A Nonlocal Symmetrization Flow on Closed Curves

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***Abstract*—We introduce a nonlocal evolution equation on smooth closed curves in R^3 based on pairing arc-length antipodal points. The induced flow is linear and generates a strongly continuous semigroup. We prove global existence, uniqueness, and exponential convergence to a symmetric configuration. An associated energy functional is shown to decay monotonically along the flow.**

MSC (2020): Primary 47D06; Secondary 35B40.

I. INTRODUCTION

Symmetry plays a central role across many areas of mathematics, often serving as a guiding principle in the study of geometric and analytic structures. A common theme is that complicated objects may be simplified by systematically removing asymmetries, either through averaging procedures or through dynamical processes.

In this paper, we introduce a simple nonlocal transformation on smooth closed curves in R^3 based on pairing arc-length antipodal points. Given a curve, each point is associated with the point located halfway along the curve, and the two are averaged. Iterating this idea leads naturally to a continuous-time evolution in which each point is drawn toward its diametric partner.

The resulting evolution equation defines a linear, nonlocal flow on the space of curves. Despite its simplicity, the flow exhibits a clear and instructive structure: it decomposes the curve into symmetric and antisymmetric components, exponentially suppressing the latter while preserving the former.

This behavior can be interpreted as a form of symmetrisation, analogous in spirit to classical procedures such as averaging or Fourier projection. However, unlike local geometric flows, the present evolution is inherently nonlocal, coupling distant points along the curve.

Our goal is to present this transformation and its associated flow in a self-contained and accessible manner. We establish existence and uniqueness of solutions, describe the underlying operator structure, and prove exponential convergence to a symmetric configuration. Several explicit examples illustrate the behavior of the flow and highlight its geometric interpretation.

The simplicity of the construction makes it a useful example of how ideas from functional analysis, symmetry, and geometry can combine to produce a structured and solvable dynamical system.

II. PRELIMINARIES

We identify S^1 with $R/\ell Z$. Let $X = L^2(S^1, R^3)$.

Definition 1. Define the shift operator $T: X \to X$ by $(Tf)(t) = f(t + \ell/2)$.

Lemma 1. The operator T is linear, bounded, and satisfies $T^2 = I$.

Proof. Linearity is immediate. Since translation preserves the L^2 norm, $\|Tf\| = \|f\|$, so T is bounded. Moreover,

$T^2f(t) = f(t + \ell) = f(t)$,

so $T^2 = I$.☐

III. DEFINITIONS OF THE FLOW

Definition 2 (Diametric midpoint flow). Let $f: S^1 \times [0, \infty) \to R^3$ satisfy

$$\frac{\partial f}{\partial s}(t,s) = \frac{1}{2}\big(f(t+\ell/2,s) - f(t,s)\big).$$

Define the operator

$$A := \frac{1}{2}(T - I).$$

Then the flow can be written as

$\partial_s f = Af$.

IV. EXISTENCE AND UNIQUENESS

Theorem 1. For any initial data $f_0 \in X$, there exists a unique global solution

$f(s) = e^{sA}f_0$ in X.

Proof. The operator A is bounded since T is bounded. Hence A generates a uniformly continuous semigroup e^{sA} on X. By standard semigroup theory for linear evolution equations in Banach spaces, there exists a unique global solution given by
$f(s) = e^{sA}f_0$.

V. SPECTRAL DECOMPOSITIONS

Definition 3. Define the symmetric and antisymmetric subspaces:
$X^+ = \{f \in X: Tf = f\}$, $X^- = \{f \in X: Tf = -f\}$.
Lemma 2. Every $f \in X$ admits a unique decomposition
$f = f^+ + f^-$,
where $f^+ \in X^+$ and $f^- \in X^-$.
Proof. Define

$$f^+ = \frac{1}{2}(f + Tf), \quad f^- = \frac{1}{2}(f - Tf).$$

Then $Tf^+ = f^+$ and $Tf^- = -f^-$. Uniqueness follows from linearity.☐

VI. LONG-TIME BEHAVIORS

Theorem 2 (Exponential convergence). Let f(s) solve the flow. Then
$f(s) = f^+ + e^{-s}f^-$,
and hence
$\|f(s) - f^+\|_{L}2 \leq e^{-s}\|f^-\|_{L}2$.
In particular, f(s) converges exponentially to the symmetric component f^+. Proof. Using the decomposition $f = f^+ + f^-$, we compute

$$Af^+ = \frac{1}{2}(T - I)f^+ = 0,$$

and

$$Af^- = \frac{1}{2}(T - I)f^- = \frac{1}{2}(-f^- - f^-) = -f^-.$$

Thus, the evolution equations decouple:
$\partial sf+ = 0$,
Hence $\partial sf- = -f-$.
$f+(s) = f+$,
The result follows.

VII. ENERGY DECAY

Definition 4. Define the energy functional $f-(s) = e-sf-$.

Z
E(f) = |f(t) − f (t + ℓ/2) |2 dt.
S1

Theorem 3. Along the flow, the energy satisfies

$$\frac{d}{ds}E(f(s)) = -8\int_{S^1} |f^-(t,s)|^2\, dt \leq 0.$$

Proof. Observe that $f(t) - f(t + \ell/2) = f(t) - Tf(t) = 2f^-(t)$.
Thus
Differentiating,
Using $\partial_s f^- = -f^-$,
Z E(f) = 4
S1

$$\frac{d}{}E = 8\int \langle f^-, \partial_s f^- \rangle dt.$$

$|f-(t)|^2dt$.
ds S1

$$\frac{d}{ds}E = -8\int_{S^1} |f^-|^2 dt \leq 0.$$

VIII. EXAMPLES

8.1 Example 1: Circle (complete collapse)

Let f(t) = (Rcost, Rsint,0), t ∈ [0,2π].
Then
f (t + π) = (−Rcost, −Rsint,0).
Hence,

$$\frac{\partial f}{\partial s}(t,s) = \frac{1}{2}(f(t+\pi,s) - f(t,s)) = (-R\cos t, -R\sin t, 0)$$

Thus, the evolution equation becomes
$\partial_s f = -f$.
Solving, we obtain $f(t,s) = e^{-s}f(t,0)$.
Therefore, the curve shrinks exponentially to the origin as s → ∞.

8.2 Example 2: Ellipse (anisotropic collapse)

Let f(t) = (acost, bsint,0).Then f(t + π) = (−acost, −bsint,0) = −f(t). Thus, the same computation as in Example 1 yields
$\partial_s f = -f$,and hence $f(t,s) = e^{-s}f(t,0)$. Therefore, any centrally symmetric curve collapses exponentially to a point under the flow.

8.3 Example 3: Non-symmetric curve

Consider f(t) = (cost + ϵcos2t, sint,0), ϵ ≠ 0.
Then f (t + π) = (−cost + ϵcos2t,−sint,0).
Compute the symmetric and antisymmetric parts:

$$f^+(t) = \frac{1}{2}(f(t) + f(t+\pi)) = (\epsilon\cos 2t, 0, 0),$$

$$f^-(t) = \frac{1}{2}(f(t) - f(t+\pi)) = (\cos t, \sin t, 0).$$

By the general solution of the flow,

$f(t,s) = f^{+}(t) + e^{-s}f^{-}(t)$.
Thus, $f(t,s) = (\epsilon\cos 2t + e^{-s}\cos t, e^{-s}\sin t, 0)$.
As $s \to \infty$, the antisymmetric component vanishes and $f(t,s) \to (\epsilon\cos 2t, 0, 0)$.
Hence the curve converges to a symmetric configuration rather than collapsing to a point.

IX. CONCEPTUAL INTERPRETATION

The diametric midpoint flow admits a natural interpretation in terms of symmetry and averaging. The operator T, defined by shifting a function by half the length of the curve, is an involution, and the transformation

$$\frac{1}{2}(I+T)$$

acts as a projection onto the subspace of functions invariant under this symmetry.
From this perspective, the flow

$$\partial_s f = \frac{1}{2}(T-I)f$$

may be viewed as continuously removing the antisymmetric component of the curve. The evolution preserves the symmetric part while exponentially damping deviations from symmetry.
This behavior is closely related to the decomposition of functions into even and odd components, familiar from Fourier analysis. In particular, the long-time limit of the flow corresponds to retaining only those components of the curve that are invariant under the antipodal symmetry.
The flow may therefore be regarded as a nonlocal symmetrisation process. Unlike classical symmetrization procedures, which are often defined geometrically or combinatorially, the present construction arises from a linear evolution equation and admits an explicit solution via operator methods.
This interpretation highlights the role of nonlocal interactions in shaping global structure. Each point on the curve evolves based on information from a distant point, yet the overall effect is to enforce a coherent symmetry across the entire curve.
Such examples illustrate how simple linear operators can generate meaningful geometric behavior, providing a bridge between functional analysis and geometric intuition.

X. CONCLUSIONS

The diametric midpoint flow defines a nonlocal linear evolution that removes antisymmetric components of a curve. The flow admits a complete spectral decomposition and converges exponentially to a symmetric configuration. This provides a simple yet structured example of a nonlocal symmetrisation flow.

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