Local Neutrosophic Rough Sets on Topological Space

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Abstract—Uncertainty and incompleteness are ongoing key challenges in the fields of data analysis, decision-making, and topological modelling. To address these concerns, this paper introduces a novel framework for local neutrosophic rough sets (LNRS) within a topological space, referred to as the local neutrosophic rough topology (LNRT). Based on the rough set theory, neutrosophic sets, and their hybrid models, LNRT successfully navigates the challenges posed by indeterminacy and the significance of attributes, particularly in large-scale data analysis. Additionally, present the definitions of LNRT along with illustrative examples.

***Index Terms*—Neutrosophic Set, Local Rough Set, LNRS, LNRT, LNRT- interior, LNRT- exterior, LNRT-closure.**

I. INTRODUCTION

The origins of topology can be traced to the 18th and 19th centuries, developing from geometric and set-theoretic ideas into a formalized field of study. Today, topology is categorized into various subfields, such as general topology, algebraic topology, differential topology, and geometric topology, each providing unique viewpoints and methodologies for analysis. It can be formally described as the study of the qualitative characteristics of mathematical spaces, known as topological spaces, that capture notions of structure and continuity. The limitations of existing studies in fields including clinical research, financial and economic sciences, image processing, bioinformatics, intelligent systems, and strategic decision-making. These challenges can be addressed using Fuzzy sets (FS), Intuitionistic FS (IFS), Rough sets, Neutrosophic sets, Rough Neutrosophic sets, soft sets, and various hybrid configurations of these sets. The idea of Rough Set theory (RST) was initially presented in the field of computer science in 1982 by the Polish mathematician and computer scientist Z. Pawlak. RST is related to fuzzy logic; it is a mathematical structure designed to solve problems with insufficient information. The RST offers a practical approach to recognizing the precise patterns in the given data. It effectively manages data reduction, generates accurate decision rules, and is also easy to understand. The RST addresses lower & upper approximations and their boundary region.

In this context, F. Smarandache employed an NS to address three types of membership functions [X]. He introduced the term "neutrosophic," which is derived from the word "neutrosophy" [neutre in French <neuter in Latin < Greek Sophia and skill/wisdom]. It indicates the primary difference between 'fuzzy' or 'intuitionistic fuzzy' sets/logics and neutrosophic sets/logics, which include a middle component. The neutrosophic component is truth, ambiguity, and indeterminacy. All of these membership function values fall between $]0^{-}, 1^{+}[$. The notion of the Neutrosophic Set (NS) is applied to resolve practical issues with indeterminacy and incompleteness. Broumi et al [II] first introduced a novel approach known as Rough Neutrosophic Set (RNS). The RNS is a combination of NS and RST. RSs and NSs both have the capacity to handle incomplete instructions and uncertainty. RNS theory serves as a robust mathematical tool for addressing issues of incompleteness. In RNSs, the assessment of alternatives is depicted using upper & lower approximation operators, which capture the uncertainty inherent in the data. Furthermore, categorizing sets according to degrees of truth, ambiguity, and falsehood provides a refined description of sufficient instruction, making RNSs an adaptable tool for decision-making (DM) in uncertain contexts. Consequently, a variety of hybrid models have been developed, including DM frameworks, RN (rough neutrosophic) Similarity Measures, rough single-valued NS, RNS in coefficient correlation, and more.

Local Rough Set (LRS) is one of the effective tools to overcome the limitations of classical RST. The RS model can handle RS data analysis effectively, but still faces challenges like inefficiency in handling huge-scale datasets. The existing models do not effectively distinguish the importance of different attributes in DM when defining and characterizing binary relations. This lack of differentiation results in the formation of vague information granules or categories, which ultimately undermines the accuracy of concept approximation. These challenges are solved by using a theoretical framework named LRS. Yuhua Qian et al [VII] reconstructed a classical RS defined as LRS, which overcomes the drawbacks of RSs. The key reason for this LRS is to solve the limited labelled data, computational inefficiency, and overfitting in attribute reduction. The LNRS concept was initiated by S. Bharathi et al [I]. This tool has been established as highly effective in dealing with and managing uncertainty-related problems.
The concept of neutrosophic topological spaces was introduced by Salama and Alblowi [IX], building upon the idea of neutrosophic sets, thereby extending the principles of intuitionistic fuzzy topological spaces. Smarandache [VI] has introduced the concepts of fuzzy soft topological space, intuitionistic fuzzy soft topological space, and neutrosophic soft topological space. In 2021, Das et al. [III] established the idea of a quadripartitioned neutrosophic topological space. The notion of a pentapartitioned neutrosophic topological space is presented by Das and Tripathy [V]. In 2022, the topology on rough pentapartitioned NS was introduced by Das and Tripathy [IV]. Riaz, Smarandache et al [VIII] introduce an innovative mathematical structure named Neutrosophic Soft Rough Topology (NSR-topology), which integrates neutrosophic, soft, and RS theories to improve the handling of uncertainty and vagueness in complicated decision environments. Tripathy et al [XI] presented and analysed a new structure referred to as single-valued Neutrosophic Rough continuous, along with its compactness within their Topological space.
This paper presents an innovative novel framework for LNRS within a topological space, referred to as the LNRT. Also introduces basic definitions of LNRT and illustrates them with examples. The advancement of LNRT enhances the relationship between neutrosophic logic and Local Rough Set theory, thus contributing to the field of mathematical topology.

II. PRELIMINARIES

2.1.Definition
Let (C,R) be a space of approximation, C be a non-zero set, and R be a relation of equivalence in C. Let K as an RNS in C, defined by the membership τ_K, indeterminacy δ_K and non-membership η_K . Let $0 \leq \beta < \alpha \leq 1$ at some $K \epsilon C$, the local α -lower and local β - upper approximations are denoted as $\underline{N}_\alpha(K)$ *and* $\overline{N}_\beta(K)$ in C respectively.
Define

$$\underline{N}_\alpha(K)=\{(l,D((\tau_{\underline{N}_\alpha(K)}(l),\delta_{\underline{N}_\alpha(K)}(l),\eta_{\underline{N}_\alpha(K)}(l))/[l]_R)) \geq \alpha \mid l \epsilon C,[l]_R \neq \emptyset\}$$

$$\overline{N}_\beta(K)=\{(l,D((\tau_{\overline{N}_\beta(K)}(l),\delta_{\overline{N}_\beta(K)}(l),\eta_{\overline{N}_\beta(K)}(l))/[l]_R)) > \beta \mid l \epsilon C,[l]_R \neq \emptyset\}$$

Where,

$$\tau_{\underline{N}_\alpha(K)}(l) = min_{m\epsilon[l]_R}\tau_K(m) \quad ,\delta_{\underline{N}_\alpha(K)}(l) = max_{m\epsilon[l]_R}\delta_K(m) \quad ,\eta_{\underline{N}_\alpha(K)}(l) = max_{m\epsilon[l]_R}\eta_K(m) \quad ,$$

$$\tau_{\overline{N}_\beta(K)}(l) = max_{m\epsilon[l]_R}\tau_K(m) \quad ,\delta_{\overline{N}_\beta(K)}(l) = min_{m\epsilon[l]_R}\delta_K(m) \quad ,\eta_{\overline{N}_\beta(K)}(l) = min_{m\epsilon[l]_R}\eta_K(m)$$

Here $\tau_K(l),\delta_K(l),\eta_K(l)$ denoted as membership, indeterminacy & non-membership of l in K.
Therefore $0 \leq \tau_{\underline{N}_\alpha(K)}(l)+\delta_{\underline{N}_\alpha(K)}(l)+\eta_{\underline{N}_\alpha(K)}(l) \leq 3$, $0 \leq \tau_{\overline{N}_\beta(K)}(l)+\delta_{\overline{N}_\beta(K)}(l)+\eta_{\overline{N}_\beta(K)}(l) \leq 3$. The functions

$$\tau_{\underline{N}_\alpha(K)}(l),\delta_{\underline{N}_\alpha(K)}(l),\eta_{\underline{N}_\alpha(K)}(l),\tau_{\overline{N}_\beta(K)}(l)+\delta_{\overline{N}_\beta(K)}(l)+\eta_{\overline{N}_\beta(K)}(l) : K \to]0^-,1^+[.$$

The pair $(\underline{N}_\alpha(K),\overline{N}_\beta(K))$ called LNRS in C. In this definition, $\underline{N}_\alpha(K)$ and $\overline{N}_\beta(K)$ are having the constant membership function on equivalence classes in C if $\underline{N}_\alpha(K)=\overline{N}_\beta(K)$.
(ie) $\tau_{\underline{N}_\alpha(K)}(l)=\tau_{\overline{N}_\beta(K)}(l),\underline{\delta}=\overline{\delta},\underline{\eta}=\overline{\eta}$
This is called a definable LNRS of the approximation (C,R).

2.2. Example
Consider $Q_z=\{Q_1,Q_2,Q_3,Q_4,Q_5,Q_6\}$ be a universal set. The truth value takes the equivalence class $R =$

$\{J_1\}$ and $X = \{Q_1, Q_3, Q_4, Q_6\}$. Assume that the parameter $\alpha = 0.6$ and $\beta = 0.2$.

Table 1: The values of truth, indeterminacy, and falsity are illustrated in the table below.

Q_z	J_1	J_2	J_3
Q_1	(0.2,0.5,0.7)	(0.6,0.4,0.9)	(0.4,0.2,0.1)
Q_2	(0.2,0.7,0.8)	(0.7,0.2,0.1)	(0.1,0.5,0.8)
Q_3	(0.7,0.3,0.4)	(0.9,0.5,0.7)	(0.6,0.7,0.1)
Q_4	(0.7,0.9,0.1)	(0.3,0.2,0.8)	(0.5,0.7,0.9)
Q_5	(0.7,0.6,0.8)	(0.5,0.9,0.8)	(0.3,0.4,0.6)
Q_6	(0.3,0.9,0.2)	(0.2,0.5,0.6)	(0.5,0.1,0.9)

From the above table, the equivalence classes are $\{Q_1, Q_2\}$, $\{Q_3, Q_4, Q_5\}, \{Q_6\}$. Using the parameter to calculate the lower and upper approximation values.

$D[Q_z|Q_1]_R = \frac{1}{2}, D[Q_z|Q_3]_R = \frac{2}{3}$

$D[Q_z|Q_4]_R = \frac{2}{3}, D[Q_z|Q_6]_R = 1$

$\underline{Q}_{0.6}(K) = [Q_3]_R \cup [Q_4]_R \cup [Q_6]_R$

$\underline{Q}_{0.6}(K) = \{Q_3, Q_4, Q_6\}$

$\overline{Q}_{0.2}(K) = [Q_1]_R \cup [Q_3]_R \cup [Q_4]_R \cup [Q_6]_R$

$\overline{Q}_{0.2}(K) = \{Q_1, Q_2, Q_3, Q_4, Q_5, Q_6\}$.

This pair $(\underline{Q}_{0.6}(K), \overline{Q}_{0.2}(K))$ is called a local $\alpha -$ lower and local $\beta -$ upper approximations in LNRS.

2.3.Remark

In reality, attaining the essential information $X \in U$ in large datasets, and managing multiple screening processes is required. The upper and lower approximations are described using a target concept, denoted as X. A target set X, and an information granule is required to define the global rough set, and its calculation is substantial and time-consuming. The LRS cannot consider information granules and only filters the data in the target set X. This idea is quite helpful for rough data analysis using enormous datasets. The elements of the upper and lower approximations are extracted from the target set $x \in X$. Previously, lower and upper approximations were characterized by a target set using approximating. Consequently, the lower approximation belongs to the target set, and the upper approximation must belong to a target set.

2.4.Definition

Let W denote the initial space, with φ being a subset of W, and $\rho = (W, Z)$ described as a neutrosophic soft approximation space, where $Z = (Q, \varphi)$ represents a neutrosophic soft set. The upper and lower approximations are established based on the neutrosophic soft approximation space and its neighbourhoods. Therefore, the collection

$\tau_{NSR}(\varphi) = \{W, \emptyset, \underline{apr}_{NSR}(\varphi), \overline{apr}_{NSR}(\varphi), B_{NSR}(\varphi)\}$

is referred to as the neutrosophic soft rough topology (NSR-T), which ensures the following principles

- Both W and $\emptyset$ are elements of $\tau_{NSR}(\varphi)$.
- Union of the elements of $\tau_{NSR}(\varphi)$ is contained within $\tau_{NSR}(\varphi)$.
- Finite intersection of the elements of $\tau_{NSR}(\varphi)$ is also part of $\tau_{NSR}(\varphi)$.

Hence, $(W, \tau_{NSR}(\varphi), P)$ is recognized as an NSR-T if $\tau_{NSR}(\varphi)$ is called Neutrosophic soft rough topology.

III. LNRS ON TOPOLOGICAL SPACE

3.1. Definition

Suppose (F, R) be an LNR approximation space. Let R be a relation of equivalence on the universal F. The LNRS is utilized to compute the lower and upper approximations. The collection

$\tau_{LNRT}(R) = \{F, \emptyset, \underline{W}(Q), \overline{W}(Q), W_B(Q)\}$,

$[\because W_B(Q) = \underline{W}(Q) - \overline{W}(Q)]$

It is referred to as the local neutrosophic rough topology (LNRT). This ensures the following conditions.

1. F and $\emptyset \in \tau_{LNRT}(R)$.
2. The union of any components in τ_{LNRT} belongs to τ_{LNRT}.
3. The finite intersection of components in τ_{LNRT} belongs to τ_{LNRT}.

Then $(F, \tau_{LNRT}(R))$ is referred to as an LNR topological space, if $\tau_{LNRT}(R)$ be an LNRT.

3.2. Example

Consider $Q = \{Q_1, Q_2, Q_3, Q_4, Q_5, Q_6\}$ be a universal set. From Example 2.2, we have

$\underline{W}(Q) = \{Q_3, Q_4, Q_6\}, \overline{W}(Q) = \{Q_1, Q_2, Q_3, Q_4, Q_5, Q_6\}$ and $W_B(Q) = \{Q_1, Q_2, Q_5\}$. Then

$\tau_{LNRT}(R) = \{F, \emptyset, \{Q_3, Q_4, Q_6\}, \ , \{Q_1, Q_2, Q_3, Q_4, Q_5, Q_6\}, \{Q_1, Q_2, Q_5\}\}$

Be a $\tau_{LNRT} -$ topology.

3.3. Definition
Suppose $(F, \tau_{LNRT}(R))$ be an LNRT- space. Then, the components of $\tau_{LNRT}(R)$ are referred to as LNR open sets (LNRO).
The components of LNRS are termed as LNR closed sets (LNRC), if its complement is a member of $\tau_{LNRT}(R)$.

3.4. Proposition
Assume $(F, \tau_{LNRT}(R))$ be an LNRT space. Then
1. F and $\emptyset$ are LNRC sets.
2. The finite union of LNRC sets is a component of the LNRC set.
3. The intersection of LNRC sets is a component of the LNRC set.

3.5. Definition
Suppose $(F, \tau_{LNRT}(R))$ be an LNRT- space such that $\tau_{LNRT}(R) = \{F, \emptyset\}$. Then, $\tau_{LNRT}(R)$ is referred to as LNR- indiscrete topology in F with respect to R and the corresponding space is referred to as LNR- indiscrete topological space.

3.6. Definition
Suppose $(F, \tau_{LNRT}(R))$ be an LNRT- space and $Q \subseteq H \subseteq F$. Subsequently, the collection
$\tau_{LNRT}(Q) = \{H_K \cap Q: H_K \in \tau_{LNRT}(R), K \in N\}$ is referred to as LNR- subspace topology on F. Then $(Q, \tau_{LNRT}(R))$ be an LNR topological subspace of $\left(F, \tau_{LNRT}(R)\right)$.

3.7. Example
From example 2.2, we have,
$$\tau_{LNRT}(R) = \left\{F, \emptyset, \{Q_3, Q_4, Q_6\}, \{Q_1, Q_2, Q_3, Q_4, Q_5, Q_6\}, \{Q_1, Q_2, Q_5\}\right\}$$
Take the set $Q = \{Q_1, Q_2, Q_3, Q_4, Q_5, Q_6\}$ and $F = H_K = \{Q_1, Q_2, Q_3, Q_4, Q_5, Q_6\}$

The LNR-subspace topology is,
$\tau_{LNRT}(Q) = \left\{F, \emptyset, \{Q_3, Q_4, Q_6\}, \{Q_1, Q_2, Q_5\}\right\}$
Since $\tau_{LNRT}(R)$ contains all sets in $\tau_{LNRT}(Q)$
$\tau_{LNRT}(Q)$ is a LNR topological subspace of $\tau_{LNRT}(R)$.

3.8. Definition
Let $(F, \tau_{LNRT}(R))$ and $(F, \tau'_{LNRT}(R))$ are two LNRT spaces. Then $(F, \tau'_{LNRT}(R))$ be finer than $(F, \tau_{LNRT}(R))$ iff $\tau'_{LNRT}(R) \supseteq \tau_{LNRT}(R)$.

3.9. Definition
Let $(F, \tau_{LNRT}(R))$ represent a LNR topological space and $\sigma_{LNRT} \subseteq \tau_{LNRT}$. If the components of τ_{LNRT} be a union of components of σ_{LNRT}. Then σ_{LNRT} is known as the LNR basis of LNRT.

3.10. Example
From example 2.2, we have,
$$\tau_{LNRT}(R) = \left\{F, \emptyset, \{Q_3, Q_4, Q_6\}, \{Q_1, Q_2, Q_3, Q_4, Q_5, Q_6\}, \{Q_1, Q_2, Q_5\}\right\}$$
And let $Q = \{Q_1, Q_2, Q_3, Q_4, Q_5, Q_6\}$
Each element of Q is contained in at least one set in $\tau_{LNRT}(R)$. The sets,
$\emptyset, \{Q_1, Q_2, Q_3, Q_4, Q_5, Q_6\}, \{Q_3, Q_4, Q_6\}, \{Q_1, Q_2, Q_5\}$
Are the basis elements of $\tau_{LNRT}(R)$. Therefore, $\tau_{LNRT}(R)$ satisfies the conditions to be a basis for an LNRT on Q.

3.11. Theorem
Consider two LNRT spaces $(F, \tau_{LNRT}(R))$ and $\left(F, \tau_{LNRT'}(R')\right)$. Let σ_{LNRT} and $\sigma_{LNRT'}$ are LNR bases in τ_{LNRT} and $\tau_{LNRT'}$ respectively. If $\sigma_{LNRT'} \subseteq \sigma_{LNRT}$, then τ_{LNRT} be finer than $\tau_{LNRT'}$ and $\tau_{LNRT'}$ is weaker than τ_{LNRT}.

3.12. Theorem
Let $(F, \tau_{LNRT}(R))$ is an LNRT space. If σ_{LNRT} be an LNRT basis for τ_{LNRT}. Then,
$\sigma_{LNRT} = \{Q_K \cap H: Q_K \in \sigma_{LNRT}(R), K \in N\}$ be an LNR basis on the LNR subspace topology on F.

Proof
Assume $Q_K \in \tau_{LNRT}(R)$.
From definition 6, write it as, $K = J \cap D$, where $J \in \tau_{LNRT}$. It follows that $J = \cup Q_K$.
Hence $K = (\bigcup_{Q \in \sigma_{LNRT}} Q_K) \cap H$
$K = \bigcup_{Q \in \sigma_{LNRT}} (Q_K \cap H)$.

3.13. Definition
Consider $(F, \tau_{LNRT}(R))$ is an LNRT- space, and $\gamma \subseteq F$. The LNR interior (LNR-I) of γ be the union of every LNRO subset on γ. It is denoted as $Int_{LNRT}(\gamma)$. Also $Int_{LNRT}(\gamma)$ be the largest LNRO set that belongs to γ.

3.14. Theorem
Let $(F, \tau_{LNRT}(R))$ is an LNRT space. Consider Z and O are the LNR sets over F. Then

1. $Int_{LNRT}(\emptyset) = \emptyset$ and $Int_{LNRT}(F) = F$.
2. $Int_{LNRT}(Z) = Z$.
3. Z be an LNRO set iff $Int_{LNRT}(Z) = Z$.
4. $Int_{LNRT}(Int_{LNRT}(Z)) = Int_{LNRT}(Z)$.
5. $Z \subseteq 0$ such that $Int_{LNRT}(Z) \subseteq Int_{LNRT}(O)$.
6. $Int_{LNRT}(Z) \cup Int_{LNRT}(O) \subseteq Int_{LNRT}(Z \cup O)$.
7. $Int_{LNRT}(Z) \cap Int_{LNRT}(O) = Int_{LNRT}(Z \cap O)$.

Proof

(i) and (ii), From the definition of 14, $Int_{LNRT}(Z) \subseteq Z$.
Assume $Z = 0$ in the equation mentioned above.
$Int_{LNRT}(0) \subseteq 0$
The condition also exists, $0 \subseteq Int_{LNRT}(0)$. Therefore,
$Int_{LNRT}(0) = 0$
Similarly, to prove $Int_{LNRT}(Z) \subseteq Z$.
(iii) Assume $Int_{LNRT}(Z) = Z$.
Since $Int_{LNRT}(Z)$ be an LNRO set. Hence Z is LNRO set.

Conversely, suppose Z be an LNRO set. Then, the largest LNRO set that exists within Z is Z itself.
Hence $Int_{LNRT}(Z) = Z$.
(iv) We know that $Int_{LNRT}(Z)$ be an LNO set from the above statement (ii).

$\therefore$ Hence $Int_{LNRT}(Int_{LNRT}(Z)) = Int_{LNRT}(Z)$.
(v) Assume that $Z \subseteq 0$. From the subdivision (ii),
$Int_{LNRT}(Z) \subseteq Z$. Then $Int_{LNRT}(Z) \subseteq 0$.
Since $Int_{LNRT}(Z)$ is contained in T.

From definition 14, hence $Int_{LNRT}(Z) \subseteq Int_{LNRT}(O)$.
(vi) We know that, $Int_{LNRT}(Z) \subseteq Z$ and $Int_{LNRT}(O) \subseteq O$
Then, $Int_{LNRT}(Z) \cup Int_{LNRT}(O) \subseteq Z \cup O$.
Therefore, $Int_{LNRT}(Z) \cup Int_{LNRT}(O)$ is open.

$\therefore Int_{LNRT}(Z) \cup Int_{LNRT}(O) \subseteq Int_{LNRT}(Z \cup O)$.
(vii) From (ii) $Int_{LNRT}(Z) \subseteq Z$ and $Int_{LNRT}(O) \subseteq O$.
To prove, $Int_{LNRT}(Z) \cap Int_{LNRT}(O) \subseteq Z \cap O$.
From $Int_{LNRT}(Z) \cap Int_{LNRT}(O)$ is open.
Therefore,
$Int_{LNRT}(Z) \cap Int_{LNRT}(O) \subseteq Int_{LNRT}(Z \cap O)$ (1)
Conversely, $Z \cap O \subseteq Z$, further $Z \cap O \subseteq O$.
Next, $Int_{LNRT}(Z \cap O) \subseteq Int_{LNRT}(Z)$ and
$Int_{LNRT}(Z \cap O) \subseteq Int_{LNRT}(0)$

$\therefore Int_{LNRT}(Z \cap O) \subseteq Int_{LNRT}(Z) \cap Int_{LNRT}(O)$ (2)
From equations (1) and (2),
$Int_{LNRT}(Z) \cap Int_{LNRT}(O) = Int_{LNRT}(Z \cap O)$.

3.15. Definition

Consider $(F, \tau_{LNRT}(R))$ is an LNRT- space. Let $Z \subseteq F$, then LNR- exterior (LNR-E) of Z is denoted as $Int_{LNRT}(Z)$, where Z^c represents the complement of Z. the LNR exterior of Z is $Ext_{LNRT}(Z)$.

3.16. Definition

Consider $(F, \tau_{LNRT}(R))$ is an LNRT- space. Let $Z \subseteq F$, then LNR-closure (LNR-C) of Z is referred to as the intersection of every LNR closed supersets of T. It is denoted as $Cl_{LNRT}(Z)$.

3.17. Example

Let us analyse the LNRT space described in Example 2.2. Let us choose the subset
$Z = \{Q_1, Q_2, Q_5\}$. The complement of Z is $Z^c = \{Q_3, Q_4, Q_6\}$.
$\therefore$ The interior of Z is $Int(Z) = \{Q_1, Q_2, Q_5\}$,
$\therefore$ The exterior of Z is $Ex(Z) = Int(Z^c) = \{Q_3, Q_4, Q_6\}$
The closure of Z is $Cl(Z) = \{Q_1, Q_2, Q_5\}$.
Here, the smallest closed set containing Z is $\{Q_1, Q_2, Q_5\}$.

3.18. Theorem

Let $(F, \tau_{LNRT}(R))$ is an LNRT space, Z and O are the LNR sets on F. Then

1. $Cl_{LNRT}(\emptyset) = \emptyset$ and $Cl_{LNRT}(F) = F$.
2. $Z \subseteq Cl_{LNRT}(Z)$.
3. Z be a LNR closed set iff $Z = Cl_{LNRT}(Z)$.
4. $Cl_{LNRT}(Cl_{LNRT}(Z)) = Cl_{LNRT}(Z)$.
5. $Z \subseteq 0$ implies $Cl_{LNRT}(Z) \subseteq Cl_{LNRT}(O)$.
6. $Cl_{LNRT}(Z \cup O) = Cl_{LNRT}(Z) \cup Cl_{LNRT}(O)$.
7. $Cl_{LNRT}(Z \cap O) = Cl_{LNRT}(Z) \cap Cl_{LNRT}(O)$.

Proof

Given $Z, O \subseteq F$ is an LNR set on F.
(i) From definition 1 of LNRT, $\emptyset$ be a member of itself, and LNRT is closed, its closure is $\emptyset$.
$\therefore Cl_{LNRT}(\emptyset) = \emptyset$.
Since the set F is obviously LNRT closed and contains itself, which means its closure is F.

$\therefore Cl_{LNRT}(F) = F$.
(ii) It is established that $Cl_{LNRT}(Z)$ represents the smallest LNRT closed set that encompasses Z, for any LNRT set Z.

(iii)A As the smallest LNR closed set that contains Z is represented by $Cl_{LNRT}(Z)$. Furthermore, M is closed. Thus, the only feasible case is,
$Z = Cl_{LNRT}(Z)$.
Conversely, assume $Z = Cl_{LNRT}(Z)$.
Based on the definition of LNR closure Z belongs to the LNR-C set.

$\therefore Cl_{LNRT}(Z)$ is also belongs to the LNR-C set.
(iv) It is known that the closure of a set represents the smallest closed set that includes the original set.

$\therefore$ Any LNR set Z, $Cl_{LNRT}(Z)$ represents the smallest LNR-C set that includes Z.

Moreover, $Cl_{LNRT}(Z) = Z$ to any LNR-C set.
$\therefore\ Cl_{LNRT}(Cl_{LNRT}(Z)) = Z$ to any LNR-C set.
$\therefore\ Cl_{LNRT}(Cl_{LNRT}(Z) = Cl_{LNRT}(Z)$.
(v) Consider Z and $O(Z \subseteq O)$ be the LNR subsets of LNRT space $\left(F, \tau_{LNRT}(Z)\right)$. Since $Z \subseteq O$.
Each LNR-C set that contains Z and also contain O, therefore, the intersection of such sets for O includes the intersection for Z. This indicates $Cl_{LNRT}(Z) \subseteq Cl_{LNRT}(O)$.
$\therefore Z \subseteq 0 \Rightarrow Cl_{LNRT}(\,Z) \subseteq Cl_{LNRT}(O)$.
(vi) Consider Z and O as the two LNR subsets of an LNRT space $\left(F, \tau_{LNRT}(R)\right)$.
Since $Z \subseteq Z \cup 0$ and $O \subseteq Z \cup 0$

$\therefore Cl_{LNRT}(Z) \subseteq Cl_{LNRT}(Z \cup 0)$ And
$Cl_{LNRT}(O) \subseteq Cl_{LNRT}(Z \cup 0)$
Hence $Cl_{LNRT}(Z) \cup Cl_{LNRT}(O) \subseteq Cl_{LNRT}(Z \cup 0)$ (3)
Subsequently, it is written as $Z \subseteq Cl_{LNRT}(Z)$ and $O \subseteq Cl_{LNRT}(O)$
Which indicates $Z \cup 0 \subseteq Cl_{LNRT}(Z) \cup Cl_{LNRT}(O)$.
In general, the union of the LNR-C sets belongs to $\left(F, \tau_{LNRT}(R)\right)$
So $Cl_{LNRT}(Z) \cup Cl_{LNRT}(O)$ is also belongs to the LNR-C set. This LNR-C set contains $Z \cup 0$.
Since $Cl_{LNRT}(Z \cup 0)$ is the smallest LNR-C set that contains $Z \cup 0$.

$\therefore Cl_{LNRT}(Z \cup 0) \subseteq Cl_{LNRT}(Z) \cup Cl_{LNRT}(0)$ (4)
Equations (3) and (4) give,
$Cl_{LNRT}(Z \cup 0) = Cl_{LNRT}(Z) \cup Cl_{LNRT}(0)$.
(vii) Consider Z and O as the two LNR subsets of an LNRT space $\left(F, \tau_{LNRT}(R)\right)$.
Since $Z \cup 0 \subseteq Z$ and $Z \cup 0 \subseteq O$.
According to a known result, $Cl_{LNRT}(Z \cap 0) \subseteq Cl_{LNRT}(Z)$ and $Cl_{LNRT}(Z \cap 0) \subseteq Cl_{LNRT}(O)$.
Hence, $Cl_{LNRT}(Z \cap O) = Cl_{LNRT}(\,Z) \cap Cl_{LNRT}(O)$.

3.19. Definition
Consider the LNRT space $\left(F, \tau_{LNRT}(R)\right)$, defined in relation to a subset $Q \subseteq U$. Let Z be any subset of F. The LNR boundary, also referred to as the LNR frontier, of the set Z is denoted by either $Fr_{LNRT}(Z)$ or $b_{LNRT}(Z)$ and is formally expressed as,
$Fr_{LNRT}(Z) = Cl_{LNRT}(Z) \cap Cl_{LNRT}(Z^c)$
This indicates that the LNR frontier of Z is the intersection of the LNR closure of Z and the LNR closure of its complement Z^c. By this definition, the LNR frontier is always an LNR closed set.

3.20. Example
Let us examine the LNRT space described in Example 2.2 and consider the subset
$Z = \{Q_1, Q_3, Q_4\}$.
The complement of Z is $Z^c = \{Q_2, Q_5, Q_6\}$.
The closure of Z is represented as $cl(Z) = \{Q_1, Q_3, Q_4, Q_6\}$, whereas the closure of its complement is denoted as $cl(Z^c) = \{Q_1, Q_2, Q_5, Q_6\}$
Consequently, the boundary of Z is defined by the intersection of the closures
$Fr(Z) = Cl(Z) \cap Cl(Z^c) = \{Q_1, Q_6\}$.

IV. CONCLUSION

This paper introduced an innovative mathematical framework called the LNRT, where the essential definitions have been developed and demonstrated with examples. The advancement of LNRT enhances the relationship between neutrosophic logic and Local Rough Set theory, thus contributing to the field of mathematical topology. This framework not only enhances theoretical knowledge but also opens new pathways for further research and applications in related fields.

REFERENCES

[1] Bharathi, S., and K. Arulmani. Local neutrosophic rough set. Indian Journal of Natural Sciences, 14(81): 66710–66717, 2023.

[2] Broumi, Said, Florentin Smarandache, and Mamoni Dhar. Rough neutrosophic sets. Infinite Study, 2014.
[3] Das, Suman, Rakhal Das, and Carlos Granados. Topology on quadripartitioned neutrosophic sets. Neutrosophic Sets and Systems, 45 (2021): 54–61.
[4] Das, Suman, Rakhal Das, and Binod Chandra Tripathy. Topology on rough pentapartitioned neutrosophic set. Iraqi Journal of Science (2022): 2630-2640. https://doi.org/10.24996/ijs.2022.63.6.28
[5] Das, Suman, and Binod Chandra Tripathy. Pentapartitioned neutrosophic topological space. Neutrosophic Sets and Systems, 45.1 (2021): 9.
[6] Parimala, M., M. Karthika, and Florentin Smarandache. A Review of Fuzzy Soft Topological Spaces, Intuitionistic Fuzzy Soft Topological Spaces and Neutrosophic Soft Topological Spaces. International Journal of Neutrosophic Science, 10(2): 96–104, 2020.
[7] Qian, Yuhua, Xinyan Liang, Qi Wang, Jiye Liang, Bing Liu, Andrzej Skowron, Yiyu Yao, Jianmin Ma, and Chuangyin Dang. Local rough set: a solution to rough data analysis in big data. International Journal of Approximate Reasoning, 97: 38–63, 2018. https://doi.org/10.1016/j.ijar.2018.01.008
[8] Riaz, Muhammad, et al. Neutrosophic soft rough topology and its applications to multi-criteria decision-making. Infinite Study, 2020.
[9] Salama, A. A., and S. A. Alblowi. Neutrosophic set and neutrosophic topological spaces. 2012. http://dx.doi.org/10.9790/5728-0343135
[10] Smarandache, Florentin. A unifying field in logics: Neutrosophic logic. In Philosophy, pages 1–141. American Research Press, 1999.
[11] Tripathy, Binod Chandra, Suman Das, and Rakhal Das. Single-Valued Neutrosophic Rough Continuous Mapping via Single-Valued Neutrosophic Rough Topological Space. Transactions on Rough Sets XXIII. Springer, Berlin, Heidelberg, 2023. 77–95. http://dx.doi.org/10.1007/978-3-662-66544-2_6