then R/p become an integral domain. i.e., R/p is a field. Hence, p is maximal ideal of R as R is Kasch ring.

(3) ⇒ (4) Let p be a prime ideal of R so R/p is simple artinian ring. Then, it can be easily check that $Hom_R(R/p, R) \models 0$.

(4) ⇒ (5) Here R is artinian and R/p is simple artinian ring. Thus, R_R is simple, so we have R_R semi-simple.

(5) ⇒ (1) Obvious from ([11] Example 2.2).

We know that direct sum of coretractable modules is coretractable module but in the case of direct product of coretractable modules the result may not be same. Next result validates our supposition.

Proposition 2.13. *The infinite direct sum of coretractable modules may not be a coretractable module.*

Proof. Let $K = K_1 \oplus K_2 \oplus ... \subset M$ be an infinite direct product of coretractable modules where K_i (i = 1, 2...) are projective submodules of M. Consider a map $f\colon M/K \to K$ with $f(K) = 0$. Also suppose that $M = M_1 \oplus M_2$. Since $f(K) = 0$, so $f(M_1) = 0$ and $f(M_2) = 0 \Rightarrow f(M) = 0 \Rightarrow f\colon M/K \Rightarrow M = 0$. Thus, an infinite direct sum of coretractable modules may not be a coretractable module.

Proposition 2.14. *Let R be hereditary ring, then* M_R *is coretractable.*

Proof. Let N be an injective submodule of M. Then there exist a submodule K of M such that $M = K \oplus N$. Consider a non-zero map $f\colon M/N \to N$. Consider a injection map $g\colon N \to M$. Clearly the composition $gf\colon M/N \to M$ is non-zero. So, M_R is a coretractable module.

A Module M is called *S-symmetric or symmetric* whenever $fgm = 0$ implies $gfm = 0$ for any $m \in M$ and $f, g \in S$ where $S = End_R(M)$. In [13, 14] vari- ous properties of symmetric modules are obtained. Now, we discuss symmetric modules and try to develop some relations between symmetric and coretractable modules.

Proposition 2.15. *Let R be a projective R-module, then symmetric modules are coretractable.*

Proof. Let M_R be a symmetric module. Consider $\pi \in S$, then we have $fg\pi \models 0 \Rightarrow gf\pi \models 0$. where f, g is non-zero elements of S. Thus $fg\pi(M/K) \models 0 \Rightarrow \pi(M/K) = mR = M$. So M is a coretractable module.

We have the following straightforward result.

Corollary 2.16. *Let M be a cyclic projective R-module then:*
1. R is a symmetric ring.
2. M is a coretractable module.

We have the following proposition related to coretractable module.

Proposition 2.17. *The following conditions are equivalent for a symmetric ring R:*
1. M is a symmetric module.
2. M is 1-epiretractable module.
3. M is a coretractable module.

Proof. (1) ⇒ (2) It followed from [13, Theorem 2.4].

(2) ⇒ (3) Follows from Proposition 2.4.

(3) ⇒ (1) Let M be a coretractable module. Let $h \in S$. So, we have $h(M/K) = mR \Rightarrow fgh(M/K) = 0$. This implies that $fgh = gfh$. So M is a symmetric module.

REFERENCES

[1] K.R. Goodearl, R.B. Warfield Jr., An Introduction to Noncommutative Noetherian Rings, Second edition, Vol. 61, London Mathematical Society, Student Texts, Cambridge Press, Cambridge, 2004.
[2] J.C. McConnel, J.C. Robson, Noncommutative Noetherian Rings, Graduate studies in Mathematics, Vol. 30, American Mathematical Society, 1987.
[3] J. Zelmanowitz, Weakly primitive rings, Communication of Algebra 9 (1) (1981) 23-45.
[4] J. Zelmanowitz, An extension of the Jacobson

density theorem, Bulletin American Mathematical Society 82 (4) (1976) 551-553.

[5] P.F. Smith, Modules with many homomorphism, Journal of pure and Applied Algebra 197 (2005) 305-321.

[6] P.F. Smith, M.R. Vedadi, essentially compressible modules and rings, Journal of Algebra 304 (2006) 812-831.

[7] V.S. Rodrigues, A.A. Sant'Ana, A note on a problem due to Zelmanowitz, Algebra and Discrete Mathematics (3) (2009) 85-93.

[8] M.R. Vedadi, essentially retractable modules, Journal of Science, Islamic Republic of Iran 18 (4) (2007) 355-360.

[9] A. K. Singh, Essentially Slightly Compressible Modules and Rings, Asian-European Journal of Mathematics 5 (2) (2012) 1-8.

[10] A. Ghorbani, M.R. Vedadi, Epi-retractable modules and some Applications, Bulletin of Iranian mathematical Society 35 (1) (2009) 155-166.

[11] B. Amini, M. Ershad, H. Sharif, Coretractable Modules, Journal of Australian Mathematical Society 86 (2009) 289-304.

[12] A.M.A. Al-Hosainy, H.A. Kadhim, Co-compressible Modules, International Research Journal of Scientific Findings 1 (6) (2014) 1-8.

[13] B. Ungor, Y. Kurtulmaz, S. Halicioglu, A. Harmanci, Symmetric Modules Over Their Endomorphism Rings, Preprint.

[14] N. Agayev, S. Halicioglu, A. Harmanci, On symmetric modules, Riv. Mat. Univ. Parma 8 (2) (2009) 91-99.

[15] S.M. Khuri, Endomorphism rings and lattice isomorphism, J. Algebra 56 (2) (1979) 401-408.

[16] S. M. Khuri, Endomorphism rings of nonsingular modules, Ann. Sci. Math. Quebec 4 (2) (1980) 145-152.

[17] J. Zemlicka, Completely Coretractable Rings, Bulletin of the Iranian Mathe- matical Society 39 (3) (2013) 523-528