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# Recent Advances in The Nonlinear Solution of Partial and Ordinary Differential Equations

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## I. INTRODUCTION

In modern science and engineering, complex phenomena are modeled by nonlinear differential equations ordinary and partial ones. In the one hand, for linear systems whose solution spaces are determined by superposition and spectral completeness properties of their operators' nonlinear equations give rise to behaviours that can not be treated from a purely diagonal point of view: bifurcation phenomena, multistable character of solutions, or deterministic chaos and partial differential equations seem better suited as mathematical tools and thus cannot be studied through the methods available linear theory. As a result, the study and solution of nonlinear ODEs/PDEs continue to present core challenges in applied mathematics. Note Added In the past period there has been a remarkable surge in research activity in this field, fueled by three interrelated developments: (i) the improvement of analytical and semi-analytical tools applicable to strong nonlinearities and intricate boundary conditions, (ii) the advancement of high-order numerical methods coupled with breakthroughs in high-throughput computing capabilities, and (iii) the fast proliferation of data-driven and hybrid computational paradigms that merge conservation laws with machine learning techniques. These two advancements together have broadened the class of nonlinear problems that can be studied with certainty and accuracy, as well as revolutionized the conceptual definition of what a "solution" in nonlinear analysis is supposed to be. The following is a critical review of what has been achieved recently in the nonlinear solution of ODEs and PDEs from an approach perspective, including methodological development, comparative performance, and unsolved theoretical challenges.

### 1.1 Context and Motivation

Nonlinear differential equations remain significant due to the inherent nonlinearity in most physical and artificial systems. Turbulence in fluid flow, non-linear wave propagation, reaction–diffusion systems, biological morphogenesis, dynamics of epidemics and quantum many-body interactions are all governed by equations where linearity is a poor approximation to describe the global behavior. Even when a local linearization is valid, the long-term dynamics are typically strongly influenced by non-linear phenomena associated with large temporal or spatial scales. Typically, nonlinear equations have been analyzed by means of perturbation techniques, asymptotic expansions and qualitative phase space characteristics. Although these methods have provided deep insights, they are often limited by assumptions that are overly restrictive, t such as small parameters or weak nonlinearity. Systems of these types, from high-Reynolds-number flows to nonlinear optical media (e.g., found in quantum electronics applications [42]), exhibit all the features described and then some; for them it seems fair to say that they are far removed in parameter space from all aforementioned regimes of applicability, calling for effective study methods that do respect strong nonlinearities [1–3]. The simultaneous development of numerical analysis means that nonlinear PDEs on complex geometries and in dimensions large are routinely solved to high accuracy. However, the numerical solvers themselves exhibit intrinsic trade-offs among stability, convergence and solution cost especially in relation to stiff or multiscale problems.

More recently, physics-informed neural networks (PINNs), sparse system identification and operator learning frameworks have pushed the state-of-the-art by directly integrating governing equations into learning architectures [4–6]. While these methods have shown remarkable flexibility, they naturally lead to questions on convergence guarantees, generalization outside the training data and mathematical interpretability.

The motivation for this review is to provide a critical synthesis that considers not only theecessity of how one solves nonlinear differential equations, but also why certain methods appear to work (or fail) on specific sets of problems. It is necessary to make such evaluation so as to better direct the further research and also minimize the distance between theoretic derivation and practical calculations.

### 1.2 The Mathematical Nature of Nonlinearity

Nonlinearity is mathematically due to the loss of additivity and homogeneity in the governing operators. These and other complex phenomena are the hallmark of nonlinear terms in ODEs, leading to dependence on initial conditions, multiple-attractor coexistence and qualitative changes due to parameter variation. In PDEs nonlinearity synchronizes spatial and temporal dynamics which leads to the formation of shocks, finite-time blow-up, coherent structures, spatiotemporal chaos. One of the key characteristics of nonlinear systems is that their solution spaces are highly structured. Non-linear equations are notably less predictable than linear ones, and we know that beyond possible some isolated solutions there can be/attract all others: distinct branches of continuous solutions, weak and distributional solution branches, breaking at bifurcation events. Extensions of these works were recently carried out: several earlier contributions emphasized a rigorous functional-analytic setting (using Sobolev spaces or variational inequalities), and energy methods, to provide for existence, uniqueness, and regularity results for nonlinear PDE's of physical relevance [7,8]. Another basic is the interaction of scales. A number of nonlinear PDEs display multi-scale behavior where small-scale transient fluctuations drive macroscopic evolution, but when this system is time-invariant, the appropriate analytical tools include multiple scale expansions, homogenization theory and reduced order modeling. These notions have gained fresh relevance in the study of turbulence, nonlinear wave, and pattern forming systems [9–11]. Hence, understanding nonlinearity relates not just to solving solutions but also the discovery of structural means that produce complexity and emergence.

### 1.3 Objectives and scope of the review

The main goal of this far-reaching review is to give a detailed treatment on recent developments in the field of solution for nonlinear ODEs and PDEs. Instead of going through methods one by one, the review focuses on comparative evaluation that shows how analytic, numerical and data-driven techniques map onto complementary aspects of nonlinear behaviour.

It covers the following:

- Basic theoretical advances: existence, uniqueness, stability and sensitivity of these systems to initial conditions;
- Analytical and semi-analytical methods of solutions, such as the recent perturbation approach, decomposition method, and nonlinear transformation;
- New methods for non-linear PDEs (symmetry analysis, soliton theory, and similarity reductions);
- Modern numerical methods and their convergence, stability and computational efficiency;
- Advancements in AI-aided solvers, such as physics-based neural networks and symbolic regression: focusing on theory bound & blank spots;
- Typical applications in fluid dynamics, nonlinear optics, quantum physics and mathematical biology.

The review attempts a synthesis of this different perspectives in an effort both to be a reference point for advanced researchers, and to also offer a coherent treatment for those who are new to the field. Each subsequent section extends from the material laid down here, with consideration made to delve deeper while keeping nonlinearity at the core of our vision of contemporary applied mathematics.

## II. FUNDAMENTAL THEORY OF NONLINEAR SYSTEMS

The backbone of methods is the mathematical theory of nonlinear systems that analytical, numeric and data-based solution routines rely on. In the nonlinear setting, unlike in linear analysis where superposition, mode decomposi- tion, and global well-posedness can be expected to hold, ordinary and partial differential equations of physical interest are known to display properties that are generically dependent on model form/featureset-up or initial/boundary conditions. Hence, for the existence, uniqueness, stability and long-term qualitative behaviour much more sophisticated tools are required.

With increasingly more detailed understanding of classical nonlinearity over the past few decades we have made considerable advances in generalising these results to such wider classes as nonsmooth systems, infinite-dimensional dynamics and strongly nonlinear regimes. These contributions are of more than theoretical interest, but rather immediately pertain to the validity and limitations of modern day computational and machine-learning-based solvers. This subsection revisits current developments in the theory of existence and uniqueness, sensitivity to initial conditions and nonlinear stability analysis with particular focus on results pertinent for applied modelling.

### 2.1 Existence, Uniqueness, and Sensitivity to Initial Conditions

Hadamard's sense of well-posedness means that the existance and uniqueness are characterising. In the case of nonlinear ordinary differential equations, a local existence theorem is attributed to Picard–Lindelöf and depends on (the notion of) Lipschitz continuity of nonlinearity terms. Nevertheless, a great deal of new models ubiquitously appearing in, e.g., nonsmooth mechanics and biological switching systems and target control do not satisfy this kind of assumptions. For this reason, recent works pursue more general frameworks under Carathéodory conditions, differential inclusions and monotone 475 operator theory to achieve existence with milder regularity requirements [12,13]. In nonlinear partial differential equations, the theory of existence has progressed thanks to refined variational formulations and compactness techniques. Galerkin approximations with uniform energy estimates remain critical, but the literature in the last few years has focused more heavily on sharper a priori bounds and entropy methods that facilitate global-in-time existence for strongly nonlinear evolution equations like reaction-diffusion systems, nonlinear Schrödinger equations, and viscoelastic models [14–16]. These results are important in the context of high-dimensional systems, in which classical smooth solutions may not always exist. By contrast, uniqueness still belongs to the most opaque aspects of nonlinear analysis. In fact, for a large class of nonlinear PDEs uniqueness is only known on admissibility classes or in restricted regularity classes. In the last five years there has been significant development in conditional as well as for weak–strong uniqueness principles, especially for fluid and transport equations. Such results clarify when weak solutions agree with those that are more regular and offer a fully justified guide for choosing physically meaningful solution branches [17].

The sensitivity to initial conditions is a characteristic feature of nonlinear dynamics and has important consequences on predictability. Even systems with smooth nonlinearities can display exponential divergence of nearby paths and chaotic dynamics. More recently, further mathematical description of sensitivity has been put forward through finite-time Lyapunov exponents (FTLE) [18], stability spectrum [19] and probability notions of predictability. For nonlinear PDEs, sensitivity appears as both instabilities and long-memory effects, in particular for the case of turbulent or dispersive systems. The former features present major obstacles for the numerical approximation and data-driven solvers, whose trustworthiness is dependent significantly on how robust they are to perturbations in initial datum.

### 2.2 Stability Analysis: Lyapunov Methods and Limit Cycles

The theory of stability is the main qualitative tool to understand behavior [l] With this in mind, static responses are still attractive potential solutions. The direct method of Lyapunov is still the most general and powerful approach, because it can be used to determine the stability without having explicit solution formulas. Classical Lyapunov theory has been widely extended in recent years for the nonsmooth, timedelay and infinite-dimensional control systems [19-20]. One major progress after 2020 is the finding out of Lyapunov functionals for nonlinear PDE's from variational and energy perspective. Such functionals allow for a rigorous description not only of the asymptotic stability and dissipativity, but also of attractor structure for a broad class of nonlinear evolution equations including (but not restricted to) damped wave equation, thermoelastic systems and nonlinear viscoelastic models [21]. It is noteworthy to mention that the Lyapunov-based method is also nested as a building block in the development of structure-preserving numerical methods by propagating stability properties from discrete approximation to the exact system. Another fundamental central phenomenon in the nonlinear stability theory is limit cycles. In systems of finite dimensions, a number of recent contributions have addressed the role of bifurcations in the onset of period two orbits including Hopf ones and homoclinic/canard 333 type in strong nonlinear limits [22]. The analytical findings are supported by numerical continuation techniques that enable us to systematically track the parameter dependence of oscillatory solutions beyond perturbative limits.

In higher dimensions, and for some classes of nonlinear partial differential equations (PDEs), these limit cycles become invariant manifolds, quasi-periodic orbits or global attractors. Modern studies have employed center-manifold theory, spectral analysis and reduced order modelling methods to understand the stability and persistence of these structures [23]. Such approaches are especially useful in applications like fluid dynamics, biological rhythmics or nonlinear optics where persistent time patterns have a major impact. However, there are still some challenges facing the SDA method. Building global Lyapunov functionals for general nonlinear PDEs (including even the system (1)) is problem-dependent overall, and stability properties of solutions obtained via machine learning-based solvers are not known in detail from a rigorous point of view. Bridging classical non-linear stability theory to emerging data-based methods will remain an important direction in the future and prepares the ground for further parts of this review.

## III. ANALYTICAL AND SEMI-ANALYTICAL SOLUTIONS

Analytical and semi-analytical techniques play an important role in the analysis of nonlinear differential equations, which often provide data for less revealing numerical methods. Although specific solutions of nonlinear ODE (over both the ordinary and partial domains) in closed form are very rare, their analytical approximations often provide insight into qualitative behavior, observed parameter dependence, or asymptotic nature. In addition, semianalytical methods are often used as a benchmark for verification of numerical methods and reduced order modeling of multiscale problems. Over these last years the range of analytical and semi-analytical techniques have widened considerably, a fact motivated by strong nonlinearities, multiscale effects or nonstandard boundary conditions. Generalizations of standard perturbation theory are applied and combined with variational principles, decomposition schemes and symbolic computations to address fundamental problems such as secular growth, loss of uniform validity and dependence on smallness of parameters. This article is devoted to the so-called perturbation methods which are one of the most systematic and widely-used instruments in nonlinear analysis.

### 3.1 Perturbation Methods (Regular, Singular, and Multiple-Scale)

Perturbation methods are based on the notion that a non-linear problem can in some sense be written as a deformation of a simpler, exactly-solvable one by means of the introduction of a small parameter. In normal perturbation theoiy, solutions are expressed as power series expansions in this parameter which are assumed to be smoothly dependent and uniform everywhere within region of interest. While the classical regular perturbation methods are well known, new results have investigated (the iterated usage of) these for nonlinear ODEs and PDEs with superior convergence analysis as well as error estimates [24]. But a lot of the decent practically interesting nonlinear problems are actually singularly perturbed. In these situations, small parameters multiply the highest-order derivatives which give rise to boundary layers, sharp gradients or fast temporal transients that cannot be described by regular expansions. The singular perturbation theory, in particular match asymptotic expansions, has received new growth in the past five years, improved matching conditions and strict justifications for nonlinear reaction–diffusion and convection dominated systems [25–26]. These progresses have improved the convergence properties of the asymptotic approximations in regimes with an extreme separation of scales. Scaling methods offer an alternative approach to treating nonlinear systems with slow–fast dynamics or with resonant interactions. Through the introduction of independent quantities having different time or space scales, these methods eliminate the secular terms and offer a uniformly valid approximation over long periods. Recent contributions have pushed multiscale analysis to the border of non-autonomous and stochastic dynamical systems [26], as well as weakly-nonlocal PDE's in fluid dynamics, nonlinear optics [27,28]. The search for such extensions is of particular interest in the context of investigations on modulation instability, envelope dynamics and long-time behavior of nonlinear waves. One interesting trend since 2020 was the fusion of perturbation methods and symbolic/numerical computation. Automated multiple scale expansions38 and computer aided asymptotics21,39 have made feasible the calculation of higher-order terms which were previously infeasible, while hybrid analytical-numerical methods for linking local asymptotics to global properties are also presented [29]. These advances vastly simplify the manual workload of perturbation analysis and extend its reach to high-dimensional and parameter-intensive models.

Despite these advantages, perturbative methods still have several limitations. Their existence often depends on the presence of a small (or asymptotically-small) separable parameter that might not exist or have no physical meaning in all models. Furthermore, it is often the case that strongly nonlinear regimes and bifurcation points (394) are outside the scope of classical perturbative expansions; this includes chaotic dynamics as well. These limitations have inspired the search for semianalytic, non-perturbative approaches, such as homotopy-based methods and decompositions schemes that are discussed in later subsections.

To conclude, perturbation methods regular [48], singular [31] and multiple-scale [47,49] are essential tools of nonlinear analysis still resolute to uncover the multiscale structure and parametric sensitivity of many systems. Their ongoing development, particularly when combined with computational tools, keeps them relevant for the study of modern-day nonlinear problems and serves as a concept bridge to the unified semi-analytical and numerical methods described in later sections of this review.

### 3.2 Integral Transform Approaches (Nonlinear Laplace and Sumudu)

Integral transform methods have a long history of serving as the basis for the analysis of differential equations under which they can be transformed into more easily manipulated algebraic forms. In the linear theory, these types of transforms such as Laplace and Fourier transformations lead to the closed-form solutions in some a regular way. Generalizing these ideas for nonlinear differential equations is more involved due to the loss of superposition, and linearity by convolution. In the last ten years, and in particular since 2020 there has been a remarkable progress in conforming classical integral transforms into nonlinear and semi-analytical methods that can tackle a large class of nonlinear ODEs and PDES.

The classical transform definition is generally kept, and nonlinearity is introduced through iterative procedures, decomposition techniques or convolution formulations. In this framework, the transform is heuristically applied to the nonlinear differential equation and one solves, in a recursive way, the corresponding nonlinear algebraic or integral equations. Recent works [30, 31] have elucidated the functional analytic setting in which such procedures are known to converge; see particularly [30, 31] for systems of nonlinear evolution equations with polynomial or weakly nonlocal nonlinearities. Their work has contributed to the mathematical rigor of Laplace-based semi-analytical methods condemned for being based on intuitive arguments.

A prominent development has been the systematic merging of the Laplace transform with decomposition procedures leading to hybrid methods which damp secular growth and enlarge regions of convergence. For instance, Laplace–Adomian and Laplace–variational iteration methods have been rigorously investigated for nonlinear reaction–diffusion and fractional systems showing more stable behavior than purely time-domain iterations [32]. These methods prove particularly powerful for initial-value problems, since the Laplace transform automatically incorporates the initial conditions and enables time-domain reconstruction.

The Sumudu transformation has become quite popular as an appealing substitute of the Laplace transform in nonlinear analysis [2, 3]. With its scale preservation and one-to-one time-domain to transformed-function mapping, the Sumudu is able to bypass some of the numerical stiffness as well as the dimensional non-dimensionalization found in Laplace-based formulations. Since 2020, there has been increasing interest in using the Sumudu transform for nonlinear ODEs nd PDEs through a number of recent literatures on viscoelastic models, population dynamics and nonlinear oscillators [33,34]. Comparative research has shown that Sumudu-based semi-analytical solutions tend to converge faster and be more accurate during short- and intermediate-time regime than those from other semi-analytical methods.

In recent years, computational methods have extended the Laplace and Sumudu transforms to fractional-order and variable-order systems. Such extensions are of particular interest for models with memory effects and anomalous diffusion, for which classical integer-order definitions do not provide adequate representations. Existence and convergence analysis in fractional PDE have been established for transform solutions of nonlinear fractional PDE which also support their applicability into modern modeling scenarios [35, 36].

Simplification of the integral transform methods, which has become quite sophisticated, still suffers from some inherent limitations. Their applicability directly relies on the analytical properties of the nonlinear terms and the explicit form of a choice for an inverse transform (either closed or computable). Highly nonlocal nonlinearities, complex geometries and strong coupling between the elements would prevent the transform domain representations to be concise. In addition, transform-based methods are usually most appropriate for problems with simple boundary conditions and hybridization with numerical schemes might be needed to reflect more general settings. It can thus be concluded that the class of non-linear Laplace and Sumudu techniques are a mature one with many more developments on the horizon. The discovery of new fundamental attributes makes them even more appealing for the solution of nonlinear differential equations when associated with hybrid analytical–numerical methods advanced recently. These methods provide a natural way for us to pass to decomposition and iterative methods considered in the next subsection, in which nonlinearity is dealt with more directly during solution construction.

### 3.3 Adomian Decomposition and Variational Iteration Methods

The Adomian Decomposition Methods (ADM) and Variational Iteration Method (VIM) stand out among semi-analytical methods for solving Nonlinear Differential Equations because of their modest philosophy, versatility as well as wide range application to Ordinary and Partial differential equation. Contrary to perturbation-based methods, these methods do not require a small parameter to be present explicitly and are consequently especially appealing for strongly nonlinear system. Significant work has been dedicated to enhance the convergence properties of these methods, to broaden their theoretical framework and also couple them with other analytical or numerical techniques in the last five years.

It is rooted in the decomposition of a nonlinear equation solution into an infinite series, and writing a counted part of the nonlinearma operator by a corresponding series of Adomian polynomials. This construction reduces the original non-linear problem to a sequence of recursive sub-linear problems, which can in general be solved explicitly or with manageable computational effort. Recent developments have been dedicated to the systematica construction of Adomian polynomials for non linearities which are complex, such as non local and fractional operator and thus enlarge the scope of acce ptable problems [37,38]. These advances have broadened the application of this method to nonlinear PDEs in fluid dynamics, plasma physics, and reaction–diffusion systems.

Convergence analysis has traditionally been a significant challenge of ADM. Recently, there are several studies offering stringent convergence conditions of fixed-point under the framework of Banach space and also demonstrating the relationship between convergence radius analysis and edginess level for nonlinear operator (see [39]). ADM has also been extended to include convergence control parameters [42, 53], or by using Padé-type rational approximations yielding improved stability and faster convergence for stiff and highly nonlinear problems

[40]. These improvements are especially useful if ADM serves as a semi-analytical benchmark for the numerical solutions.

The VIM, a kind of expansion method proposed as an alternative correction- functional method (2), establishes successive approximations through variational principles and Lagrange multipliers applications. One of the features which distinguish VIM is its capability to embed disturbance as well as initial conditions into iterative formalism naturally. The theoretical justification of VIM gets well verified recently, via showing an equivalency between it and some fixed-point iterations in suitable functional spaces [41]. This has bolstered its mathematical legitimacy, and shed light on its scope of applicability.

In the past five years, VIM has considered nonlinear systems with fractional time-delays [4], fractional derivatives interactions [5] and stochastic forcing [6]. These extensions have been Further complemented by enhanced methods to determine the optimal Lagrange multipliers, a key factor in convergence properties [42,43]. Some comparisons show that faster convergence and lower number of iterations are reached by VIM for some classes of nonlinear evolution equations when compared with ADM, particularly in problems where the nonlinearity shows a strong coupling.

Combination with the ADM (or VIM) methodology and usage of integral and spectral discretization methods or numerical continuation methods have been also received attention [6, 7]. For instance, Laplace–ADM and VIM–spectral hybrids were proven to enhance the accuracy of time-dependent nonlinear PDEs while keeping semi-analytically interpretability [44]. This kind of hybridization is symptomatic for a general trend to combine analysis and computation rather than keep any methods separate.

However, ADM and VIM also have their own limitations. Both approaches may exhibit slow convergence or even diverge when they are used in extremely stiff systems, or problems with sharp spatial gradients. Furthermore, no universal error estimates and dependency on problem dependent tuning parameters are still open problems. These considerations are particularly important when applying the decomposition-based methods to high-dimensional or chaotic systems, where long-time accuracy is essential.

In conclusion, AD and VIM continue to be powerful tools in the semi-analytical arsenal for nonlinear differential equations. Their wavelengths and convergence rates, as well as hybrid schemes, have been recently improved using theory. Their further evolution thus constitutes a natural transition from classical geometric analysis to pure numerical or data-driven methods, and lays the ground for the more advanced nonlinear PDE methods.

## IV. ADVANCED TECHNIQUES FOR NONLINEAR PDES

Nonlinear partial differential equations (NPDEs) occupy a central place in the framework of contemporary applied mathematics, because they have found popularity as models for spatially extended dynamical systems. Phenomena including the formation of shock waves, the propagation of solitons, pattern formation and turbulence, as well as the generation of finite-time singularities are fundamentally described by nonlinear PDEs whose analysis (both from both analytical and numerical viewpoints) is a deeply challenging subject. Although perturbative and semi-analytical studies are intriguing within certain regimes, many nonlinear PDEs demand much more sophisticated methods that make use of structure. During last 5-year period (2020–2025), research on nonlinear PDEs has been characterized by a fast-growing area focusing on structure-exploiting methods, in the sense of using symmetry, invariance, integrability and scaling properties to simplify complexity or extract exact or reduced-form solutions. Such advanced methods are not local approximations, they are very efficient in revealing the global behavior of solutions, the conservation laws and invariant manifolds. They are also rigorous testing grounds for numerical and machine-learning solvers.

A central topic in modern analysis for nonlinear PDEs is the reduction of governing equations using symmetry and group theoretic techniques. In particular, lie group method has developed into a powerful methodology for recognizing continuous symmetries, constructing invariant solutions and detecting hidden conservation laws of nonlinear evolution equations [45, 46]. Furthermore, the classical symmetry approach has been recently generalized to nonlocal, fractional and variable-coefficient PDEs [9–23], which makes this method more applicable in practice to the real models.

Second major technical progress topic is soliton theory and integrable systems. Complete integrability is, however, an exceptional property and nonlinear PDEs with solitonsolutions still play a central role in mathematical physics and applied analysis. Development of inverse scattering techniques, Hirota bilinear formulations and Riemann–Hilbert methods has allowed us to acquire deeper insight into the interaction between nonlinear waves, stability of coherent structures and long-time asymptotics [47,48]. Such results for us are especially interesting for nonlinear optics, fluid dynamics and plasma physics. Similarity methods and dimensional analysis are a related class of techniques by which scaling invariance is used to cast non-linear PDEs as lower-dimensional ODEs. A recent effort has been focused on justifying rigorously the existence of similarity reductions and combining them with numerical continuation and asymptotic analysis to investigate solution families beyond classical self-similar regimes [49]. These types of techniques are particularly useful when investigating blow-up phenomena, spreading dynamics and intermediate asymptotics. Some of these recent developments have been crucial for further progress: in particular, today advanced methods in non-linear PDEs are no longer developed independently. One interesting trend for the latter since 2020 is numerical and computational hybrids. Symbolic computation allowed for automatic symmetry detection; numerical solvers were linked to analytical reductions in order to track parameter dependent solution landscapes [50]. This synergy is representative of a larger trend toward analysis-informed computation, where rich mathematical structure informs the design of algorithms.

Despite these achievements, plenty of challenges still exist. Most practical nonlinear PDEs do not posses' exploitable symmetry or integrability, and the application to high-dimensional strongly coupled systems of practitioners having dealt with advanced analytical tools is limited. Furthermore, connections between classical analytical formations and neuro-AI-based solvers are not well established but rather lead to challenging questions about their interpretability and coherence with theoretical results. In the three following subsections we describe in detail, three major components of advanced nonlinear PDE analysis symmetry methods, soliton theory and similarity techniques. Together they represent the current state of the art in terms of both synthesis and development, and reveal how contemporary mathematical analysis remains an ever-capable tool for tackling the most complex aspects associated with nonlinear PDEs while profoundly influencing future research into stronger solution concepts.

### 4.1 Symmetry Analysis and Lie Groups

Based on the theory of Lie group of continuous transformation, symmetry analysis has been widely known as one of the most effective methods for studying nonlinear partial differential equations. This originated in the seminal work of Sophus Lie and forms an organized framework for discovering the symmetries of differential equations and using them as a tool to generate exact solutions, conservation laws and low-order models. Symmetry methods are especially useful for nonlinear PDEs because they do not rely on linearity, so that even hidden symmetries beyond perturbative analyses can be discovered. Essence of Lie group analysis is to find smooth groups of point transformations which a PDE admits as the symmetry. These symmetries give rise to infinitesimal vector fields, who's corresponding to determining equations reflect the structural restrictions that the nonlinearity has on these fields. In the past years, we have witnessed remarkable development in generalizing classical Lie symmetry theory to nonlinear partial differential equations (PDEs) with formulation dependent forces, nonlocalities and fractional derivatives which are fundamental ingredients of various realistic mathematical models [51, 52]. These extensions have very much increased the range of applicability of symmetry-based methods far beyond idealized canonical equations.

The most important results of symmetry method are systematic converts nonlinear PDEs into O.D.E by invariant and reduction. Taking advantage of admitted symmetries, the problem dimension can usually be lowered without loosing essential dynamics. The classification of symmetry algebras of nonlinear evolution equations and the construction and use of optimal systems of subalgebras, leading to complete derivation of nonequivalent similarity reductions are considered in [53]. These advances have been particularly viable for nonlinear diffusion–convection equations, reaction–diffusion systems and nonlinear wave models. In addition to the classical point symmetries, there has followed the study of generalized symmetries such as contact, higher derivative, nonclassical and conditional ones. The lagrangian approach to the study of exact solutions in field theories tends to lose its consistency and nonclassical symmetry methods for this reason have been recently rediscovered, since it appears that they can lead us further to the computation of exactsolutions than classical Lie analysis. Advanced nonclassical symmetry computing algorithms have also been implemented in the last year [54,55], which were supported by symbolic computation, and succeeded to be applied for strong nonlinear PDEs from fluid dynamics as well as nonlinear optics. Another very important development is about the interplay between symmetries and conservation laws. With the help of Noether type correspondences and multipliers, symmetries are employed to get conserved quantities which are essential part in stability analysis and numerical discretization. Recent work has made clear the form of conservation laws for non-variational nonlinear PDEs and systematic procedures to derive it, even when there exists no Lagrangian formulation underneath [56]. These results are all the more pertinent in long-time numerical simulations, for which conservation properties is critical to accuracy and stability. The union of symmetry analysis and computation is an important direction in the past 5 years. Automatic wave type determination by (computer) algebraic methods dramatically lowers the technical barrier to such an approach, and numerical continuation methods have been linked with symmetry reductions to probe bifurcation structure in reduced groups [57]. These mixed analytical–computational approaches therefor illustrate the contemporary face of symmetry analysis as both a theoretical and numerical tool. These methods, however, have the disadvantage that symmetry analysis is not applicable to all systems. Many applied nonlinear PDEs possess only trivial symmetries, curbing the possibility of exact reductions. Furthermore, the understanding of symmetry-reduced solutions in complex geometries or realistic boundary conditions is still in its infancy. For the cases in which they can be applied, however, Lie group-based approaches provide unsurpassed analytical insights as well as provide strict tests for numerical and data-based solvers. To summarize, the symmetry reduction and Lie group methods are still fundamental tools in nonlinear PDEs theory. These properties can be generalized by: nonlocal, fractional and other related forms of calculus extensions of traditional ei and that due to the growth in symbolic computation new interest has been spurred for their relevance in recent scientific research. These techniques naturally lead toward soliton theory and integrabilities.

### 4.2 Soliton Theory and Inverse Scattering Transforms

Soliton theory is a remarkable achievement in the study of nonlinear PDEs that gives exact, stable and particle-like solutions which are preserved under nonlinear interactions. While generic nonlinear waves do not exist on their own account, solitons are dispersive structures occurring in a particular type of non-linear PDE s which admit hidden integrable structure, infinite hierarchies of conservation laws and generation mechanisms for exact solutions. Although complete integrability is rare, soliton theory still has significant impact in the context of theoretical analysis and applied modeling, especially in the systems governed by coherent structures. At its core are what is called the inverse scattering transform (IST), a nonlinear version of the Fourier transforms which turns certain classes of evolution equation into linear spectral problems. Remarkable examples where IST facilitates the construction of exact solutions are provided by classical integrable equations, such as KdV [1], NLS [2] or sine–Gordon equations. In 2020, there has been a revival of interest in the study of IST based analysis to long time asymptotics, soliton resolution conjectures and stability properties with respect to multi-soliton interactions [58,59]. An important progress in the last years is the detailed understanding of long-time properties for integrable nonlinear PDEs based on Riemann–Hilbert problem formulations. These tools allow the accurate asymptotics of dispersive shock waves, burst formation, and radiation damping in nonlinear wave equations [60]. Such results have added to the basis of soliton theory and helped to explain why localized coherent structures govern asymptotic dynamics. Apart from classical integrable systems, attention is now focused on near-integrable and perturbedintegrablesystemswhere solitonrlike behavior can bemaintained even in the presence of weak perturbations. Analytical methods, such as adiabatic perturbation theory, multiscale expansions and numerical IST have been used to investigate soliton deformation, interaction with external potentials and stability in the presence of dissipation [61 62]. These are especially important for physical realizations since in most cases the ideal integrability is broken due to forcing, damping or inhomogeneity. A further new area opened up since 2020 is the extension of these soliton ideas to higher-dimensional and nonlocal nonlinear PDEs. For higher dimensions, it is very rare that the system (2.1) admits complete integrability but it has been reported that the dromions, lumps and vortex solitons can be examined by employing generalised bilinear forms and when symmetry reductions were imposed [63]. On the other side, nonlocal integrable equations have also become of interest, motivated by parity–time symmetry and nonlocal interactions with new solitons dynamics and stability (Ref. [64], and references therein).

Any progress in computer technology and symbol computation can be welcome for soliton theory as well. Automated Hirota bilinearization, Darboux transformations and symbolic computation of multi-soliton solutions have made it possible to investigate systematically solution families which were hitherto unavailable. [65] Such features not only enable exact solution generation but may also serve as benchmarks for verifying numerical methods as well as machine learning enhanced solvers for nonlinear wave equations. Soliton theory has its own ceiling of glass despite its success. The condition of integrability makes it applicable in a direct way only to a limited class of nonlinear PDEs, so that the generalization of IST approach to the nonintegrable cases is quite unclear. Besides, for a lot of physically relevant nonlinear PDEs the behaviour is mixed, i.e., soliton-like structures are accompanied by chaotic or turbulent regimes. It is also an open and ongoing area of investigation to understand the interplay between integrable cores and nonintegrable perturbations. In conclusion, Soliton theory and inverse scattering transforms still play the leading role in advanced studies of nonlinear PDEs. Recent developments in asymptotic theory, perturbation analysis and computational implementation have underscored their relevance outside classical settings. These results are naturally complementary to symmetry-based approaches, and offer a conceptual link between similarity methods and scaling analysis.

### 4.3 Similarity Solutions and Dimensional Analysis

Similitude and dimensional analysis provide a classical, but still developing systematic approach for the study of nonlinear PDEs. They remain relevant because they can take advantage of scaling invariance and dimensional consistency to simplify complex PDEs into lower-dimensional problems, for which exact or semi-analytic solutions can be found that capture key features of nonlinear physics. Unlike those based on the full Lie group classification, symmetry methods involving scaling transformations and self-similarity have been especially successful for problems with spreading processes, blow-up behaviour and intermediate asymptotics. Mathematically, similarity solutions occur whenever a nonlinear partial differential equation (PDE) is invariant under a one-parameter scaling group with a combination of independent and dependent variables. This invariance allows the introduction of similarity variables which transform the PDE into an ordinary differential equation by eliminating space-time interdependence. In the past five years sharp criteria for the existence and admissibility of similarity reductions have been developed and it was made clear that there are two types of self-similar solutions: on one hand those characterised only by scaling, i.e., purely dimensional solution (self-similarity of the first kind), but also those where rescaling exponents can be encoded in non-linear eigenvalue problems [66,67].

In recent research, special attention has been focused on similarity techniques for nonlinear diffusion, porous media and fast-diffusion equations which are considered as canonical models of transport phenomena in heterogeneous materials. Recent progress, after 2020, has allowed a more refined analysis of similarity profiles, uniqueness, regularity and asymptotic stability of self-similar solutions [68]. These results help to improve the theoretical basis for similarity solutions as attractors determining their long-time behavior, rather than simply isolated exact solutions. Dimensional analysis is complementary to the similarity methods of due its systematic way to determine some physically or mathematically admissible scaling laws. More general treatments (see, for example that extend the traditional Buckingham–Π arguments introduced in 2 to the study of nonlinear PDEs exhibiting multiple interacting scales, a varying background and tensorial coefficients, as well as space-dependent anomalous diffusion. So much so that this kind of dimensional analysis has, for instance, been extremely useful in finding reduced models for non-linear wave propagation, combustion theory and geophysical flows [69]. With asymptotic analysis, dimensional considerations commonly lead the choice of dominant balances within which self-similar regimes emerge. There has been an important development in this area of application during the past five years, namely, associated similarity analysis with numerical and computational methods. By exploiting numerical continuation techniques, families of similarity solutions have been more broadly connected through parameter space and bifurcation structures, as well as relation between regimes of scaling exponents verified [70]. In addition, similarity variables have been applied to the construction of adaptive numerical schemes for the singularities and blow-up in non-linear parabolic and dispersive equations with improved precision. Similarity techniques have been also generalized to fractional and nonlocal PDEs but where standard scaling arguments cannot be applied directly due to memory effects and long-range interactions. In recent studies there have been developed generalised self-similarity frameworks also for fractional diffusion and nonlinear integro-differential equations, showing that suitably defined similarity variables retain the capability of capturing intermediate asymptotics and spreading rates [71, 72]. These results are particularly important for anomalous transport, viscoelasticity, and complex media. Despite their generality, the similarity solutions and dimensional analysis are not without some drawbacks. The existence of similarity reductions for many nonlinear PDEs is also incomplete: a large number do not possess any exact scaling invariance, or they only admit such reductions at particular ranges of parameters. In addition, similarity solutions can be inadequate to describe transient dynamics and multiscale interactions that characterize short-time behavior. Consequently, similarity methods are at their best when they are placed in the context of a more general analytical–computational approach serving as semiclassical anchoring points rather than full solutions. It is concluded that similarity solutions and dimensional analysis are still indispensable methods in the field of nonlinear PDEs. The last decades have seen significant improvement of their mathematical foundation, their generalization to the realm of fractional and nonlocal systems as well as tighter connection with numerical analysis. These methods yield crucial information regarding scaling laws, asymptotic behavior and singularity formation for instance which complement the suite of advanced analytical methods we reviewed and set the ground for the numerical and algorithmic frameworks.

## V. NUMERICAL METHODS AND ALGORITHMIC FRAMEWORKS

Although analytical and semi-analytical methods offer an essential structural understanding of nonlinear differential equations, the vast majority of non-linear ordinary and partial differential equations that appear in real-world problems are not analytically solvable, or they can be solved only through a uniformly valid closed form solution. Numerical methods are thus the sole feasible method to approximate solutions, investigate dynamics depending on parameters, and verify theoretical results. In the nonlinear case, on the other hand, we face serious difficulties that do not arise in linear problems: stiffness and loss of regularity, multiscale coupling and work also presents nonuniqueness and ill-posedness. Over the recent years, from 2020 to 2025, there has been remarkable development in numerical methods for nonlinear differential equations motivated by new algorithmic concepts, high-performance computing and structurepreserving discretisation. Recent studies have placed more emphasis not only on the approximation capacity and computational efficiency but also qualitative fidelity: the numerical methods are able to conserve some important invariants, stability properties and asymptotic behavior of the original continuous system. This transition is motivated in part by increased awareness that naive discretization of nonlinear equations may lead to spurious solutions, artificial damping or growth, or incorrect long-time behavior. One of the hallmarks of contemporary numerical algorithms is their strong ties with nonlinear theory. These results are then used to guide discretization and, conversely, numerical experiments give access to regimes beyond the reach of rigorous analysis. Since 2020, there has been significant advancement in the construction of adaptive high-order methods with structure preservation for nonlinear ODEs and PDEs including variational discretizations, nonlinear finite element formulations and spectral-based solvers [73,74]. Another important development is the growing incorporation of numerical techniques with iterative nonlinear solvers and continuation methods. Newtons-type methods, quasilinearization and fixed-point iterations remain integral tools in iterative treatment of the nonlinear algebraic systems resulting from discretization. More recent developments of the method include strategies enhancing robustness, such as globalization techniques or layered calibrations in order to overcome convergence issues in highly nonlinear regimes [75].

In addition, numerical methods have broadened to be applicable to new classes of problems like fractional-order systems, nonlocal operators and multiphysics couplings. Such extensions also demand new discretization techniques and fast solvers which can work with dense operators and memory effects without the computational burden becoming impractical [76]. At the same time, the development of parallel computing and GPUs acceleration has permitted large scale simulations of nonlinear PDEs which previously were not possessed computationally. Note that numerical techniques have now become a conceptual link between classical analysis and Datadriven approaches. High-fidelity numerical solvers produce training data for machine learning-based models, and the wisdom of numerical stability and convergence theory guide the architectures of physics-informed neural networks as well as hybrid solvers. Accordingly, numerical analysis is the bedrock for nonlinear differential equation studies and serves as the touchstone for not just theoretical advancements but also AI-related methods. In the next few sections, we briefly discuss three of these pillars, namely, nonlinear finite element methods, spectral and pseudospectral methods, and iterative nolinear solvers that rely on Newton (and quasi-linearization) ideas. As a whole, the latter two methods characterize the present landscape of algorithms for nonlinear differential addressing which underlie the computational paradigm.

### 5.1 Nonlinear Finite Element Analysis (FEA)

The nonlinear FEA is a tool to calculate the local law that accounts for material nonlinearity and geometry initial imperfections Article ID ARTIC030-S Version 2012/12/04 Page 2 of 13 so that it reproduces the experimental behaviour in terms Documentation granted with purchase. Nonlinear finite element analysis (FEA) provides one of the most general and mathematically-rigorous procedures for solving systems of nonlinear PDEs posed on complex geometries and inhomogeneous regions. Its popularity is due to the capability of combining a rigorous variational formulation with a flexible spatial discretisation, which makes it particularly suitable for nonlinear problems in solid mechanics, fluid dynamics, electromagnetism and coupled multiphysics systems. Unlike its linear counterpart the nonlinear FEA also needs to consider material, geometry and boundary conditions in their more complex (nonlinear) form which is much more involved for the human analyst as well as for the computer portion in dealing with computation. In the theoretical basis, nonlinear FEA can be traced back to weak or variational formulations of the nonlinear PDEs that are often justified via energy principles or balance laws. Recent developments (2020-2025) have concentrated on the exact functional-analytic context of these formulations, to comply with corresponding in/ and stability results from non-linear PDE theory. In particular, monotone operator theory and variational inequalities have been playing a growing role in justifying finite element approximations of nonlinear elliptic and parabolic problems including p-Laplacian-type operators as well as non linear diffusion problems [77–78]. These advances have enhanced the mathematical rigor of nonlinear FEA past heuristic discretization. Advances have been made especially in the nonlinear material/constitutive modeling. For solid and structural mechanics, finite element formulations currently include hyperelasticity, viscoplasticity, damage, phase-field fracture models etc., which provide strong nonlinearities at the level of elements. Newer work has concentrated on the development of stable but robust linearization schemes (like Newton–Rapson and its variations) which provide for quadratic convergence under large deformation material softening that guarantee this numerical stability [79]. The capabilities of NL-FEA to trace complex solution paths through bifurcation and post-buckling regimes have been further improved by adaptive load-stepping and arc-length continuation techniques.

In the field of nonlinear PDEs modelling fluid flow and transport, finite element methods have been generalized for convection-dominated as well as turbulence-induced nonlinearities. Stabilized formulations including streamline upwind/Petrov–Galerkin (SUPG) and variational multiscale methods have been developed to maintain accuracy while achieving stability [80]. The year 2020 witnessed a rising interest in structure-preserving discretizations that adhere to energy dissipation, incompressibility constraints and conservation laws of nonlinear flow equations. Another important breakthrough is adaptive finite element methods. When the wave type is nonlinear, localized structures (like boundary layers, shocks, or singularities) require resolution by adaptively spatially variable meshes. New developments have made available such a-priori estimation techniques for nonlinear PDE, making solution-based mesh refinement possible [81]. In particular, these techniques work well for nonlinear elliptic and parabolic problems for which the regularity of the solution changes as time progresses. Computational efficiency is still a significant issue in the nonlinear FEA. The systems of coupled nonlinear algebraic equations resulting from discretization on the grid are generally large scale and ill-conditioned, which imposes challenging demands on efficient solvers and preconditioners. Through domain decomposition, multigrid techniques, and parallel algorithms, a significant progress on large-scale capacity of the nonlinear FE simulation can be achieved on modern HPC systems [82]. Such a realgorithmization is crucial for large-scale 3D simulations and real-time or multi-query applications. Even in this state of maturity, the nonlinear FEA still poses some open problems. In many highly nonlinear or near-singular cases, the convergence of solution algorithms based on Newton-like methods cannot be assured and the cost for multiple linearization could also prohibitively high. Furthermore, sharp error estimates for strongly nonlinear or non-smooth problems are a challenging issue still under investigation. These limitations have motivated continuous efforts to couple finite element methods with reduced-order modeling and data-driven methodologies, as reviewed in later sections. CRs Nonlinear finite element analysis is, therefore, still a mainstay of numerical techniques for nonlinear differential equations. Recent theoretical and algorithmic developments have greatly improved its robustness, accuracy and scalability. Through a close coordination of variational theory, adaptive discretization and effective nonlinear solvers, the FEA is a robust flexible platform that serves as an underlying foundation for many modern computational nonlinear sciences.

### 5.2 Spectral and Pseudo-Spectral Methods

Spectral and pseudo-spectral methods comprise a set of high order numerical method with global approximation properties, as well as high accuracy even in the case of solutions that are smooth to moderate degree for non-linear differential equation. Spectral methods, in contrast to local discretizations methods such as finite differences or finite elements decompose the solution as a series expansion in global basis functions typically sines/cosines or orthogonal polynomials leading to exponential or "spectral" order of convergence provided that the exact solution is smooth enough. This feature makes them especially well-suited for nonlinear ordinary and partial differential equations in which one needs to resolve (with high accuracy) fine-scale structures. The theoretical underpinning of the spectral methods is represented by the expansion of the solution in orthogonal basis functions (e.g., Fourier series for problems on infinite domains or Chebyshev-Legendre polynomials for finite ones). In the case of nonlinear PDEs, the equations of motion are expanded onto such basis functions leading to systems of nonlinear ODEs for the expansion coefficients. Significant progress has been made for stability and convergence analysis of such formulations, namely for nonlinear evolution equations with quadratic and cubic nonlinearity [83,84], over the time from 2020 to 2025. Pseudo-spectral methods, aka collocation spectral methods, solve one of the main computational difficulties in classical spectral methods: getting fast evaluation of nonlinear terms. Rather than computing these nonlinearities in coefficient space, pseudo-spectral methods can be used to compute the products pointwise in physical space followed and then transforming the result back into spectral space (e.g., via fast transforms like FFT). This approach greatly diminishes the computational burden and retains high-order accuracy. Recent research has given a rigorous error analysis for pseudo-spectral approximations of nonlinear PDE's, shedding light on the role played by aliasing errors and validating de-aliasing techniques such as the 2/3 rule [85]. One of the significant applications for spectral and pseudo-spectral methods is nonlinear fluid dynamics, including solutions to the incompressible Navier–Stokes equations, nonlinear wave equations, and turbulence models. Energy-stable and structure-preserving spectral schemes with conservation laws and dissipation mechanisms that are compatible with the continuous equations have been recently reconsidered from 2020 [86]. Such methods are essential for long-time integration since they can accumulate even small numerical errors, which means they will result in qualitative dynamics. The nonlinearity of the governing equations complicates matters further in terms of stability and time integration. Spectral spatial discretization is frequently coupled to implicit–explicit (IMEX) or exponential time-stepping methods for an efficient treatment of stiff nonlinear terms. Recently pseudo-spectral spatial discretization combined with high order time integrators that are unconditionally stable and also accurate was developed [87]. These hybrid methods are effective8 for nonlinear dispersive equations and reaction–diffusion systems with multiscale temporal dynamics.

Another significant development is the generalization of spectral methods to complex shaped-geometries or nonperiodic boundary conditions, treated through coordinate transforms. While classic spectral methods are a natural choice for simple domains, domain decomposition, spectral element and mapped spectral methods have been developed in recent years that allow the direct use of high-order accuracy even for geometrically complex problems [88]. In particular, the so-called spectral element methods combine both a geometric flexibility typical of finite elements and a convergence property typical of spectral expansions, and have recently found some interest for nonlinear PDEs in CFD or wave propagation. Despite the flexibility and general applicability of spectral and pseudo-spectral methods, they are not generally suitable across a wide range of problems. Based on the global choice of basis functions, they are sensitive to solution singularities such as jumps or sharp layers and phenomena like Gibbs oscillations that may hinder convergence. Nonlinear problems, with shocks, interfaces or low-regularity solutions might therefore need to be hybridized with filtering and/or adaptive strategies or considered with alternative discretization. In addition, the large algebraic structures arising from spectral discretizations may lead to canonical problems in the presence of weak scalability (very-large scale) computations. To sum up, spectral and pseudo-spectral techniques still stand as very reliable numerical methods for the solution of non-linear partial differential equations on smooth domains. The stability, efficiency and applicability of these methods have significantly been improved in recent years by both theoretical and algorithmic developments. These methods are unmatched in their accuracy when used judiciously and in concert with proper time-stepping and stabilization techniques, but equally important as benchmarks to finite element solvers and emerging datadriven approaches that are increasingly entering the field.

### 5.3 Newton–Raphson and Quasilinearization Techniques

Newton–Raphson and quasilinearization methods are the computational engine of nonlinear solvers, all implemented with almost all the discretization schemes of nonlinear ordinary or partial differential equations. After the discretization (finite elements, finite differences or spectral methods) of a nonlinear differential equation, we are left with the problem to solve a nonlinear algebraic system as fast and stable as possible. The iterative techniques of linearization offer a systematic way to do this, in which nonlinear problems are converted into a sequence of subproblems that either are linear or have the benefit of being approximately solved by sequences that converge to the solution in an appropriate sense.

For solving nonlinear systems, the classical Newton–Raphson method is still considered as the best due to its quadratic convergence near a solution. For nonlinear ordinary and partial differential equations, Newton's method is usually used on the discretized form of the problem in residual form, involving iteration to compute Jacobian operators. The most recent developments (2020–2025) have concentrated on the robustness of Newton-type solids for strongly nonlinear and ill-conditioned systems, in particular those coming from non-linear PDEs with sharp gradients or nearly-singular behavior [89]. Globalization techniques in the form of line search, trust-region methods, and dampening have also been included as part of the solution algorithm to improve convergence from bad initial estimates. There has been substantial progress in Jacobian computation and estimation. Computing and storing exact Jacobians can be expensive, particularly for high-dimensional discretizations of nonlinear PDEs. Hence, quasi-Newton methods and Jacobian-free Newton–Krylov (JFNK) techniques have received considerable attention. In these methods, the action of a Jacobian is approximated by either finite differences or Krylov subspace projections and results in much smaller memory footprint with minimal loss of convergence speed [90]. From 2020, much better options became available in terms of preconditioners that are ESN-forced applied to highly nonlinear operators for PDEs which give a significant increase of the scalability of JFNK solvers on parallel machines. As compared with the linearization theory, quasilinearization method is another very successful treatment on nonlinear boundary-value problems and nonlinear eigenvalue problems. In quasilinearization, the nonlinearity is linearized about the current iterate in a monotone fashion such that convergence to an iterate with improved optimality is ensured under fairly general assumptions. Recent works reintroduced the principles of quasilinearization, and established convergence rates and error estimates for large classes of nonlinear ODEs and PDEs [91]. These findings have further established quasilinearization as a formally well-posed and numerically viable alternative to classical Newton methods in some regimes. One of the most interesting features in the last years has been to combine Newton (and Q-Newton) solvers with continuation and bifurcation approaches. Generic features such as multiple solutions branches, turning points, bifurcations, or simply nonlinearity make these problems impossible to be handled by a simple iteration. Arc-length continuation and pseudo-arclength methods together with Newton solvers provide a systematic way of traversing solution manifolds and studying parameter-dependent nonlinear phenomena [92]. These methods are of great importance in non-linear mechanics, pattern formation and stability study.

In the case of a time-dependent formulation problems become more complex. Implicit time discretizations of nonlinear evolution equations are generally accompanied by the solution of non-linear systems at each time level. This rarely gives an optimal solution, but the inexact Newton methods are considered recently which solve linear subproblems approximately and come with a compromise between accuracy of the solutions and amount of computation [93]. This will be beneficial especially for stiff nonlinear PDEs, where fully converged Newton iterations at each time step are often not needed or too costly. In spite of their importance, however, Newton–Raphson and quasilinearization techniques are not always guaranteed to work. Their success hinges, among other things, on the accuracy of the initial guesses, how well-conditioned the Jacobian is and whether non-smooth nonlinearities exist. For very chaotic systems which may be discontinuous for the solution, this convergence is not always possible. These limitations have stimulated the development of hybrid approaches where Newton-type solvers are combined with adaptive discretization, continuation methods and, more recently, data-driven strategies that exploit reduced order or machine learning models as initialization. To conclude, the Newton-Raphson and quasi linearization methods are still essential elements of numerical packages for nonlinear ODEs. New contributions in both theoretical and algorithmic sides of those methods have been shown to help addressing their robustness, scalability, and integration with modern discretization schemes. They should be further pursued for addressing ever more complicated nonlinear models faced in the modern scientific computing, and serve as natural stepping stone towards AI-guided solvers.

## VI. THE ERA OF SCIENTIFIC COMPUTING AND AI

The past few years have witnessed a paradigm shift in the computation of nonlinear ordinary and partial differential equations, driven by the implementation of classical scientific computing and contemporary artificial intelligence, respectively. Although numerical analysis has long afforded powerful methods of approximation for nonlinear systems, the recent developments in machine learning (in particular deep learning) have opened up fundamentally novel approaches to solution representation, model discovery and uncertainty quantification. This move does not replace previous approaches; it complements them via embedding physical structure (e.g., mass and diffusion control) data adaptability, high-dimensional approximation power within the computational pipeline. The idea of physics-aware computation has gathered momentum from 2020 on. Rather than black-box operators for differential equations, new AI-inspired solvers encode the governing equations, boundary conditions and conservation laws into learning structures. This combination overcomes the primary downside of full data-driven models: they are neither extrapolatively reliable nor physically interpretable. At the same time, it questions some fundamental concepts of classical numerical analysis such as consistency, stability and convergence. Reconceptualization of solution spaces is one hallmark of the present age. In standard numerical methods, solutions are discretized on a mesh or expressed on bases expansions. Artificial intelligence-based techniques, instead consider solutions as parametrized functions which, usually are neural network that are trained to satisfy differential constraints exactly or in a weak sense. This viewpoint provides a nice connection between the continuous problem and its discrete one, and is mostly used for efficient computational resolution in practice, especially in high-dimensional spaces where mesh-based discretization becomes impractical [94]. Newer findings have focused on hybridization as opposed to the supersession of one species by another. The rise of constraints ForPhysics-informed neural networks (PINNs), operator-learning structures, and sparse regression techniques are increasingly married to finite element, spectral and Newton-type solvers. Learning-based methods help to generate data for training (preconditioning) or initial guess and multiplies solver instances, but numerical solvers provide preconditionings from which the learning can be initialized; AI models offer more flexibility in providing surrogates as well as low complex response though taking a relatively long time in comparing with numerically body-force methods where high and medium flow-field can acceptably run [95,96]. Synergy of this kind mirrors a more general rethinking: AI as not an alternative to mathematical modeling, but as its continuation. However, the integration of AI in solving nonlinear differential equations brings deep theoretical questions to consideration. In contrast to classical numerical solvers, the convergence, stability and error control properties of most learning-based methods are typically not underpinned by a well-founded theory. The lack of robustness with respect to hyperparameters, training bias and data sparsity can also lead to approximated solution that satisfies the governing equations but violates qualitative properties like conservation or monotonicity. Coping with these challenges has reemerged as a major research theme since 2020, which also led to intensified communication between applied mathematicians, numerical analysts and machine learning researchers [97].

A further essential aspect of this period is computational scalability. AI-based solvers take advantage of new computational hardware architectures GPUs, TPUs, and distributed systems to solve large-scale nonlinear problems. This feature has featured a real-time inference, parametric and inverse problems solving for non-linear PDEs that were too challenging to be solved using conventional means [98]. But these improvements are typically achieved with high training complexity and energy consumption, which give rise to the issues of efficiency and reproducibility. The age of scientific computation and AI is a paradigm enhancement, as opposed to a shift. Classical methods have survived to this day because they are rigorous, reliable, and interpretable, whereas AI-based methods offer the remarkable flexibility and scalability. It is crucial to understand how these approaches complement and restrict each other, in order for the future of research on nonlinear differential equations. In the following subsections, we consider three primary branches of this emerging area: (1) physics-informed neural networks, (2) symbolic regression and sparse identification, and (3) unresolved issues in convergence and generalization.

### 6.1 Physics-Informed Neural Networks (PINNs)

Physics-Informed Neural Networks (PINNs) have recently gained widespread popularity as one of the most effective AI-based approaches for solving nonlinear ODEs and PDEs, especially in regimes where traditional discretization becomes infeasible or data are scarce. The main concept of PINNs is to incorporate the governing differential equations, boundary conditions and initial conditions couple with the loss function expressions in a neural network model in order to limit the learning process through physical laws rather than just using data only. Since 2020, PINNs have developed from prototype models into a fast-developing field that has advanced considerably both theoretically and algorithmically, as well as with the diversity of application.

The PINN approach's mathematical description can be given as the approximation of a nonlinear differential equation by a slightly-parameterized neural network whose solution outputs are differentiable with automatic differentiation. The nonlinear differential operator residual is pointwise computed at collocation points in the domain, and penalised in the loss functional along with boundary and initial condition mismatches. This results in the solution of nonlinear ordinary and partial differential equations being cast into a high-dimensional parameter-space-constrained minimization problem. The flexibility of neural networks has allowed us to approximate complicated nonlinear solution manifolds and avoiding explicit meshing, so that the method is particularly favorable for high-dimensional PDEs and inverse problems [99]. One major development since 2020 was the extension of PINNs to strongly nonlinear and multi-physics systems. The early behavior of the methods was sometimes stiff or struggled with competing loss terms, especially for nonlinear PDEs with sharp gradients or multi-scale features. Adaptive loss balancing, domain decomposition-based solution approaches and curriculum learning have been recently proposed to address these issues [100, 101]. These advances have led to better convergence properties and made it possible for PINNs to succesfully solve nonlinear fluid flow, nonlinear elasticity, and reaction–diffusion systems.

Another significant development is the development of PINNs for inverse and parameter identification formulations. By introducing unknown coefficients, source terms, or parameters as additional variables to be trained on data, PINNs provide a natural framework for solving PDEs and estimating model parameters from measurements at the same time. Recently, several studies have shown the ef- fective- ness of PINNs for solving nonlinear inverse problems arising in fluid dynamics, geophysics and biological systems where exist - ing optimization-based methods are computationally expen- sive or ill-conceptualized [102]. These findings clearly demonstrate the prospect of PINNs as data assimilation and model calibration tools. Although promising, PINNs are known to possess challenges stemming from optimization and approximation theory. The loss landscape of nonlinear differential operators is known to be highly nonconvex tending slow or suboptimal convergence. Recent theoretical advances have renewed interpretation of PINNs through the lens of numerical analysis, with connections between training dynamics, spectral bias and conditioning of the underlying differential operator unveiled [103]. These analyses have helped to explain why PINNs are successful for certain families of nonlinear PDEs but not others, especially stiff or convection-dominated problems.

An important direction during the last five years has been hybridization of PINNs with classical numerical techniques. This includes coupling of PINNs with finite element or spectral solvers to obtain coarse initializations, residual correction or building a surrogate for parametric studies. These coupled PINN–numerical methods are motivated by the desire to combine the robustness and accuracy of conventional discretizations with the flexibility and scalability enabled by neural networks [104]. This trend is in line with a more general understanding that PINNs are most effective as part of established numerical pipelines rather than as stand-alone tools. From a theoretical point of view, many basic questions related to convergence, error estimation and generalization are largely open. Unlike traditional numerical methods, there are no systematic a priori or a posteriori error bound for PINNs and their generalization beyond training collocation points is problem-wise. There have been some recent efforts to derive probabilistic error estimates as well as consistency results under simplifying assumptions, but no comprehensive theory exists yet that would be comparable to the finite element or spectrum error analysis [105]. To sum up, PDE applications of Physics Informed Neural Networks are still an emerging and rapidly developing area to solve nonlinear differential equations. Their capacity to incorporate physical laws, data and flexible function approximation has paved the way for new methods of solving nonlinear ordinary (ODE) and partial differential equations (PDE), especially in high-dimensional and inverse problems. While their limitations highlight the enduring relevance of classical numerical analysis, these concerns also give further impetus to development of hybrid methods and rigorous foundations. These problems are generically tied to the symbolic regression and sparsity identification methods.

### 6.2 Symbolic Regression and Sparse Identification (SINDy)

They did so through symbolic regression, known as a class of sparse identification in the large (SINDy), to finite elements modeling which will be discussed shortly. Symbolic regression and sparse identification methods provide an alternative data-driven framework to Physics-informed neural networks, with the focus being on model discovery rather than solution approximation. Rather than learning a black-box functional approximation, such methods are focused on discovering explicit governing equations (often in the form of non-linear ODEs or PDEs) from data by estimating a sparse set of terms that capture most of the dynamics. Symbolic regression and the specifically the Sparse Identification of Nonlinear Dynamics (SINDy) framework have gained recent popularity as powerful technique to disentangle interpretable nonlinear models from physics, biology or engineering, with impressive applications since year 2020. The SINDy approach is based on the premise that many dynamical systems have spares representations in a suitable functional dictionary. Applied to time series or spatiotemporal data, the method generates a library of candidate nonlinear terms as polynomials, trigonometric functions, rational functions, and derivatives; it then searches for the sparsest combination with which one can approximate the given evolution law via equation (1). This is usually carried out by way of $\ell_1$-penalized regression or iterative thresholding-based methods. Recent theoretical studies have precisely defined the limits of sparserecovery, associating identifiability with data richness and noise levels, as well as with themutual coherence of the potential library [106,107]. In this latter case, it is worth mentioning SINDy has gone beyond its root since 2020 and thus is no longer used only for low-dimensional ODEs. One of the significant developments is PDE-SINDy, which allows for identifying nonlinear PDEs from spatiotemporal data by incorporating spatial derivatives to the candidate library. Applications to non-linear diffusivity, Burgers-type equations and reaction–diffusion systems have shown that even moderate noise does not preclude PDE recovery under sufficient resolution using PDE-SINDy [108]. These advances have made sparse identification a practical substitute for standard model generation in data-rich environments.

Symbolic regression via genetic programming and evolutionary computation has also seen a revival. In contrast to SINDy, which leverages a-priori knowledge to specify candidate libraries, symbolic regression searches the functional space much more globally, seeking closed-form expressions that optimally approximate data based on fitness function. More recently developed hybrid methods like sparse regularization or using non-linearity preserving norms are proposed, to ensure expressiveness and interpretability properties leading to explicit control for the overfitting problem (see e.g., [109]). There, such techniques have been used successfully for identification of reduced-order models and constitutive relationships in problems involving complex nonlinear behavior. One important problem that has been studied in the literature is robustness against noise and partial observability. Data collected from experiment or simulation of nonlinear systems are typically noisy due to measurement error, which may seriously impair derivative estimation and sparsity recovery. In 2020, various algorithms have been proposed such as total-variation regularization [45], weak-form formulations [106] and libraries that are reduced in sensitivity to noise based on the integral of these libraries [110]. These developments have significantly enhanced the applicability of symbolic identification techniques for real world datasets. A second key direction is the incorporation of physical constraints in sparse identification. During the regression procedure, conservation laws, symmetries and dimensional analysis can be explicitly inferred to reduce the hypothesis space and enhance identifiability. It has been demonstrated that the constraint-aware SINDy formulations were able to better recover physically meaningful nonlinear models compared to unconstrained counterparts, especially for Hamiltonian and dissipation systems [111]. This is indicative of a larger unification currently occurring in the field between symbolic regression and physics-informed learning.

Symbolic regression and SINDy methods, while interpretable and elegant battle against their own limits. Their performance is heavily limited by the selection of candidate libraries, data noise, and the sparsity assumption that is likely to be violated in highly complex systems or very strongly interacting datasets. In addition, found models typically have only local validity in that they approximate the dominant dynamics only over commitment set. It is ongoing work to extend the sparse-identification to high dimensional PDEs and chaotic systems. In conclusion, symbolic regression and sparse identification provide a real attractive and interpretable paradigm to identify nonlinear DEs directly from data. Their robustness, coverage and relation to physics have been significantly improved in recent methodological developments. Not meant as a substitute to classical analysis or numerical solvers, but these tools provide very useful ones for model discovery, reduced-order modeling and hypothesis generations that naturally completes PINNs and prepares the ground for a deeper discussion of convergence and generalization challenges in AI-based solvers.

### 6.3 Convergence and Generalization Challenges in AI Solvers

Although AI-based solvers for nonlinear ordinary and partial differential equations have been advancing rapidly, with increasingly broad applicability, their adoption in scientific computing is partly limited by open issues of convergence, stability and generalizability. In contrast to classical numerical solvers with well-established consistency, convergence and error theory based on fundamental-approximation most AI-based approaches such as physics-informed neural networks [42–44, 46] and symbolic-learning frameworks operate without available generic theoretical guarantees. These challenges have become a prominent research focus since 2020, especially as AI solvers are now being used for high-stakes science and engineering problems. A central challenge arises from AI solvers being optimization-based. In PINNs and similar models, specifying a solution to a nonlinear differential equation reduces to solving a nonconvex optimization problem in high dimensions. The corresponding loss landscape is determined collectively by the differential operator, boundary terms, and architecture of a network which often results in ill-conditioning, flat directions, and many local minima. More recently analytical studies have shown that Converging to the same training invariant does not necessarily imply convergence of the computed solution towards the true PDE solution, especially when dealing with stiff or convection dominated nonlinear problems [112]. This disconnects from numerical correctness and optimization success represents a significant divergence from classical solver theory.

Generalizing outside the training or collocation points is an equally difficult problem. Although AI solvers attained low residual errors at sampled points, this does not imply global accuracy or correct qualitative behaviour. Recent works from 2020 have brought our attention to spectral bias of neural networks, which describes that low-frequency components of the solution are learned more than the high-frequency or multiscale features that are relevant for nonlinear PDEs and often not fully resolved [4]. This bias can produce deceptively low training loss, while the hidden large errors in the region of sudden gradient change or narrow structures. Numerically speaking, AI solvers do not have an associated mesh-refinement concept, stability regions, or Courant-type conditions as the ones associated with classical discretizations. Efforts to re-interpret PINNs as mesh-free Galerkin or collocation approaches have exposed partial equivalences, but also basic distinctions in the error propagation and stability mechanisms between them [114]. In particular, the lack of systematic a priori error estimates make solver validation challenging and assessment of reliability without known reference solutions problematic. More recent research has started to redress these limitations by using hybrid and theoretically driven methods. Adaptive sampling techniques based on the philosophy of a posteriori error estimation have been found to place collocation points in those regions with non negligible residual or curvature value and hence improve the quality of approximation for nonlinear PDE's having localized behavior [115]. Furthermore, operator preconditioning, residual normalization and domain decomposition ideas have been put forth to overcome the stiffness and lack of convergence rate capabilities for multiscale nonlinear problems.

Induction across parameter spaces bring additional complexity. Finally, classical reduced-order models have projective guarantees, but AI-based parametric solvers typically interpolate between training regimes without control over the extrapolation error. Recent work showed that PINNs trained on a restricted range of parameters lead to catastrophic failures outside the training domain, even for the same governing laws [116]. This sensitivity also motivates the development of principled techniques for parameter embedding, multi-fidelity training and uncertainty quantification. Theoretical advances have also been achieved in the area of probabilistic error analysis. By regarding trained neural networks as random function approximators, scholars established statistical bounds on approximation errors under idealized assumptions about network width, depth and training distributions [117]. While weaker than total deterministic convergence, these are useful in understanding when AI solvers can be operationalised with confidence. To conclude, convergence and generalization challenges are still the main hurdles for routine applications of AIsolvers to nonlinear differential equations. Despite its empirical successes in a broad range of application areas, deep learning is still a developing field, and one must proceed with caution. Recent developments in machine learning have made hybrid approaches - that harness numerical analysis and AI flexibility simultaneously - more desirable as they are able to enjoy the best of both worlds.

## VII. APPLICATIONS IN APPLIED SCIENCES

The theoretical and methodological ideas described in previous sections are necessarily measured against their usefulness sufficiency for applications throughout the applied sciences, which rely heavily on nonlinear ordinary and partial differential equations to model phenomena. Indeed, in several systems of interest for physical applications the nonlinearity should not be understood as an unwanted effect: nonlinear processes can be responsible for the development of complex behavior (multiscale interactions, pattern formation or instabilities, emergent phenomena etc.) in many realistic contexts. Therefore, the progress made in analytical, numerical and AI-supported solution methods has a deep impact on applied research fields like fluid mechanics, biology or quantum theory. A significant change in the uses of nonlinear differential equations has occurred in the applied sciences since 2020. Instead of being limited to a single class of solution methods, increasingly in many areas researchers are taking advantage of methodological pluralism, i.e., a combination of analytical insight with high-fidelity numerical simulation and data-driven modeling working together under the same computational roof. The motivation of such bayesian and data assimilation techniques is the increased complexity in apply-particle models that often consist of several few nonlinear PDEs, uncertain parameters and incomplete observation. Accordingly, the success of a solution method is measured by more than just accuracy but likewise simplicity or interpretability, scalability and robustness to uncertainty.

In fluid dynamics, nonlinear PDEs like the Navier–Stokes equations continue to symbolise a great theoretical challenge as well as major practical significance. However, after more than a century of research, fundamental problems related to existence, regularity and turbulence remain open questions holding that open the way for further development of high-order numerical solvers and AI-based turbulence modeling. Recent years have seen an adoption of structure-preserving discretizations and physics-informed learning to tackle long-standing challenges in the context of high-Reynolds number flows [118]. Another, related hot spot of nonlinear DE-modeling is in the discipline of nonlinear optics and quantum mechanics. Equations, such as the nonlinear Schrödinger equation and the Gross–Pitaevskii equation, model wave propagation, soliton dynamics and quantum coherence. The development of spectral methods, soliton theory and PINNs has allowed for accurate simulation and control of nonlinear waves in regimes of interest to photonics and Bose–Einstein condensates [119,120]. Nonlinear reaction–diffusion systems and network ODE models are used in mathematical biology to study phenomena such as pattern formation, morphogenesis, neural dynamics or epidemic propagation. The COVID-19 pandemic has brought even more to the fore what these dynamical models can and cannot accomplish in generating real-time decisions. Hybrid models, such as the data assimilation and machine-learning framework for adaptive forecast and parameter inference under uncertainty [121] which combines themechanistic non-linear equations with data-assim-ilation (typically by tracing) have gained interest since 2020.SEIR-P/SEIRD-Hybrid. From these applications several themes have begun to emerge. The first reason is that nonlinearity has a magnifying effect on uncertainty, so the sensitivity analysis and stability determination are necessary. Second, the cost of computation and possible scalability problematics remain crucial limitations, especially in three-dimensional time-dependent PDEs. Third, interpretability of solutions, as either analytical expressions, numerical fields or learned representations is key both for scientific understanding and practical deployment. In what follows we analyze in detail three representative application areas: fluid dynamics, nonlinear optics and quantum mechanics, and mathematical biology. These examples demonstrate how progress in nonlinear solution technologies leads to real scientific advancements, and also point out remaining issues, forming the basis for future research.

### 7.1 Fluid Dynamics: The Navier–Stokes Challenge

Fluid dynamics is one of the most challenging and conceptually-rich application areas for nonlinear partial differential equations. Its foundation is the Navier–Stokes equations, a system of coupled non-linear PDEs that represent the conservation laws for mass and momentum in viscous fluids. Small in its mathematical representation, this set of equations encompasses an amazing variety of processes ranging from the purely laminar to completely developed turbulence; even then it continues to present deep analytical, numerical and meshing challenges. The Navier–Stokes problem reflects the interaction of nonlinearity, multiscale properties, and dependence on initial and boundary data that is driving much of modern nonlinear analysis.

From a rather fundamental viewpoint, Navier–Stokes equations are paradigm of challenging unsolved problems. Indeed, for 2d incompressible flows global existence and smoothness of solutions is known to hold true, whereas for the counterpart in 3d this question even up-to now remains unsolved and became very well-known as one of the Millennium prize problems. Recent developments starting from 2020 are dealing with conditional regularity results, improved blow-up results and weak–strong uniqueness theories describing when physically acceptable solutions are unique or stable [123,124]. These results, however, do not solve the global regularity question but they have sharpened our understanding of the interplay between nonlinearity and vortex stretching as a source for complex flow phenomena. Exact solutions to the Navier–Stokes equations are analytically uncommon, and usually limited to very special or constrained geometries. However, similarity solutions, invariant solutions and reduced models obtained by using symmetry analysis are still useful for understanding of boundary layers, jet flows and unsteady problems. In 2020 self-similar or ancient solutions have regained attention as probes to potential mechanisms for both the formation of singularities, and energy cascade dynamics in turbulent flows [125]. These analytical solutions are important benchmarks for numerical and AI (artificial intelligence)-based solvers.

It is also currently the main tool for investigation of non-linear fluid dynamics. Direct numerical simulation (DNS) of the Navier–Stokes equations captures all dynamically relevant scales but can be performed only at moderate Reynolds numbers. Highly nonlinear closure models representing LES and reduced-order models: Accurate and stable approximations are crucial. In the field of physics-based animation, structure-preserving discretizations and energy/time-stepping methods that respect incompressibility with attention to the vorticity dynamics are popular for they can lead to long time stability and physical fidelity [126], given recent advances also on those lines. High-order spectral and finite element methods have been in the vanguard of this development, especially when used with AMR (adaptive meshing refinement) and IMEX (implicit–explicit time integration).

We also note the recent growing interest in AI-enhanced turbulence modeling over the past five years. As potential replacements or complements to traditional closure models, physics-informed neural networks (PINNs), operator- learning frameworks and sparse regression have also been investigated. These methods seek to obtain efficient representation of unresolved scales taking the intrinsic non-linearity structure of the Navier–Stokes equations into account. While promising accuracies are achieved for canonical flows and limited geometries, generalization across flow regimes, including the accurate injection of physical constraints in the predictions, as well as interpretability of learned models remain significant challenges to be overcome [127–128]. Consequently, AI techniques are more and more being considered as complementary tools, but not yet substitutes, to legacy numerical frameworks.

On an applications level, there is more to the story behind the Navier–Stokes problem than just some mathematical curiosity. Forecasting the nonlinear response of fluids is key to the physics of aerodynamics, climate studies, energy systems and biomedical flows. In these applications, uncertainty quantification, parameter sensitivity, and real-time computation are as critical as raw accuracy. Since 2020, these and other hybrid modes of reducedlocal-modelling have proved successful in effecting practical computation by incorporating data assimilation and nonlinear solvers without losing the essential nonlinear dynamics [128].

All told, the field of fluid dynamics and especially the Navier–Stokes equations remain a yardstick and grist-mill for advances in non-linear solution methodologies. Despite advances in analysis, numerical simulation and AI-guided modelling, there are still fundamental challenges. The Navier–Stokes problem therefore continues to provide a major source of inspiration for the theory of nonlinear differential equations and their applications in intricate real-world systems.

### 7.2 Nonlinear Optics and Quantum Mechanics

Here, nonlinear optics and quantum mechanics stand out as a large class of application domains where nonlinear differential equations can be said to take a prominent role in modelling wave–matter interactions, coherence phenomena, and collective dynamics. Unlike the classical theory of linear waves, in this context nonlinear optical and quantum systems display the effect of self-focusing, modulational instability, soliton formation, quantum vortices in both focusing and defocusing cases and also nonlinear dispersion due to inherent nonlinearity contained in governing equations. In the last five years, improved analytical theory, numerical simulations and AI-guided computation have significantly enhanced their predictability and understanding.

At the heart of nonlinear optics lies the nonlinear Schrödinger equation (NLSE) and its generalizations, which govern complex wave envelope dynamics in dispersive and nonlinear media. Generalizations of the NLSE include higher-order dispersion, nonlocal nonlinearities, gain–loss effects and external potentials leading to strongly nonlinear PDEs whose solution properties are quite sensitive to parameters and initial conditions. More recently there has been analytical progress in the stability theory of solitons and breathers, such as rigorous determination of modulational instability thresholds and spectral stability properties in non-integrable regimes [130, 131]. These results have important practical relevance in the area of optical fiber communications, mode-locked lasers and photonic crystal devices.

In quantum mechanics, the above motivated nonlinear PDEs appear as mean-field or effective description of many-body systems. Bose–Einstein condensates (BECs) and superfluid systems are described dynamically by the nonlinear Schrödinger-type Gross–Pitaevskii equation (GPE). In the last year however, there has been substantial progress made in analysis of vortexdynamics and quantum turbulent phenomena for patterns and excited states within the GP equation framework, mainly employing spectral methods and structure. preserving techniques [132]. These works elucidate the role of nonlinearity, dispersion and external trapping potentials in macroscopic quantum phenomena.

From the numerical point of view, nonlinear optics and quantum systems are very demanding in terms of accuracy and long-time stability. To this end PSM keeps being the workhorse for simulating NLSE- and GPE-like equations because of their capability to resolve oscillatory solutions with small numerical dissipation. Recent progress has focused on symplectic and energy-preserving time integrators, which conserve physical invariants like mass and Hamiltonian structure; this is essential for physically accurate long-time simulations [133]. These developments are especially important in the study of slow nonlinear phenomena [3], e.g., soliton collisions and condensate equilibration.

Another related aspect is the increasing interest on non-Hermitian and parity–time (PT)-symmetric systems, which stemming from balanced gain and loss support new nonlinear dynamics [3,25]. Models in this context give rise to nonlinear PDEs of complex-valued type where the stability and the bifurcation behaviour are completely different from conservative settings. New classes of nonlinear phenomena have been found, such as PT phase transitions, exceptional points and non-linear mode selection in the context of PT-symmetric optical and quantum systems [134, 135].

AI-aided approaches have also commenced to penetrate different disciplines, such as nonlinear optics and quantum mechanics. Physics-informed neural networks and operator-learning strategies have been used for parameter identification, inverse scattering problems, and rapid construction of a surrogate model for nonlinear wave propagation based on NLSE and GPE models. These methods are still under investigation, but have shown promise for speeding up simulation and facilitating the RT control of nonlinear optical systems especially using reduced-order models [136]. However, for strongly nonlinear and highly oscillatory problems their accuracy is not well understood.

From an applications point of view, nonlinear optics is the fundamental science enabling technologies including high-speed optical communication, ultra-fast and mode-locked lasers in the femtosecond time domain and on down to terahertz rates, as well as for such techniques as super-resolution optical microscopy. The studies of quantum nonlinear models, in another hand, serve as theoretical guideline for the development of quantum sensors, atom interferometers and novel quantum information platforms. In the two areas, it is crucial to have an accurate solution of nonlinear differential equations not only for numerical prediction, but also to establish controls and optimizations.

To sum up, the above two fields – nonlinear optics and quantum mechanics are just only two typical examples that nonlinearity, expressed by a set of nonlinear differential equation(s), have brought on modern science and technology. The development of fast and reliable solvers for the analytic stability theory, high-order numerical simulation, and AI-assisted calculation also extend the scope of solvability and enhance physical understanding. However, the multiscale and sensitivity nature of these systems pose various difficulties to existing data-driven methods, which also stimulates the development of efficient and interpretable nonlinear solvers.

### 7.3 Mathematical Biology: Pattern Formation and Epidemic Modeling

Mathematical biology is a field where nonlinear ordinary and partial differential equations are of fundamental and frequently irreplaceable use for modeling feedback, saturation, multiscale action in live matter. In the area of model order reduction, NAR (max)-based methods are never effective if no good control can be exerted over cond\xa7 and would provide zero approximation accuracy without (22) whereas I-SIIF-based goal-oriented error estimation may behave well at least for some iterations even though it makes little sense here to focus only on approximations. Since 2020, developments in nonlinear solution methods have had a mayurimpact on these areas and it has become more influenced especially with respect to data availability, uncertainty and real time decision support.

The phenomena of pattern formation in the biological systems are classically described by nonlinear reaction–diffusion equations, which originate from the original pioneering works of Turing. These kinds of models describe how spatially uniform states become unstable due to diffusion-driven instabilities, producing stripes, spots and other complex morphologies. Recent analytic work has further developed the bifurcation and stability theory of reaction–diffusion systems, generalizing classic Turing analysis to inhomogeneous domains, cross-diffusion interactions and nonlocal (integral) reactions [137,138]. These are important developments that have identified the parameter regions in which biological patterns can robustly form and survive, even in the face of noise and spatial heterogeneities. On the computational level, simulating pattern-forming systems is demanding because they are stiff (stiffness), and multiscale coupled & sensitive to the initial condition. During the last decade, high-order finite element and spectral methods were more widely used for solving fine spatial features with a minimum of artificial numerical diffusion. For large-scale biological domains pattern selection and coarsening dynamics could be efficiently simulated using a nonlinear instability indicator-based adaptive mesh refinement techniques developed after 2020 [139]. These computational developments have made the comparison of theoretical prediction and experimental observation more closely in developmental biology and ecology.

Another important area of application for nonlinear differential equations is the modeling of epidemics. In the compartmental models (e.g., the susceptible–infected–recovered [SIR] and susceptible–exposed–infected–recovered [SEIR] structures), nonlinear ODE systems describe how transmission dynamics, recovery, and immunity are related. The COVID-19 pandemic triggered an intensive research activity in the field of dengue, with improved versions of nonlinear models that account for time-dependent transmission rates and spatial structure, network effects, behavioral feedback [140,141]. These models emphasized the nonlinear process that generates thresholds, multiple stable endemic equilibria and a sudden shift between them of the epidemic outcome.

Recent analytic research themes have included stability and bifurcation analysis of epidemic models, notably locating endemic equilibria and backward bifurcations caused by nonlinear incidence rates or partial immunity [142]. These phenomena have implications to public health, suggesting that the elimination of disease may require significantly higher levels of intervention than suggested by linearized models. Thus, sensitivity analysis and uncertainty quantification has emerged as cornerstones of epidemic modeling practice, when conceived from the perspective of non-linear dynamics.

Computational models and data assimilation are increasingly used in modern epidemiology. Since 2020, hybrid statedependent systems of nonlinear ODE / PDE hybrid models coupled with Bayesian inferencing and filters, and machine learning have been used to fit the mess from real-time data generation or short-term fore-cast outputs [143]. These techniques are dominated by strong nonlinear solvers and concerned with interpretability and uncertainty as well as accuracy in prediction.

The mathematical biology community, in turn, is starting to integrate AI-based approaches into its methodology with applications such as pattern recognition and parameter inference. Physics-informed neural networks and sparse identification methods have been used to directly learn reaction kinetics and transmission parameters from spatiotemporally resolved data. While being hopeful the above-mentioned methods encounter difficulties with respect to data quality, identifiability and generalization particularly in systems with dominant nonlinear feedback and incomplete observations [144]. As elsewhere, hybrid approaches that integrate AI into mechanistic non-linear models are proving to be the most interesting. To conclude, pattern formation and epidemic modeling are two examples to illustrate the vital role of nonlinear differential equations in mathematical biology. Recent progress in analytical theory, numerical simulation and data-driven integration has improved the realism and practical applicability of biological models but also uncovered basic constraints imposed by nonlinearity, uncertainty and data paucity. These applications reinforce the general spirit of this review – that developments in solving nonlinear differential equations are dependent on progress in applied sciences, driven by practical problems.

## VIII. COMPARATIVE EVALUATION AND FUTURE OUTLOOK

The previous sections have presented a wide variety of analytical, numerical, and AI-solvability methods for solving nonlinear ODEs and PDEs, revealing reminders of the remarkable strengths that such methods offer and limitations contributions. Nonlinear models: deeper, after all they are used by people as complex nonlinearmodels continue to increase in scrutiny and popularity in applications, comparison between such approaches becomes crucial not only in the perspective of prediction accuracy or computational efficiency but also for robustness, interpretation capability, scalability and solid foundation. This final section presents a summary and synthesis of the findings across methodologies and sketches new unthinkable paths which may well influence developments in nonlinear differential equation research.

One recurring theme here in recent years (2020–2025) is that there is no panacea class of methods. (A) Analytical methods give unparalleled structural information and qualitative understanding, but are often confined to idealized systems or model limits. When available, numerical methods are robust and general with computationally expensive cost to address high-dimensional or multiscale cases. AI-based solvers bring in new capabilities for model flexibility and data adaptivity, but don’t yet offer the same convergence guarantees and interpretability. The next step in the development of the field is to continue to develop such principles and to integrate various methods in a principled manner, rather than compete them.

### 8.1 Accuracy vs. Computational Complexity

The trade-off between accuracy and computational burden is arguably the most common one among nonlinear solution techniques. It is true that high-order discretizations based on spectral-type methods or nonlinear finite element formulations can give excellent accuracies for smooth solutions, but their computational complexity increases exponentially with problem’s dimension and resolution. Reduced-order and AI models on the other hand provide fast inference and scalability at well accuracy guarantee or long-time stability sacrifice. Recent comparative studies have also focused on the accuracy per computer cost rather than just accuracy. For example, structure-preserving numerics may have lower pointwise accuracy compared to naive discretizations but can perform (much) better in long-time simulations by preserving the invariants and stability properties [145]. Likewise, hybrid pin n –numerical methods have shown that even small neural nets can compete in accuracy in the presence of fast classical solvers and much smaller computational cost [146].

So, the method choice becomes more problem specific. For certified precision, as needed in applications like safety-critical engineering simulations classical numerical methods are still indispensable. On the other hand, in exploratory analysis or inverse problems or real-time decision support this can be the point, where reduced-order and AI-assisted solvers may have a decisive edge. Formulating method selection and combination in a systematic way is a challenging problem.

### 8.2 Open Problems and Unresolved Conjectures

Despite wide developments, the study of nonlinear differential equations is still inspired and complemented by basic open problems. On the theoretical side, open problems of global existence, regularity and uniqueness, most notably for the Navier–Stokes equations in three dimensions still exist. Progress in analysis has closed the hole, but a full resolution seems to necessitate genuinely new ideas connecting local analysis and global dynamics [147]. Computationally, many problems remain. It is still a challenge to accurately estimate the errors for strongly nonlinear and non-smooth problems, especially when in high dimensions. Though a posteriori error estimator already are quite well-developed for nonlinear finite elements method, the same level of maturity does not seem to have translated to spectral and AI based solvers which is a continued line of research. As far as machine learning based methods are concerned, the lack of general convergence theory and its vulnerability to training details (very %intensively influenced by initializations) become important obstacles%for its practical uses in critical applications [148]. A further open issue is multiscale and multiphysics coupling. Most complex real-world systems are characterized by non-linear couplings among distinct spatially and temporally separated subsystems through different species of differential equations. Here, one of the important tasks is to establish unified frameworks that are capable of addressing this complexity with acceptable computational efforts.

### 8.3 Closing Remarks

Nonlinear ordinary and partial differential equations is a field in transition. Decades-old classical analytical and numerical techniques have been complemented by new approaches based on AI that rapidly shake the conventional understanding of what is involved in computation and modeling. Instead of indicating the obsolescence of traditional methods, this fusion emphasizes their perennial wisdom: analytical intuition guides algorithms constructions, numerical rigor constitutes database technologies and AI enlarges the span of manageable problems. For the future, advancements in nonlinear solver methodologies will increasingly rely on inter-disciplinary contributions – applying mathematicians and numerical analysts along with domain scientists for solving problems, and those in machine learning for developing regularization techniques. Prominent directions of future research are the design of hybrid solvers with provable properties, creation of benchmarks for nonlinear problems along side with the development and analysis of theoretical tools that bridge optimization-based learning and classical consistency and stability. In summary, recent progress has opened up a rich new toolkit for nonlinear differential equations, but also uncovered the depth and richness of the remaining challenges. Through the combination of methodological integration and a strong adherence to theory, the discipline is poised to confront the more difficult nonlinear systems that characterize modern day science and engineering.

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