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A Study on Zweier Ideal Statistical Convergent λ – Sequence Spaces

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Abstract- In this paper, we have generalized Zweier I–Convergent λ-sequence spaces to Zweier Ideal Statistical Convergent λ-sequence spaces. The new classes of Zweier Ideal Statistical Convergent λ-sequence spaces $\mathbb{Z}^{IS}(\lambda)$ and $\mathbb{Z}_0^{IS}(\lambda)$ were introduced. Also, we have established some of the topological properties of these spaces.

Keywords: Ideal Statistical Convergence, Completeness, nowhere dense.

Mathematics Subject Classification: 46A45.

I.INTRODUCTION

The theory of sequence spaces has been an important area of research in functional analysis and summability theory. Classical sequence spaces such as l_∞, c, and c_0 play a significant role in understanding the behavior of sequences and their convergence properties.

One such generalization is statistical convergence, introduced by Henry Fast. Statistical convergence allows the study of sequences whose terms may deviate from the limit on a set of indices having density zero.

Another important extension is ideal convergence, developed by Pratulananda Kostyrko and his collaborators. Ideal convergence replaces the notion of negligible sets with an ideal of subsets of natural numbers, providing a more flexible framework that includes statistical convergence as a special case.

Matrix transformations also play a crucial role in the construction of new sequence spaces. Among them, the Zweier matrix generates the class of Zweier sequence spaces, which have been studied with respect to different types of convergence and summability methods. These spaces possess interesting algebraic and topological structures and have been investigated by several researchers.

Motivated by above defined, the concept of Zweier ideal statistical convergent λ sequence spaces is introduced in this paper. We investigate their fundamental properties and structural characteristics, including linearity, solidity, monotonicity, and closedness. Furthermore, several inclusion relations and related results are established, which contribute to the study of generalized sequence spaces in summability theory. In this paper ideal statistical convergent is denoted as IS-convergent.

II.PRELIMINARIES

Definition 2.1.1
I is said to be ideal if,
i) I is non-empty.
ii) if a, b ∈ I ,then a ∪ b ∈ I.
iii) if a ∈ I and b ≤ a, then b ∈ I.

Definition 2.1.2
Let X be a sequence space. A filter F in X as a subset satisfying;
i) F is non-empty.
ii) A ∈ F, and B contains A implies B ∈ F.
iii)A, B ∈ F implies A∩B ∈ F.
iv)A ∈ F and x ∈ X implies A+x ∈ F.

Definition 2.1.3
Let X be a sequence space. An ideal I ∈X is admissible if
i) I is a closed ideal.

ii) I is non- trivial (I≠{0} and I≠X).
iii) For every x ∈ I and y ∈ X with │y│≤│x│, we have y ∈ I
iv) For every x ∈ I there exists y ∈ X such that
a) $x \leq y$ and b) $y \in I$

Definition 2.1.4
Let X be a sequence space and I an ideal in X.
A sequence (x_n) in X is said to be I- Convergent to x if,
For every $\varepsilon > 0$,{n ϵ N: | $x_n - x$ | ≥ ε } ∈ I.

Definition 2.1.5
Let X be a sequence space and I an ideal in X.
A sequence (x_n) in X is said to be I-Cauchy if,
For every $\varepsilon > 0$,{n ϵ N: | x_m - x_n| ≥ ε } ∈ I ∀ m,n ≥ N.

Definition 2.1.6
Let X be a sequence space and I an ideal in X.
A sequence (x_n) in X is said to be I-Statistically Convergent to x if,
i) For every $\varepsilon > 0$,{n ∈ N: | $x_n - x$ | ≥ ε} ∈ I.
ii) $\lim_{n\to\infty} |\{k \in N: | x_n - x | \geq \varepsilon\}|/n = 0 \ \forall \varepsilon > 0$.

Definition 2.1.7
Let X be a sequence space and I an ideal in X.
A sequence (x_n) in X is said to be I-Statistically Cauchy if,
i) For every $\varepsilon > 0$,{n ∈ N: | $x_m - x_n$ | ≥ ε} ϵ I ∀ m,n ≥ N.
ii) $\lim_{n\to\infty} |\{k \in N: | x_n - x| \geq \varepsilon\}|/n = 0 \ \forall \varepsilon > 0$ and m, n ≥ N.

3.1 Zweier IS-Convergent λ-Sequence Space
In this section, by following several authors we define some new classes of Zweier IS – Convergent λ-Sequence Spaces as follows:
$\mathbb{Z}^{IS}(\lambda) = \{(x_k)\epsilon\omega: \{k\epsilon\mathbb{N}: \frac{1}{n}|\{k \leq n: |\mathcal{Z}^p\Lambda_n\ (x) - L| \geq \varepsilon\}| \geq \delta\} \ \epsilon\ I,$ for some L}
$\mathbb{Z}_0^{IS}(\lambda) = \{(x_k)\epsilon\omega: \{k\epsilon\mathbb{N}: \frac{1}{n}|\{k \leq n: |\mathcal{Z}^p\Lambda_n\ (x)| \geq \varepsilon\}| \geq \delta\} \ \epsilon\ I.$
Theorem: 3.1.1
The class of sequences $\mathbb{Z}^{IS}(\lambda)$, $\mathbb{Z}_0^{IS}(\lambda)$ these spaces are linear spaces.
Proof:
Let (x_k), (y_k) ∈ $\mathbb{Z}^{IS}(\lambda)$ and α, β be any scalars.
$IS - \frac{1}{n}|x'_k - L_1| = 0$ for some $L_1 \in c$.
$IS - \frac{1}{n}|y'_k - L_2| = 0$ for some $L_2 \in c$.
Then for given $\varepsilon > 0$, we have $A_1 = \left\{k \in N: \frac{1}{n}|x'_k - L_1| > \frac{\varepsilon}{2}\right\} > \frac{\delta}{2} \in I.$

$$A_2 = \left\{k \in N: \frac{1}{n}|y'_k - L_2| > \frac{\varepsilon}{2}\right\} > \frac{\delta}{2} \in I.$$

Consider $\frac{1}{n}|(\alpha x'_k + \beta y'_k) - (\alpha L_1 + \beta L_2)| \leq \frac{1}{n}|\alpha|(|x'_k - L_1|) + \frac{1}{n}|\beta|(|y'_k - L_2|)$

$$\leq \frac{1}{n}|x'_k - L_1| + \frac{1}{n}(|y'_k - L_2|) \ \geq \ \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \geq \varepsilon$$

Thus, $\{k \in N: \frac{1}{n}|(\alpha x'_k + \beta y'_k) - (\alpha L_1 + \beta L_2)| > \varepsilon\} \geq \delta$
This implies that $(\alpha x_k + \beta y_k) \in \ \mathbb{Z}^{IS}(\lambda)$
Hence $\mathbb{Z}^{IS}(\lambda)$ forms a linear space.
Similarly, the space $\mathbb{Z}_0^{IS}(\lambda)$ is also linear.

Theorem:3.1.2
The spaces $\mathbb{Z}^{IS}(\lambda), \mathbb{Z}_0^{IS}(\lambda)$ are complete normed linear space.
Proof:
Consider a Cauchy sequence $\{x^{(n)}\}$ in $\mathbb{Z}^{IS}(\lambda)$, where $x^{(n)} = (x_1^{(n)}, x_2^{(n)}, \ \ldots.)$.
We need to show that the sequence $\{x^{(n)}\}$ approaches a point $x \in \mathbb{Z}^{IS}(\lambda)$.
Since,$\{x^{(n)}\}$ satisfies the Cauchy condition, for every $\varepsilon > 0$, there is an $n \in \mathbb{N}$ holds

$\|x^{(n)} - x^{(m)}\| = \sup_{k \in \mathbb{N}} \{|x_k^{(n)} - x_k^{(m)}| : k \in \mathbb{N}\} < \varepsilon$ for all $n, m \in N$.

For each $k \in \mathbb{N}$, $|x_k^{(n)} - x_k^{(m)}| < \varepsilon$ for $n, m \in N$.

This shows that for each k, $\{x_k^{(n)}\}$ possesses a Cauchy property in $\mathbb{Z}$.

For some $x_k \in \mathbb{Z}$ such that $(x_k^{(n)}) \to x_k$ as $n \to \infty$

Now, define $x = (x_1, x_2, \dots)$.

We claim that $x \in \mathbb{Z}^{IS}(\lambda)$ and $x^{(n)} \to x$ in $\mathbb{Z}^{IS}(\lambda)$.

To show this, note that

$$\begin{aligned} \|x^{(n)} - x\| &= \sup_{k \in \mathbb{N}} \{|x_k^{(n)} - x_k| : k \in \mathbb{N}\} \\ &\leq \sup_{k \in \mathbb{N}} \{|x_k^{(n)} - x_k^{(m)}| : k \in \mathbb{N}\} + \sup_{k \in \mathbb{N}} \{|x_k^{(n)} - x_k| : k \in \mathbb{N}\} \\ &< \varepsilon + \sup_{k \in \mathbb{N}} \{|x_k^{(n)} - x_k| : k \in \mathbb{N}\} \text{ for all } n \geq N. \end{aligned}$$

Since $\varepsilon > 0$ was arbitrary, we conclude that $x^{(n)} \to x$ in $\mathbb{Z}^{IS}(\lambda)$.

Therefore $\mathbb{Z}^{IS}(\lambda)$ is complete.

Similarly, the space $\mathbb{Z}_0^{IS}(\lambda)$ is also complete.

Note: The norm on $\mathbb{Z}^{IS}(\lambda)$ and $\mathbb{Z}_0^{IS}(\lambda)$ can be defined as $||x|| = \sup \{|x_k|\}$.

Theorem:3.1.3

$\mathbb{Z}^{IS}(\lambda) \subset c$ and $\mathbb{Z}^{IS}(\lambda)$ is closed in c.

Proof:

Consider a sequence $\{x^{(n)}\}$ in $\mathbb{Z}^{IS}(\lambda)$ that converges to $x \in c$.

Then we need to show that the limit $x \in \mathbb{Z}^{IS}(\lambda)$.

Since $x^{(n)} \in \mathbb{Z}^{IS}(\lambda)$ for all n, we have $x^{(n)} = (x_1^{(n)}, x_2^{(n)}, \dots) \in \mathbb{Z}^{IS}(\lambda)$

here $x_k^{(n)} \to 0$ as $k \to \infty$ for all n. Since $(x^{(n)}) \to x$ in c, we have $(x_k^{(n)}) \to x_k$ as $n \to \infty$ for each k.

Fixing $k \in \mathbb{N}$, since $\left(x_k^{(n)}\right) \to 0$ as $k \to \infty$ for all n, we have

$(x_k) \to 0$ as $k \to \infty$

Thus, $x = (x_1, x_2, \dots) \in \mathbb{Z}^{IS}(\lambda)$. Hence, $\mathbb{Z}^{IS}(\lambda)$ is a closed subspace of c.

Note: A similar proof shows that $\mathbb{Z}_0^{IS}(\lambda)$ is a closed subspace of c.

Theorem:3.1.4

$\mathbb{Z}^{IS}(\lambda), \mathbb{Z}_0^{IS}(\lambda)$ are nowhere dense sets.

Proof:

Let $x = (x_1, x_2, \dots)$ be any element of $\mathbb{Z}^{IS}(\lambda)$.

To show that there exists an element $y \in c$ (the space of convergent sequences) such that

$||x - y|| = \sup \{|x_k - y_k| : k \in \mathbb{N}\} \geq 1$.

This will yield that x is not an interior point of $\mathbb{Z}^{IS}(\lambda)$.

Let $\varepsilon > 0$ be given. Since $x \in \mathbb{Z}^{IS}(\lambda)$, there is an $N \in \mathbb{N}$ which

$|x_k| < \varepsilon$ for each $k \geq N$.

Define $y = (y_1, y_2, \dots) \in c$ by

$y_k = x_k$ for $k < N$

$y_k = \varepsilon$ for $k \geq N$

Then,

$$\begin{aligned} ||x - y|| &= \sup \{|x_k - y_k| : k \in \mathbb{N}\} \\ &= \sup \{|x_k - \varepsilon| : k \geq \mathbb{N}\} \\ &\geq 1 \end{aligned}$$

Since $\varepsilon > 0$ arbitrary. Therefore, x is not an interior point of $\mathbb{Z}^{IS}(\lambda)$.

Then for every $x \in \mathbb{Z}^{IS}(\lambda)$, we conclude that $\mathbb{Z}^{IS}(\lambda)$ has no interior points.
Hence, $\mathbb{Z}^{IS}(\lambda)$ is nowhere dense set.
Similarly, $\mathbb{Z}_0^{IS}(\lambda)$ is nowhere dense set can be proved.

Theorem:3.1.5
The spaces $\mathbb{Z}^{IS}(\lambda), \mathbb{Z}_0^{IS}(\lambda)$ are not separable.
Proof:
Assume, if possible, that $\mathbb{Z}^{IS}(\lambda)$ is separable.
Then, there exists a countable dense subset $\{x^n\}$ of $\mathbb{Z}^{IS}(\lambda)$.
Let $x = (x_1, x_2, \dots)$ be any element of $\mathbb{Z}^{IS}(\lambda)$.
Since $\{x^n\}$ is dense in $\mathbb{Z}^{IS}(\lambda)$, there exists a subsequence $\{x^{n_k}\}$ of $\{x^n\}$ such that
$(x^{n_k}) \to x$ in $\mathbb{Z}^{IS}(\lambda)$ as $k \to \infty$
This implies that
$(x_k^{n_k}) \to x_k$ as $k \to \infty$ for each fixed $k \in \mathbb{N}$.
However, since $x \in \mathbb{Z}^{IS}(\lambda)$, we have
$(x_k) \to 0$ as $k \to \infty$
which is a contradiction. Since the sequence $\{x^{n_k}\}$ cannot converge to both x_k and 0 as $k \to \infty$.
Hence, $\mathbb{Z}^{IS}(\lambda)$ is not separable.
Similarly, the space $\mathbb{Z}_0^{IS}(\lambda)$ is not separable.

Theorem:3.1.6
The inclusions $\mathbb{Z}_0^{IS}(\lambda) \subset \mathbb{Z}^{IS}(\lambda) \subset \mathbb{Z}_\infty^{IS}(\lambda)$ these spaces are proper.
Proof:
Consider $(x_k) \in \mathbb{Z}^{IS}(\lambda)$.
Then there is an $L \in c$ which $I\text{-}\lim|x_k'\text{-} L| = 0$
$|x_k'| \leq \frac{1}{2}|x_k'\text{-} L| + \frac{1}{2}|L|$
Taking supremum we get $(x_k) \in \mathbb{Z}_\infty^{IS}(\lambda)$
The inclusions $\mathbb{Z}_0^{IS}(\lambda) \subset \mathbb{Z}^{IS}(\lambda)$ is obvious.

Theorem:3.1.7
The function $\hbar{:}m\mathbb{Z}^{IS}(\lambda) \to \mathbb{R}$ obeys Lipschitz condition and thus is uniformly continuous where $m\mathbb{Z}^{IS}(\lambda) = \mathbb{Z}^{IS}(\lambda) \cap \mathbb{Z}_\infty$.
Proof:
By definition, $\hbar$ is Lipschitz if, for a positive number k holds for all $x, y \in m\mathbb{Z}^{IS}(\lambda)$

$$|\ \hbar(x) - \ \hbar(y)| \leq k|x - y|$$

Let assume this condition holds for $\hbar$.
Given $\varepsilon > 0$, choose $\delta = \frac{\varepsilon}{k}$
Then for each $x, y \in m\mathbb{Z}^{IS}(\lambda)$ with $\|x - y\| < \delta|$

$$|\ \hbar(x) - \ \hbar(y)| \leq k\|x - y\|$$

$$< k\delta$$

$$= k\frac{\varepsilon}{k}$$

$$= \varepsilon$$

$$|\ \hbar(x) - \ \hbar(y)| < \varepsilon$$

This shows $\hbar$ is uniformly continuous.

Theorem:3.1.8
If $x, y \in m\mathbb{Z}^{IS}(\lambda)$, then

a) $xy \in m\mathbb{Z}^{IS}(\lambda)$
b) $\hbar(xy) = \hbar(x)\,\hbar(y)$

Proof:
Let $x, y \in m\mathbb{Z}^{IS}(\lambda)$

a) Let $x = (x_n)$ and $y = (y_n)$ be elements of $m\mathbb{Z}^{IS}(\lambda)$.

Then $xy = (x_n y_n)$ is also an element of $m\mathbb{Z}^{IS}(\lambda)$.
(Since, the product of sequences in $m\mathbb{Z}^{IS}(\lambda)$ remains in $m\mathbb{Z}^{IS}(\lambda)$ due to its algebraic structure)

b) For the function $\hbar: m\mathbb{Z}^{IS}(\lambda) \to \mathbb{R}$ we have $\hbar(xy) = \hbar((x_n y_n)) = (\hbar(x_n y_n))$

using the property of $\hbar$, we can write
$\hbar(x_n y_n) = \hbar(x_n)\,\hbar(y_n)$

$$\hbar(xy) = (\hbar(x_n)\,\hbar(y_n))$$
$$= \hbar(x)\,\hbar(y)$$

Theorem:3.1.9
If I is a non-maximal ideal together with I differ from I_f, then the space $\mathbb{Z}^{IS}(\lambda), \mathbb{Z}_0^{IS}(\lambda)$ are not symmetric.
Proof:
Suppose I is not maximal
Let $A \in I$ be infinite.
If $x'_k = \begin{cases} 1, & for\ k \in A \\ 0, & otherwise \end{cases}$
[Lemma: If $I \subset 2^N$ and M is included in N. Whenever $M \notin I$, then common elements of $M, N \notin I$]
It follows that by lemma, $x_k \in \mathbb{Z}_0^{IS}(\lambda) \subset \mathbb{Z}^{IS}(\lambda)$.
Take $K \subset N$ be holds $K \notin IS$, $N - K \notin IS$
Suppose $\varphi: K \to A$, $\omega: N - K \to N - A$ are related by a bijection then the map $\pi: N \to N$ defined by

$$\pi(k) = \begin{cases} \varphi(k), & for\ k \in K \\ \omega(k), & otherwise \end{cases}$$

represents a permutation defined on N, but $x_{\pi(k)} \notin \mathbb{Z}^{IS}(\lambda)$, $x_{\pi(k)} \notin \mathbb{Z}_0^{IS}(\lambda)$
Thus, $\mathbb{Z}^{IS}(\lambda), \mathbb{Z}_0^{IS}(\lambda)$ are not symmetric.

Theorem:3.1.10
$\mathbb{Z}^{IS}(\lambda)$ is linearly isomorphic to c^I and $\mathbb{Z}_0^{IS}(\lambda)$ is linearly isomorphic to c_0^I respectively,
i.e) $\mathbb{Z}^{IS}(\lambda) \cong c^I$, $\mathbb{Z}_0^{IS}(\lambda) \cong c_0^I$

Proof:
Define $\varphi: \mathbb{Z}^{IS}(\lambda) \to c^I$ and
$\omega: \mathbb{Z}_0^{IS}(\lambda) \to c_0^I$

i. φ and ω are linear:
$$\varphi(ax + by) = a\,\varphi(x) + b\,\varphi(y)$$
$$\omega\,(ax + by) = a\,\omega(x) + b\,\omega(y)$$
ii. φ and ω are bijective:
Since $\varphi(x) = \varphi(y) \Longrightarrow x = y$ and similarly for ω
For any $y \in c^I$, for some $x \in \mathbb{Z}^{IS}(\lambda)$ holds $\varphi(x) = y$
Similarly,
For any $y \in c_0^I$, for some $x \in \mathbb{Z}_0^{IS}(\lambda)$ holds $\omega(x) = y$.
iii. Isomorphism:
φ and ω are linear isomorphisms
$\mathbb{Z}^{IS}(\lambda) \cong c^I$ and $\mathbb{Z}_0^{IS}(\lambda) \cong c_0^I$.

Theorem:3.1.11

Let (x_n) and (y_n) be a real sequences. Suppose that $\mathbb{Z}^{IS}(\lambda)\lim_{n\to\infty} x_n = L_1$ and $\mathbb{Z}^{IS}(\lambda)\lim_{n\to\infty} y_n = L_2$ then $L_1 = L_2$.

ie) $\mathbb{Z}^{IS}(\lambda)$ of sequence is unique, if it exists.

Proof:

Suppose $L_1 \neq L_2$

Let $\varepsilon = \frac{|L_1 - L_2|}{3} > 0$

Now define the zweier means of the sequence (x_n) as

$$\mathcal{Z}_n = \frac{x_n + x_{n+1}}{2}$$

Assume that $\mathcal{Z}_n$ is $IS(\lambda)$ convergent to L_1

For each $\delta > 0$, the following set is in the ideal I.

$$A = \{k \in \mathbb{N}: \frac{1}{n}\{k \leq n: |\mathcal{Z}_k - L_1| \geq \varepsilon\} \geq \delta\} \in I$$

Similarly,

Define the zweier means of the sequence (y_n) as

$$\mathcal{Z}_n = \frac{y_n + y_{n+1}}{2}$$

$$B = \{k \in \mathbb{N}: \frac{1}{n}\{k \leq n: |\mathcal{Z}_k - L_2| \geq \varepsilon\} \geq \delta\} \in I$$

Consider,

$C = \{k \in \mathbb{N}: |\,\mathcal{Z}_k - L_1| < \varepsilon \text{ and } |\,\mathcal{Z}_k - L_2| < \varepsilon\}$

For $k \in \mathbb{C}$, apply the triangle inequality

$$|L_1 - L_2| \leq |L_1 - \mathcal{Z}_k| + |L_2 - \mathcal{Z}_k| < \varepsilon + \varepsilon = 2\varepsilon$$

But we choose $\varepsilon = \frac{|L_1 - L_2|}{3}$

So $|L_1 - L_2| < 2\frac{|L_1 - L_2|}{3} = \frac{2}{3}|L_1 - L_2|$

This is a contradiction

Since $|L_1 - L_2| < \frac{2}{3}|L_1 - L_2| \Longrightarrow 1 < \frac{2}{3}$

Hence $L_1 = L_2$

Definition:

For every $\varepsilon > 0$ and $\delta > 0$, for some index $N \in \mathbb{N}$ holds

$\{n \in \mathbb{N}: \frac{1}{\lambda_n}|\{k \in I_n: |\mathcal{Z}_k - \mathcal{Z}_N| \geq \varepsilon\}| \geq \delta\} \in I$

Where $\mathcal{Z}_k = \frac{x_k + x_{k+1}}{2}$ is the zweier mean of x.

Theorem:3.1.12 ($\mathbb{Z}^{IS}(\lambda)$ convergent sequence is Cauchy)

Let (x_n) be a real sequence, $\lambda = (\lambda_n)$ be a positive integers that is non-decreasing holds $\lambda_n \to \infty$. If $\mathbb{Z}^{IS}(\lambda)\lim_{n\to\infty} x_n = L$, then (x_n) is $\mathbb{Z}^{IS}(\lambda)$ is Cauchy.

Proof:

Let $\varepsilon > 0$ and $\delta > 0$ be arbitrary.

Since $\mathbb{Z}^{IS}(\lambda)\lim_{n\to\infty} y_n = L$,

This means that the zweier-transformed sequence $\mathcal{Z}_k = \frac{x_k + x_{k+1}}{2}$ satisfies.

$A = \{n \in \mathbb{N}: \frac{1}{\lambda_n}|\left\{k \in I_n: |\mathcal{Z}_k - L| \geq \frac{\varepsilon}{2}\right\}| \geq \frac{\delta}{2}\} \in I$

Because I is an ideal, $\mathbb{N}\backslash A \notin I$, so for large enough $n \notin A$, we have

$$\frac{1}{\lambda_n}\left|\left\{k \in I_n: |\mathcal{Z}_k - L| < \frac{\varepsilon}{2}\right\}\right| > 1 - \frac{\delta}{2}$$

Now choose such a large $N \notin A$. Then for $\mathcal{Z}_N$, we have

$|\mathcal{Z}_k - \mathcal{Z}_N| \le |\mathcal{Z}_k - L| + |\mathcal{Z}_N - L| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$

For must $k \in I_n$, since both $\mathcal{Z}_k$ and $\mathcal{Z}_N$ are close to L.

Hence, for sufficiently large n, the set $B = \{k \in I_n : |\mathcal{Z}_k - \mathcal{Z}_N| \ge \varepsilon\}$ has less than $\delta(\lambda_n)$.

ie) $\frac{1}{\lambda_n}|B| < \delta$

Therefore, $\{n \in \mathbb{N} : \frac{1}{\lambda_n}|\{k \in I_n : |\mathcal{Z}_k - \mathcal{Z}_N| \ge \varepsilon\}| \ge \delta\} \in I$ is a subset of A, and hence belongs to I.

Theorem:3.1.13

Let (x_n) be a bounded real sequence, $\lambda = (\lambda_n)$ be a positive integers that is non-decreasing holds $\lambda_n \to \infty$. If (x_n) is Cauchy then (x_n) is convergent in $\mathbb{Z}^{IS}(\lambda)$ to some real number L.

ie) $\mathbb{Z}^{IS}(\lambda) \lim_{n\to\infty} x_n = L$

Proof:

Let the zweier transform of (x_n) be $\mathcal{Z}_n = \frac{x_n + x_{n+1}}{2}$

Because (x_n) is bounded, $|x_n| \le M$, it follows that $|\mathcal{Z}_n| \le \frac{|x_n| + |x_{n+1}|}{2} \le M$

So (z_n) is also a bounded sequence.

Since (x_n) is $\mathbb{Z}^{IS}(\lambda)$ is Cauchy then for every $\varepsilon > 0$ and $\delta > 0$, there is an $N \in \mathbb{N}$ holds

$$\{n \in \mathbb{N} : \frac{1}{\lambda_n}|\{k \in I_n : |\mathcal{Z}_k - \mathcal{Z}_N| \ge \varepsilon\}| \ge \delta\} \in I$$

A bounded sequence that is $IS(\lambda)$-cauchy is also $IS(\lambda)$-convergent (this is a know result in IS convergence theory).

Hence, the zweier-transformed sequence (z_n) is $IS(\lambda)$-convergent.

ie) $\mathbb{Z}^{IS}(\lambda) \lim_{n\to\infty} x_n = \lim_{n\to\infty} z_n = L$ for some $L \in \mathbb{R}$.

Theorem:3.1.14

The space $\mathbb{Z}_0^{IS}(\lambda)$ is solid and monotone.

Proof:

Let $x \in \mathbb{Z}_0^{IS}(\lambda)$ and given $\varepsilon > 0$.

Put $A_x(\varepsilon) = \{n : |x_n| \ge \varepsilon\}$

$A_y(\varepsilon) = \{n : |y_n| \ge \varepsilon\}$

If $|y_n| \le |x_n|$ for every n, then whenever $|y_n| \ge \varepsilon$.

We must also have $|x_n| \ge \varepsilon$.

Hence $A_y(\varepsilon) \subseteq A_x(\varepsilon)$.

Any sequence z produced from x by replacing some co-ordinates by numbers of smaller modulus satisfies $|z_n| \le |x_n|$ for all n.

By solidity $z \in \mathbb{Z}_0^{IS}(\lambda)$.

In particular all truncations of x belong to the space.

Thus $\mathbb{Z}_0^{IS}(\lambda)$ is monotone.

Corollary:

The space $\mathbb{Z}^{IS}(\lambda)$ does not possess the properties of solidity and monotonicity.

Theorem:3.1.15

If a space $x = (x_k)$ λ-converges to L, then it is also $\mathbb{Z}^{IS}(\lambda)$-convergent to L.

Proof:

Assume (x_k) λ-converges to L.

Now, consider $A_\varepsilon = \{k \in \mathbb{N} : x_k - L| \ge \varepsilon\}$

From λ-convergence, no index $k \ge K$ belongs to .

Therefore, $A_\varepsilon \subseteq \{1,2, \dots\dots\dots, k-1\}$

Thus A_ε is finite.

Since I contains $\mathbb{N}$.

Thus $A_\varepsilon \in I$

The $\mathbb{Z}^{IS}(\lambda)$-density of a finite set is zero, because $\frac{1}{\lambda_n}|A_\varepsilon \cap \{n - \lambda_{n+1}, \dots\dots, n\}| \to 0$

Thus the ideal λ-density of A_ε is zero.

Since $\varepsilon > 0$, $A_\varepsilon = \{k \in \mathbb{N}: x_k - L| \geq \varepsilon\}$

Therefore, (x_k) $\mathbb{Z}^{IS}(\lambda)$-converges to L.

Thus every λ-convergent sequence is $\mathbb{Z}^{IS}(\lambda)$-convergent.

Theorem:3.1.16

There exists a sequence that is $\mathbb{Z}^{IS}(\lambda)$-convergent but not bounded.

Proof:

Let $A \in I$.

Now, $x_k = \begin{cases} k, & k \in A, \\ 0, & k \notin A \end{cases}$

Whenever, $k \in A$ and $x_k = k$

Since A contains arbitrarily large integers $\sup_{k \in A}|x_k| = \infty$

Thus the sequence (x_k) is not bounded.

Take any $\varepsilon > 0$, $A_\varepsilon = \{k \in \mathbb{N}: |x_k - 0| \geq \varepsilon\} = \{k \in \mathbb{N}: |x_k| \geq \varepsilon\} = A$

Thus $A_\varepsilon = A$

Hence the λ-ideal density of A_ε is zero.

Therefore, (x_k) $\mathbb{Z}^{IS}(\lambda)$-converges to zero.

Theorem:3.1.17

Let $x = (x_k)$ and $y = (y_k)$ be sequences such that $|x_k - x_N| \leq |y_k - y_N|$ for all k, N. If (y_k) is $\mathbb{Z}^{IS}(\lambda)$-Cauchy then (x_k) is also $\mathbb{Z}^{IS}(\lambda)$-Cauchy.

Proof:

Assume $y = (y_k)$ is $\mathbb{Z}^{IS}(\lambda)$-Cauchy.

Then for every $\varepsilon > 0$, $B_\varepsilon(N) = \{k: |y_k - y_N| \geq \varepsilon\} \in I$.

We have the inequality, $|x_k - x_N| \leq |y_k - y_N|$.

Thus, $|x_k - x_N| \geq \varepsilon$, $|y_k - y_N| \geq \varepsilon$.

Therefore, $A_\varepsilon(N) = \{k: |x_k - x_N| \geq \varepsilon\} \subseteq \{k: |y_k - y_N| \geq \varepsilon\} = B_\varepsilon(N)$.

So $A_\varepsilon(N) \subseteq B_\varepsilon(N)$

Since $B_\varepsilon(N) \in I$, the ideal property gives, $A_\varepsilon(N) \in I$

Thus the set of sequence x also has ideal λ-density zero.

For every $\varepsilon > 0$, we found an N such that $\{k: |x_k - x_N| \geq \varepsilon\} \in I$

Thus $x = (x_k)$ is also $\mathbb{Z}^{IS}(\lambda)$-Cauchy.

Theorem:3.1.18

The space $\mathbb{Z}^{IS}(\lambda)$ is a linear space. Moreover, $x \in \mathbb{Z}^{IS}(\lambda)$ with limit L if and only if $x - L \in \mathbb{Z}_0^{IS}(\lambda)$.

Proof:

By definition, a sequence $x = (x_k)$ belongs to $\mathbb{Z}^{IS}(\lambda)$.

$\{n \in \mathbb{N}: \frac{1}{\lambda_n}\{k \in \mathbb{N}: |x_k - L| \geq \varepsilon\} \geq \delta\} \in I$ for all $\varepsilon > 0$.

We know that $\mathbb{Z}^{IS}(\lambda)$ is a linear space.

To Prove: $x \in \mathbb{Z}^{IS}(\lambda)$ with limit L if and only if $x - L \in \mathbb{Z}_0^{IS}(\lambda)$.

If $x \longrightarrow L$, then $x - L \longrightarrow 0$

Let $z_n = x_n - L$. Then for any $\varepsilon > 0$ $\{n: |z_n - 0| \geq \varepsilon\} = \{n: |x_n - L| \geq \varepsilon\} \in I$

Therefore, $x - L \in \mathbb{Z}_0^{IS}(\lambda)$.

Conversely,

If $x - L \in \mathbb{Z}_0^{IS}(\lambda)$.

Let $z = x - L$. If $z \to 0$ in $\mathbb{Z}^{IS}(\lambda)$, then $x = z + L$

Hence $x \longrightarrow L$ in $\mathbb{Z}^{IS}(\lambda)$.

Theorem:3.1.19

For a sequence $x = (x_k)$, each of the following conditions implies the others.

i. x is $\mathbb{Z}^{IS}(\lambda)$-Convergent to L.
ii. There exists a set $M \in F(I)$ such that $\lim_{\substack{n\to\infty \\ n\in M}} \frac{1}{\lambda_n}\{k \in \mathbb{N}: |x_k - L| \geq \varepsilon\} = 0$.
iii. For every $\varepsilon > 0$, the set $A_\varepsilon = \{n \in \mathbb{N}: \frac{1}{\lambda_n}\{k \in \mathbb{N}: |x_k - L| \geq \varepsilon\} \geq \delta\}$.

Proof:

(i)⇒(ii)

Assume that $x \longrightarrow L$ in $\mathbb{Z}^{IS}(\lambda)$.

Then, $\{n \in \mathbb{N}: \frac{1}{\lambda_n}\{k \in \mathbb{N}: |x_k - L| \geq \varepsilon\} \geq \delta\} \in I$ for all $\varepsilon > 0$.

Choose $\delta = \frac{1}{m}, m = 1,2, \dots \dots \dots$

Define, $M = \bigcup_{m=1}^{\infty}\{\mathbb{N} \setminus \{n \in \mathbb{N}: \frac{1}{\lambda_n}\{k \in \mathbb{N}: |x_k - L| \geq \frac{1}{m}\} \geq \frac{1}{m}\}\}$

Each removed set belongs to I, hence their union belongs to I, so the complement $M \in F(I)$.

Thus the convergence holds on M, proving (ii).

(ii)⇒(iii)

Assume the convergence holds for some $M \in F(I)$.

Given, any $\varepsilon > 0$, define $A_\varepsilon = \{n \in \mathbb{N}: \frac{1}{\lambda_n}\{k \in \mathbb{N}: |x_k - L| \geq \varepsilon\} \geq \delta\}$.

Thus only finitely many elements of M can lie in A_ε.

But outside M the set belongs to the index I, because $\mathbb{N}\setminus M \in I$.

Hence $A_\varepsilon \subset (\mathbb{N}\setminus M) \cup$ (finite set) $\in I$.

(iii)⇒(i)

Fix any $\varepsilon > 0$ and any $\delta > 0$. Take $\varepsilon' = \min(\varepsilon, \delta)$

Then by assumption, $A_{\varepsilon'} \in I$, where $A_{\varepsilon'} = \{n \in \mathbb{N}: \frac{1}{\lambda_n}\{k \in \mathbb{N}: |x_k - L| \geq \varepsilon'\} \geq \varepsilon'\}$

Since $A_{\varepsilon'} \in I$, this is exactly the definition of $\mathbb{Z}^{IS}(\lambda)$-Convergence. Thus (i) holds.

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