Characterizations of $(gp)^*$- R_1 Spaces

R. Saranya[1], R. Vijayalakshmi[2]
[1]Assistant Professor, Sri Ramakrishna College of Arts & Science for Women – Coimbatore
[2]Assistant Professor, Shri Nehru Maha Vidyalaya College of Arts and Science – Coimbatore

Abstract—In this paper we introduce $(gp)^*$-R_1 spaces and we study some characterization of $(gp)^*$-R_1 spaces. We analyse the relation between $(gp)^*$- closed sets with already existing closed sets.

Index Terms—**$(gp)^*$- closed sets, $(gp)^*$- open sets, $(gp)^*$-closure, $(gp)^*$-R_1 spaces.**

I. INTRODUCTION

Levine [7] introduced generalized closed sets (briefly g-closed sets) in topological spaces and studied their basic properties. Mahi.H [12] introduced and studied gp -closed sets. Veerakumar [10] introduced g*-closed sets in topological spaces and studied their properties. Veerakumar [13] introduced and studied g*p -closed sets in topological spaces. The aim of this paper is to introduce a $(gp)^*$-R_1spaces and we investigate some characterization of $(gp)^*$-R_1spaces.

II. PRELIMINARIES

Definition 2.1 A subset A of a topological space (X,τ) is called

(i) a semi-open set if A ⊆ cl(int(A)) and a semi-closed set if int(cl(A)) ⊆A,
(ii) a preopen set if A ⊆ int(cl(A)) and a preclosed set if cl(int(A)) ⊆ A,
(iii) an α- open set if A ⊆ int(cl(int(A))) and an α -closed set if cl(int(cl(A))) ⊆ A,
(iv) a semi-preopen set if A⊆ cl(int(cl(A))) and a semi-preclosed set if int(cl(int(A))) ⊆A
(v) a regular open set if A=int(cl(A)) and a regular closed set if cl(int(A)) =A.

The semi-closure (resp.preclosure , semi-preclosure) of a subset A of a space (X,τ) is the intersection of all semi-closed (resp. preclosed, α-closed, semi-preclosed) sets that contain A and is denoted by scl(A) (resp.pcl(A), Acl(A), spcl(A)).

Definition 2.2 A subset A of a space (X,τ) is called

(i) a generalized closed (briefly g-closed) set [10] if cl(A)⊆U whenever A⊆U and U is open in (X,τ); the compliment of a g-closed set is called a g-open set,
(ii) a semi-generalized closed (briefly sg-closed) set [2] if scl(A)⊆U whenever A⊆U and U is semi-open in(X,τ); the compliment of sg-closed set is called a sg-open set,
(iii) a generalized semi-closed (briefly gs-closed) set if scl(A)⊆U whenever A⊆U and U is open in (X,τ)
(iv) an α-generalized closed (briefly αg-closed) set [3] if αcl(A)⊆U whenever A⊆U and U is α-open in (X,τ),
(v) a generalized α-closed (briefly gα-closed) set [3] if αcl(A)⊆U whenever A⊆U and U is α-open in (X,τ)
(vi) a g*- closed set [10] if cl(A)⊆U whenever A⊆U and U is g-open in (X,τ),
(vii) a g^{**}-closed set [8] if cl(A)⊆U whenever A⊆U and U is g*-open in (X,τ),
(viii) a generalized preclosed (briefly gp-closed) setif pcl(A)⊆U whenever A⊆U and U is open in (X,τ),
(ix) a generalized semi-preclosed (briefly gsp-closed) set [5] if spcl(A)⊆U whenever A⊆U and U is open in(X,τ)
(x) a generalized pre regular closed (briefly gpr-closed) set [6] if pcl(A)⊆U whenever A⊆U and U is regular open in (X,τ),
(xi) a $g^{\#}$-closed set [11] if cl(A)⊆U whenever A⊆U and U is αg-open in (X,τ),
(xii) a generalized α**-closed (briefly gα**-closed) set [3] if αcl(A)⊆int(cl(U)) whenever A⊆U and U is α-open in (X,τ),
(xiii) a g*s- closed set [9] if scl(A)⊆U whenever A⊆U and U is gs-open in (X,τ).
(xiv) A generalized pre closed set [12] if pcl (A)⊆U whenever A⊆U and U is open in (X,τ).

(xv) A g^*-preclosed set [13] if pcl(A)) ⊆U whenever A⊆U and U is g-open in (X,τ).

The compliment of the above-mentioned sets is called their respective open sets.

III.$(gp)^*$-R_1spaces

Definition 3.1. A topological space (X, τ) is said to be $(gp)^*$-R_1spaces if for x, y in X with $(gp)^*$-Cl({x}) ≠ $(gp)^*$-Cl({y}), there exist disjoint $(gp)^*$-open sets U and V such that $(gp)^*$ −Cl({x}) is a subset of U and $(gp)^*$-Cl({y}) is a subset of V.

Theorem 3.1: If (X, τ) is $(gp)^*$-R_1, then (X, τ) is $(gp)^*$-R_0. Proof: Let U be (gp) ∗-open and x ∈ U. If y ∉ U, then since x ∉ (gp) ∗-Cl({y}), (gp) ∗-Cl({x}) ≠ (gp) ∗-Cl({y}). Hence, there exists a (gp) ∗--open v_y such that (gp) ∗-Cl({y}) ⊂ v_y and x ∉ v_y, which implies y ∉ (gp) ∗-Cl({x}). Thus (gp) ∗-Cl({x}) ⊂ U. Therefore (X, τ) is (gp) ∗-R0.

Theorem 3.2: A topological space (X, τ) is (gp)* - R1 if and only if for x, y ∈ X, (gp)* - Ker({x}) ≠ (gp)* - Ker({y}), there exist disjoint (gp)* - open sets U and V such that (gp)* - Cl({x}) ⊂ U and (gp)* - Cl({y}) ⊂ V. Proof: It follows from Definition.

Theorem 3.3: If (X, τ) is (gp)* -T_2, then (X, τ) is (gp)* - R1. Proof. Since X is (gp)* -T_2, then X is (gp)* -R1. If x, y ∈ X such that (gp)* -Cl({x}) ≠ (gp)* -Cl({y}), then x ≠ y. There exists disjoint $(gp)^*$-open sets U and V such that x ∈ U and y ∈ V; hence (gp)*-Cl({x}) = {x} ⊂ U and (gp)*-Cl({y}) = {y} ⊂ V. Hence X is (gp)*-R1. Definition 3.2: A topological space (X, τ) is called (gp)*-T_2 if for any distinct pair of points x and y in X, there exist (gp)* - open sets U and V in X containing x and y, respectively, such that U ∩ V = ∅.

Theorem 3.4. For a topological space (X, τ), the following statements are equivalent : (1) (X, τ) is (gp)* -R1 ; (2) If x, y ∈ X such that (gp)* -Cl({x}) ≠ (gp)* -Cl({y}), then there exists (gp)* -closed sets F1 and F2such that x ∈ F1, y ∉ F1 , y ∈ F2 , x ∉ F2 and X = F1 ∪ F2 . Proof. (1) ⇒ (2): Let x, y ∈ X such that (gp)* -Cl({x}) ∉ (gp)* -Cl({y}), and hence x ≠ y. Therefore, there exists disjoint (gp)* -open sets U1 and U1such that x ∈ (gp)* -Cl({x}) ⊂ U1 and y ∈ (gp)* -Cl({y}) ⊂ U1. Then F1 = X – U2 and F2 = X − U1 are (gp)* -closed sets such that x ∈ F1, y ∉ F1, y ∈ F2, x ∉ F2 and X = F1 ∪ F2. (2) ⇒ (1): Suppose that x and y are distinct points of X, such that (gp)* -Cl({x}) ∉ (gp)* Cl({y}). Therefore, there exist (gp)* -closed sets F1 and F2 such that x ∈ F1, y ∉ F1, y ∈ F2, x ∉ F2 and X = F1 ∪ F2. Now, we set U1 = X – F2 and U2 = X − F1, then we obtain that x ∈ U1, y ∈ U2, U1 ∩ U2 = ∅ and U1, U2 are (gp)* -open. This shows that (X, τ) is (gp)* -T2. Therefore (X, τ) is μ^{**}- R1.

Theorem 3.5. A topological space (X, τ) is (gp)* -R1 if and only if for x, y ∈ X, Ker({x}) ≠ (gp)* -Ker({y}), there exist disjoint (gp)* - open sets U and V such that (gp)* - Cl({x}) ⊂ U and (gp)* -Cl({y}) ⊂ V. Proof. Proof follows from the definition. Definition 3.3: A point x of a topological space (X, τ) is a (gp)* -θ-accumulation point of a subset A ⊂ X, if for each (gp)* -open U of X containing x, (gp)*-Cl(U)∩A ≠ ∅. The set (gp)*-Cl(A) of all (gp)*-θ-accumulation points of A is called the (gp)*-θ-closure of A. The set A is said to be (gp)*-θ-closed if (gp)*-Clθ(A) = A. Complement of a (gp)*-θ-closed set is said to be (gp)*-θ-open. Lemma 3.1: For any subset A of a topological space (X, τ), (gp)*-Cl(A) ⊂ (gp)*-Clθ(A). Lemma 3.2: Let x and y are points in a topological space (X, τ). Then y ∈ (gp)*-Clθ({x}) if and only if x ∈ (gp)*-Clθ({y}).

Theorem 3.6. A topological space (X, τ) is (gp)*-R1 if and only if for each x ∈ X, (gp)*-Cl({x}) =(gp)*-Clθ({x}). Proof: Necessity: Assume that X is (gp)*-R1 and y ∈ (gp)*-Clθ({x}) −(gp)*-Cl({x}). Then there exists a (gp)*-open set U containing y such that (gp)*-Cl(U) ∩ {x} ≠ ∅ but U ∩ {x} = ∅. Thus (gp)*-Cl{y}) ⊂ U, (gp)*-Cl({x}) ∩ U = ∅. Hence (gp)*-Cl({x}) ≠ (gp)*-Cl({y}). Since X is (gp)*-R1, there exist disjoint (gp)*-open sets U1 and U2 such that (gp)*-Cl({x}) ⊂ U1 and (gp)*-Cl({y}) ⊂ U2. Therefore X − U1 is a (gp)*-closed (gp)*-neigbourhood at y which does not contain x.Thus y ∉ (gp)*-Clθ({x}). This is a contradiction. Sufficiency: Suppose that (gp)*-Cl({x}) = (gp)*-Clθ({x}) for each x ∈ X. We first prove that X is (gp)*-R0. Let x belong to the (gp)*-open set U and y ∉U. Since (gp)*-Clθ({y}) = (gp)*-Cl({y}) ⊂ X − U, we have x ∉ (gp) ∗ −Clθ({y}) and y ∉ (gp)*-Clθ({x}) = (gp)*-Cl({x}). It follows that (gp)*-Cl({x}) ⊂ U. Therefore (X, τ) is (gp)*-R0. Now, let a, b ∈ X with (gp)*-Cl({a}) ≠ (gp)*-Cl({b}). (X, τ) is (gp)*-T1 and b ∉ (gp)*-Clθ({a}) and hence there exists a (gp)*-open set U containing b such that a ∉ (gp)*-Cl(U). Therefore, we obtain b ∈ U, a ∈ X − (gp)*-Cl(U) and

U ∩ (X − (gp)*-Cl(U)) = ∅. This shows that (X, τ) is (gp)*-T2. It follows that (X, τ) is (gp)*-R1.

Theorem 3.7: For a topological space (X, τ) the following are equivalent: (1) (X, τ) is (gp)*-R1; (2) (X, τ) is (gp)*-symmetric. Proof: (1) ⇒ (2). If x∉ (gp)*-Cl({y}). Then there exist a (gp)*-open set U containing x such that y ∉ U. Hence y ∉ (gp)*-Cl(U). The converse is similarly shown. (2) ⇒ (1) Let U be a (gp)*-open set and x ∈ U. If y ∉ U, then x ∉ (gp)*-Cl({y}) and hence y ∉ (gp)*-Cl({x}) This implies that (gp)*-Cl({x}) ⊂ U.Hence (X, τ) is (gp)*-R1

Theorem 3.8:For a topological space (X, τ), the following statements are equivalent: (1) (X, τ) is a (gp)*-R1 space; (2) If x, y ∈ X, then y ∈ (gp)*-Cl({x}) if and only if every net in X (gp)*-converging toy (gp)*-converges to x. Proof: (1)→(2): Let x, y ∈ X such that y ∈ (gp)*-Cl({x}). Suppose that {Xα} α ∈ ∧ be a net in X such that {Xα} α ∈∧ (gp)*-converges to y. Since y ∈ (gp)*-Cl({x}), we have (gp)*-Cl({x}) = (gp)*-Cl({y}). Therefore x ∈ (gp)*-Cl({y}). This means that {Xα} α ∈∧ (gp)*-converges to x. Conversely, let x, y ∈ X such that every net in X (gp)*-converging toy (gp)*-converges to x. Then x ∈ (gp)*-Cl({y}). we have (gp)*-Cl({x}) = (gp)*-Cl({y}). Therefore y ∈ (gp)*-Cl({x}). (2) (1): Assume that x and y are any two points of X such that ge-Cl({x}) ∩ geCl({y})≠ ∅. Let z ∈ (gp)*-Cl({x}) ∩ (gp)*-Cl({y}). So, there exists a net {xα} α∈∧ in (gp)*-Cl({x}) such that {xα} α∈∧ (gp)*-converges to z. Since z ∈ (gp)*-Cl({y}), then {xα} α∈∧ (gp)*-converges to y. It follows that y ∈ (gp)*-Cl({x}). By the same token we obtain x ∈ (gp)*-Cl({y}). Therefore (gp)*-Cl({x}) = (gp)*-Cl({y}) and (X, τ) is (gp)*-R1.

IV. OTHERS PROPERTIES OF (gp)*-OPEN SETS

Definition 4.1: A subset A of a topological space X is called a (gp)*-D -set if there are two U, V ∈ (gp)*- D0 (X, τ) such that U ≠ X and A=U − V. One can observe that every (gp)*-open set U different from X is a (gp)*D -set if A = U and V = ∅.

Definition 4.2: A topological space (X, τ) is called: (i) (gp)*-D0 if for any distinct pair of points x and y of X there exists a (gp)*D -set of X containing x but not y or a (gp)*D -set of X containing y but not x. (ii) (gp)*D1 if for any distinct pair of points x and y of X there exists a (gp)*D -set of X containing x but not y and a (gp)*D -set of X containing y but not x. (iii) (gp)*D 2if for any distinct pair of points x and y of X there exists disjoint (gp)*D -sets G and E of X containing x and y, respectively. (iv) (gp)*-T0 if for any distinct pair of points in X, there is a (gp)*open set containing one of the points but not the other.

Remark 4.1: (i) If (X, τ) is (gp)*- Ti , then it is (gp)*- Ti −1, i = 1, 2. (ii) If (X, τ) is (gp)*- Ti , then (X, τ) is (gp)*- Di , i = 0, 1, 2. (iii) If (X, τ) is (gp)*- Di , then it is (gp)*- Di −1 , i = 1, 2.

Theorem 4.1: For a topological space (X, τ) the following statements are true: (1) (X, τ) is (gp)*- D0 if and only if it is (gp)*- T0. (2) (X, τ) is (gp)*- D1 if and only if it is (gp)*- D2. Proof:(1) We prove only the necessity condition since the sufficiency condition is stated in Remark 4.1(ii). Necessity. Let (X, τ) be (gp)*- D0. Then for each distinct pair x, y ∈ X, at least one of x, y, say x, belongs to a (gp)*- D -set G but y ∉ G. Let G = U1 \ U2 where U1 ≠ X and U1, U2 ∈ (gp)*-O (X, τ). Then x ∈ U1, and for y ∉ G we have two cases: (a) y ∉ U1; (b) y ∈ U1 and y ∈ U2. In case (a), U1 contains x but not y; In case (b), U2 contains y but not x. Hence X is (gp)*- T0. (2) Sufficiency. Remark 5.1(iii). Necessity. Let X be a (gp)*- D1 topological space. Then for each distinct pair x, y ∈ X, we have (gp)*D -sets G1, G1 such that x ∈ G1, y ∉ G1; y ∈ G2, x ∉ G2. Let G1= U1\ U2, G2 = U3\ U4. From x ∉ G2, we have either x ∉ U3or x ∈ U3 and x ∈ U4. Now we consider the following two cases separately (1) x ∉ U3. From y ∉ G1we have two subcases: (a) y ∉ U1. From x ∈ U1\ U2 we have x ∈ U1\ (U2∪ U3) and from y ∈ U3\ U4we have y ∈ U3\ (U1 ∪ U4). Therefore, (U1\ (U2 ∪ U3)) ∩ (U3\ (U1 ∪ U4) = ∅. (b) y ∈ U1and y ∈ U2. We have x ∈ U1\ U2, y ∈ U2. (U1\ U2) ∩ U2 = ∅. (2) x ∈ U3 and x ∈ U4. We have y ∈ U3\ U4, x ∈ U4. (U3\ U4) ∩ U4 = ∅. From the discussion above we know that the space X is (gp)*- D2.

Theorem 4.2: For a (gp)*-T0 topological space (X, τ) each pair of distinct points x, y of X, (gp)*Cl({x}) ≠ (gp)*Cl({y}). Proof:Let x, y be any two distinct points of X. Since, X is (gp)*- T0, there exists a (gp)*- open set G containing x or y, say x but not y. Then GC is a (gp)*- closed set which does not contain x but contains y. Since (gp)*- Cl({y}) is the smallest (gp)*- closed set containing y, Clb({y}) ⊂ GC, and so x ∉ (gp)*-

Cl({y}). Consequently (gp)*- Cl({x}) ≠ (gp)*- Cl({y}).

Theorem 4.3: A topological space X is (gp)*- T2 if and only if the intersection of all (gp)*- closed (gp)*- neighbourhood of each point of X is reduced to that point. Proof: Necessity: Let X be (gp)*- T2and x ∈ X. Then for each y ∈ X which is distinct from x, there exist (gp)*-open sets G and H such that x ∈ G, y ∈ H and G ∩ H = ∅. Since x ∈ G ⊂ HC, hence HC is a (gp)*- closed (gp)*- neighbourhood of x to which y does not belong. Consequently, the intersection of all μ^{**}-closed (gp)*-neighbourhood of x is reduced to {x}. Sufficiency: Let x, y ∈ X and x ≠ y. Then by hypothesis there exists a (gp)*- closed (gp)*- neighbourhood U of x such that y ∉ U. Now there is a (gp)*- open set G such that x ∈ G ⊂ U. Thus, G and GC are disjoint (gp)*- open sets containing x and y respectively. Hence X is (gp)*-T2. Definition 4.3: A function f: (X, τ) → (Y, σ) is (gp)*- irresolute if the inverse image of each (gp)*- open set is (gp)*- open.

Theorem 4.4: If f: (X, τ) → (Y, σ) is a (gp)*- irresolute surjective function and E is a (gp)*- D -set in Y, then the inverse image of E is a (gp)*D -set in X. Proof: Let E be a (gp)*D -set in Y. Then there are (gp)*-open sets U1 and U2in Y such that E = U1\ U2 and U1 ≠ Y. By the(gp)*- irresoluteness off, f-1 (U1) and f-1 (U2) are (gp)*- open in X. Since U1 ≠ Y, we have f-1 (U1) ≠ X. Hence f-1 (E) = f-1 (U1) \ f-1 (U2) is a (gp)*D -set. Theorem 4.5: If (Y, σ) is (gp)*- D1 and f: (X, τ) → (Y, σ) is g˜-irresolute and bijective, then (X, τ) is (gp)*- D1. Proof: Suppose that Y is a (gp)*- D1 space. Let x and y be any pair of distinct points in X. Since f is injective and Y is (gp)*- D1, there exist (gp)*D -sets GX and GY of Y containing f(x) and f(y) respectively, such that f(y) ∈/ GX and f(x) ∈/ GY, f-1 (GX) and f-1 (GY) are (gp)*D -sets in X containing x and y respectively. This implies that X is a (gp)*- D1space.

Theorem 4.6: A topological space (X, τ) is (gp)*- D1 if and only if for each pair of distinct points x, y ∈ X, there exists a g˜-irresolute surjective function f: (X, τ) → (Y, σ), where Y is a (gp)*- D1space such that f(x) and f(y) are distinct. Proof: Necessity. For every pair of distinct points of X, it suffices to take the identity function on X. Sufficiency. Let x and y be any pair of distinct points in X. By hypothesis, there exists a (gp)*-irresolute, surjective function f of a space X onto a (gp)*- D1 space Y such that f(x) ≠ f(y). Therefore, there exist disjoint (gp)*D -sets GX and GY in Y such that f(x) ∈ GX and f(y) ∈ GY. Since f is (gp)*- irresolute and surjective, f-1 (GX) and f-1 (GY) are disjoint (gp)*D -sets in X containing x and y, respectively. Hence X is (gp)*- D1space.

REFERENCES

[1] D. Andrijevic, "Semi-pre-open sets," *Mat. Vesnik*, vol. 38, no. 1, pp. 24–32, 1986.

[2] P. Bhattacharya and B. K. Lahiri, "Semi-generalized closed sets in topology," *Indian Journal of Mathematics*, vol. 29, no. 3, pp. 375–382, 1987.

[3] R. Devi, H. Maki, and K. Balachandran, "α-generalized closed maps and generalized α-closed maps," *Indian Journal of Pure and Applied Mathematics*, vol. 29, no. 1, pp. 37–49, 1998.

[4] R. Devi, C. Vanitha, and V. Kokilavani, "μ*-closed sets in topological spaces," *Bulletin of Pure and Applied Sciences*, vol. 26, no. 1, pp. 81–87, 2007.

[5] J. Dontchev, "On generalizing semi-pre open set," *Memoirs of the Faculty of Science, Kochi University, Series A, Mathematics*, vol. 16, pp. 35–48, 1995.

[6] Y. Gnanambal, "On generalized pre-regular closed sets in topological spaces," *Indian Journal of Pure and Applied Mathematics*, vol. 28, no. 3, pp. 351–360, 1997.

[7] N. Levin, "Semi-open sets and semi-continuity in topological spaces," *The American Mathematical Monthly*, vol. 70, pp. 36–41, 1963.

[8] P. M. H. Mary, V. Vijayan, and P. Selvarani, "g**-closed sets in topological spaces," *International Journal of Mathematical Archive (IJMA)*, vol. 3, no. 5, pp. 1–15, 2012.

[9] Pushpalatha and K. Anitha, "g*s-closed sets in topological spaces," *International Journal of Contemporary Mathematical Sciences*, vol. 6, no. 19, pp. 917–921, 2011.

[10] M. K. R. S. Veera Kumar, "Between closed sets and g-closed sets," *Memoirs of the Faculty of Science, Kochi University, Series A, Mathematics*, vol. 21, pp. 1–19, 2000.

[11] M. K. R. S. Veera Kumar, "g#-closed sets in topological spaces," *Memoirs of the Faculty of Science, Kochi University, Series A, Mathematics*, vol. 24, pp. 1–13, 2003.

[12] H. Maki, J. Umehara, and T. Noiri, "Every topological space is pre T1/2 space," *Memoirs

of the Faculty of Science, Kochi University, Series A, Mathematics*, vol. 17, pp. 33–42, 1996.
[13] M. K. R. S. Veera Kumar, “g*-preclosed sets,” *Acta Ciencia Indica (Mathematics)*, vol. 28, no. 1, pp. –, 2002.