A Comprehensive Review of Spectral Characterization and Structural Transformations in Pythagorean Fuzzy Graphs

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Abstract — **Pythagorean fuzzy graphs (PFGs) have emerged as an advanced extension of intuitionistic fuzzy graphs, offering greater flexibility in modeling uncertainty through the squared sum condition of membership and non-membership degrees. In recent years, significant attention has been devoted to the spectral characterization of PFGs, particularly in analysing adjacency and Laplacian matrices under Pythagorean fuzzy environments. Spectral properties such as eigen values, spectral radius, and energy play a crucial role in understanding structural behaviour and stability of uncertain networks. This review systematically examines the theoretical developments in spectral analysis of PFGs, highlighting recent results on eigenvalue bounds, spectral invariants, and matrix representations. Furthermore, the study explores various structural transformations, including graph operations, isomorphisms, homomorphisms, and complement structures within the Pythagorean fuzzy framework. Comparative insights between classical, intuitionistic, and Pythagorean fuzzy graph models are also discussed. Applications in decision-making, social network analysis, engineering systems, and uncertain data modeling are summarised to demonstrate practical relevance. Finally, existing research gaps and potential future directions are identified to encourage further advancements in spectral theory and structural analysis of Pythagorean fuzzy graphs.**

Index Terms — **Pythagorean fuzzy graph; Spectral characterization; Eigen values; Laplacian matrix; Structural transformations; Graph operations; Uncertainty modeling; Fuzzy set theory; Network analysis; Decision-making systems**.

I. INTRODUCTION

Graph theory plays a fundamental role in modeling relationships and interactions in complex systems such as communication networks, transportation systems, biological structures, and social networks. However, classical graph models are often insufficient when dealing with uncertainty and imprecision inherent in real-world problems. To address this limitation, fuzzy set theory introduced by Lotfi A. Zadeh has been widely applied to extend classical graph concepts into fuzzy graph models.

Among the various extensions, intuitionistic fuzzy graphs provided an improved framework by incorporating both membership and non-membership degrees. Nevertheless, certain practical scenarios require greater flexibility in representing uncertainty. This need led to the development of Pythagorean fuzzy sets, where the squared sum of membership and non-membership degrees is constrained to be less than or equal to one, allowing a broader representation space compared to intuitionistic fuzzy sets. Based on this concept, Pythagorean fuzzy graphs (PFGs) have emerged as a powerful tool for modeling uncertain and ambiguous network structures.

In recent years, increasing research attention has been directed towards the spectral characterization of Pythagorean fuzzy graphs. Spectral graph theory, which studies graph properties through eigen values and eigenvectors of associated matrices such as adjacency and Laplacian matrices, provides deep insights into structural behaviour, connectivity, stability, and robustness of networks. Extending spectral concepts to the Pythagorean fuzzy environment introduces new theoretical challenges and opportunities for analysis.

Simultaneously, structural transformations of Pythagorean fuzzy graphs—including operations such as union, join, complement, isomorphism, and homomorphism—have gained importance in understanding how graph properties evolve under modifications. These transformations are essential for analysing dynamic systems and complex decision-making models where network structures frequently change.

II. REVIEW OF LITERATURE

2.1. Naz et al. (2018) – Pythagorean Fuzzy Graphs: Some Results

Naz and co-authors formally introduced Pythagorean fuzzy graphs by extending fuzzy graph concepts into the Pythagorean domain. They defined vertices and edges with membership and non-membership degrees satisfying the squared sum condition. Fundamental concepts such as degree, strong PFGs, complete PFGs, and regular PFGs were explored. The authors established structural properties and provided illustrative examples. This study served as one of the earliest systematic treatments of PFG theory. It created a platform for analysing connectivity and graph operations under Pythagorean uncertainty. Subsequent research on spectral and structural aspects of PFGs builds upon these definitions.

2.2. Akram and Naz (2019) – Operations on Pythagorean Fuzzy Graphs

This study examined graph operations including union, join, complement, and Cartesian product in the Pythagorean fuzzy environment. The authors analysed how membership and non-membership functions transform under these operations. The structural preservation of graph properties under transformations was discussed. Conditions for isomorphism between PFGs were also presented. The work provided important insight into structural transformations, which are essential for dynamic network modeling. The algebraic characterization of graph operations in this study supports further spectral investigations. It remains a key reference for structural analysis in PFG research.

2.3 F. Smarandache (2019) – Pythagorean Fuzzy Sets: Extensions and Applications

Smarandache provided a formal mathematical foundation for Pythagorean fuzzy sets (PFS), extending intuitionistic fuzzy sets by introducing the squared sum constraint of membership and non-membership degrees. This extension significantly enlarges the domain of uncertainty representation. The study discussed algebraic operations, aggregation operators, and decision-making frameworks under Pythagorean environments. The flexibility of PFS allows more expressive modeling compared to traditional fuzzy and intuitionistic fuzzy models. This foundational work laid the theoretical basis for later developments in Pythagorean fuzzy graphs. It also highlighted the advantages of PFS in multi-criteria decision-making problems. The theoretical structure introduced in this work directly supports graph-based extensions such as Pythagorean fuzzy graphs (PFGs).

2.4. Akram et al. (2020) – Planar Pythagorean Fuzzy Graphs and Associated Properties

This research introduced the concept of planar Pythagorean fuzzy graphs and extended classical planarity conditions to uncertain graph models. The authors examined embeddings of PFGs in the plane without edge intersections. Necessary and sufficient conditions for planarity under Pythagorean constraints were derived. The study also analyzed structural invariants preserved under planar transformations. Applications in network modeling were briefly discussed. This work broadened structural graph theory into the Pythagorean fuzzy framework. It contributes significantly to transformation-based structural studies.

2.5. Akram and Dudek (2020) – Energy of Pythagorean Fuzzy Graphs with Applications

In this significant contribution, Muhammad Akram and Sumera Naz introduced the concept of energy for Pythagorean fuzzy graphs by extending classical graph energy theory into the Pythagorean fuzzy environment. The study defined the adjacency matrix of a Pythagorean fuzzy graph by incorporating membership and non-membership values satisfying the squared sum constraint. Based on this matrix representation, the authors developed the notion of graph energy as the sum of absolute eigen values. They also introduced Laplacian energy and analysed its structural implications. Several theoretical properties, bounds, and inequalities related to energy were established. The authors further demonstrated practical applications of PFG energy in decision-making and network evaluation problems. This work represents one of the earliest and most important steps towards spectral characterization of Pythagorean fuzzy graphs. It provides a strong mathematical foundation for subsequent research on eigen values, spectral radius, and structural stability in uncertain networks.

2.6. Akram et al. (2021) – Degree Analysis and Similarity Measures in PFG's under Threshold Conditions

This study focused on degree-based measures and similarity indices for Pythagorean fuzzy graphs under threshold conditions. The authors introduced new definitions of vertex degree that incorporate both membership and non-membership values. Structural comparisons between graphs were made using similarity measures. The results were applied to clustering and pattern recognition problems. This contribution strengthened analytical tools for structural assessment of PFGs. Although not directly spectral, the work supports matrix-based representations. It plays an important role in quantitative structural analysis.

2.7. Akram and Naz (2021) – Interval-Valued Pythagorean Fuzzy Graphs

The researchers extended PFGs to interval-valued Pythagorean fuzzy graphs, allowing uncertainty to be expressed in interval form. Definitions of operations, connectivity, and regularity were reformulated. The interval approach increased modeling flexibility in uncertain systems. Applications in decision support systems were provided. Structural properties under transformation were analysed. This extension broadens the theoretical depth of PFG research. It also provides potential scope for interval-based spectral investigations.

2.8. Liu and Chen (2022) – Homomorphisms in Pythagorean Fuzzy Graphs

Liu and Chen studied graph homomorphisms within the Pythagorean fuzzy environment. They defined mappings preserving structural and membership properties. Necessary and sufficient conditions for PFG homomorphism were derived. The work analysed structural invariants under such mappings. Applications in network reduction and simplification were discussed. This study is important for understanding transformation behaviour and provides foundational results for structural equivalence analysis.

2.9. Kumar and Gupta (2023) – Spectral Radius and Eigenvalue Bounds

This study investigated eigen value bounds and spectral radius of Pythagorean fuzzy adjacency matrices. The authors derived inequalities relating degree sequences to spectral parameters. Theoretical comparisons with classical spectral graph results were provided. Applications in network stability analysis were explored. The research highlighted the role of eigen values in assessing robustness. It significantly advances spectral characterization in PFGs. The findings encourage further matrix-theoretic exploration.

2.10. Zhang et al. (2023) – Laplacian Spectrum in Fuzzy Graph Models

Although focused on general fuzzy graphs, this study developed Laplacian spectral bounds that are extendable to PFGs. The authors analysed connectivity and algebraic multiplicity of eigen values. The relationship between Laplacian eigen values and graph invariants was explored. Their framework provides a methodological basis for Pythagorean extensions. The work bridges classical spectral theory and fuzzy graph environments. It is highly relevant for spectral characterization research.

2.11. Murugappan et al. (2023) – Neighbourly Edge Irregularity on Interval-Valued Pythagorean Neutrosophic Graphs

This study examines irregular properties of interval-valued Pythagorean neutrosophic graphs, including neighbourly edge irregular and highly irregular structures. By analysing conditions under which such irregularities occur, the work extends understanding of structural variability in fuzzy graph environments. This work complements spectral and degree-based analyses by highlighting structural irregularity as a significant theme in fuzzy graph research.

2.12. Energy of Fuzzy Graphs (2024) – Sujatha et al.

Although not exclusively PFGs, this study discusses graph energy concepts—including energy and Laplacian energy—applied to Pythagorean fuzzy graphs, extending classical fuzzy graph energy methodologies. It highlights computational approaches to eigen value computation and compares energy measures across fuzzy graph types. Additionally, applications in communications, AI, and environmental modeling are emphasised, making this work relevant for spectral analysis contexts.

2.13. Govindan et al. (2024) – Complement Properties of Pythagorean Co-Neutrosophic Graphs

This article explores complement and co-complement operations in Pythagorean co-neutrosophic graphs—a generalization where membership roles are reversed but the aggregated membership constraint remains intact. Structural properties and algebraic implications of these operations are examined, with applications discussed. The work contributes to understanding transformational behaviour in extended Pythagorean fuzzy environments and is relevant to structural transformation analyses.

2.14. Singh and Sharma (2024) – Structural Transformations in PFG's

This recent work examined advanced structural operations such as contraction, subdivision, and graph products in PFGs. The authors analysed how these transformations affect degree sequences and connectivity. Conditions for structural preservation were derived. Applications in communication networks were demonstrated. The study provides updated insights into transformation theory. It strengthens the structural component of PFG research.

2.15. Shahin& Mohammed Jabarulla (2025) – Degree Analysis and Similarity Measures in PFG's

This recent work develops a unified mathematical framework to analyse structural properties of Pythagorean fuzzy graphs (PFGs) under dual threshold conditions on membership and non-membership values. It introduces a novel reduced-degree concept to capture structural variations when vertices and edges are filtered by threshold values. The authors derive rigorous theorems characterizing how degrees behave under such structural simplifications, providing insight into monotonicity and boundedness. In addition, they introduced numerous similarity measures based on degree vectors—including cosine, Euclidean, Gaussian and Manhattan similarities that are useful for structural comparison of PFG's. This study also includes illustrative examples and tabular evaluations demonstrating the impact of threshold variation on similarity indices, making it a significant contribution to both theoretical structure and practical fuzzy graph comparison methodologies.

2.16. Ramesh Obbu et al. (2025) - Planarity of Intuitionistic Pythagorean Fuzzy Graphs

Although focusing on intuitionistic variants, this work extends the concept of planarity to Intuitionistic Pythagorean Fuzzy Graphs (IPFGs) and explores weak and strong planarity based on intersecting values of edges. It introduces novel definitions of IPFG planarity value and derives upper/lower bounds through several theorems. The authors investigated structural properties related to planarity and provided discussion on applications in decision-making contexts, which strengthens the theoretical foundation for structural transformation analysis. This work reflects the ongoing interest in Pythagorean-related graph planarity research.

2.17. Anitha Saravanakumar et al. (2025) – Energy and Laplacian Energy of Pythagorean Intuitionistic Fuzzy Graphs

This study expands spectral energy analysis into Pythagorean intuitionistic fuzzy graphs by defining new formulations for energy and Laplacian energy tailored to this class of uncertain graphs. The authors derived mathematical bounds and apply their framework to medical diagnosis networks, where associations between symptoms, tests and diseases are expressed as fuzzy connections. By utilizing energy and Laplacian energy as uncertainty quantifiers, this work demonstrates how spectral techniques can inform decision-making under imprecision. The work broadens the spectral analysis literature beyond classic PFGs by incorporating intuitionistic extensions.

2.18. Takaaki Fujita &Florentin Smarandache (2025) – Pythagorean, Fermatean, and Complex Turiyam Neutrosophic Graphs

This publication introduces and compares Pythagorean, Fermatean, and Complex Turiyam Neutrosophic Graphs, offering a richer framework for modeling uncertainty beyond traditional fuzzy approaches by incorporating independent truth, indeterminacy and falsity memberships. While not exclusively spectral, the study discusses structural properties of these generalized graph forms and suggests potential extensions in graph theory. Its inclusion reflects the growing interest in neutrosophic generalizations of Pythagorean fuzzy structures.

2.19. L. Fernandez, et al. (2025) - Spectral properties and structural dynamics in fuzzy graphs: A comprehensive survey

This review paper offers a broad and systematic analysis of spectral graph theory as applied to various classes of fuzzy graphs, including general fuzzy graphs, intuitionistic fuzzy graphs, and their Pythagorean extensions. The authors begin by revisiting the theoretical underpinnings of fuzzy set-based graph models and describe how classical spectral concepts such as adjacency matrices, Laplacian matrices, eigen values, eigen vectors, and spectral radius have been adapted for uncertain networks. A significant portion of the survey is dedicated to comparing the spectral characteristics of different fuzzy graph types and discussed how structural transformations impact spectral behaviour. The work also explores applications in network clustering, synchronization analysis, pattern recognition, and decision support systems. Importantly, it identifies emerging research areas such as dynamic fuzzy graphs, interval-valued spectral models, and optimization techniques leveraging spectral parameters. While not exclusively focused on Pythagorean fuzzy graphs, this review offers valuable contextual insights, especially regarding the place of Pythagorean models within the broader spectral fuzzy graph landscape. It highlights current challenges in generalizing spectral methods for highly uncertain structures and suggests future directions, including probabilistic and hybrid approaches.

2.20. Chellamani& Ajay (2026) – Irregular Pythagorean Neutrosophic Graphs

This recent research formalizes irregularity concepts in Pythagorean neutrosophic fuzzy graphs by defining classes such as neighbourly irregular and highly irregular graphs. It provides theoretical results relating irregular structures to neighbourhood properties and demonstrates applications in pattern recognition and biological networks. While focused on neutrosophic variants, the work emphasizes structural classification and deepens the theoretical base for analyzing structural behaviour under complex uncertainty.

III. FUTURE RESEARCH DIRECTIONS

Although significant progress has been made in the development of spectral characterization and structural transformations of Pythagorean fuzzy graphs (PFGs), several important research directions remain open for further investigation. One major area requiring deeper exploration is the development of advanced spectral invariants tailored specifically to the Pythagorean fuzzy environment. While concepts such as graph energy and basic eigen value bounds have been introduced, more refined measures such as spectral entropy, spectral moments, and structural energy indices could provide richer insights into network behaviour under uncertainty. In addition, a comprehensive Laplacian spectral theory for PFGs is still lacking. Establishing rigorous relationships between Laplacian eigen values and graph connectivity, robustness, spanning structures, and algebraic multiplicity would significantly strengthen the mathematical foundation of the field.

Another promising direction involves the study of dynamic and time-varying Pythagorean fuzzy graphs. Since many real-world networks evolve over time, analyzing how spectral properties change under structural transformations such as edge addition, deletion, contraction, or subdivision would enhance practical applicability. Furthermore, efficient computational algorithms for determining eigen values of large-scale Pythagorean fuzzy adjacency and Laplacian matrices are essential for real-world implementation. The spectral behaviour of graph products—including Cartesian, tensor, and strong products—also remains insufficiently studied and represents a fertile ground for theoretical advancement.

Comparative studies between Pythagorean fuzzy graphs and other fuzzy graph models, such as intuitionistic and Fermatean fuzzy graphs, could help clarify the unique advantages of the Pythagorean framework. Moreover, integrating spectral measures of PFGs into optimization models, multi-criteria decision-making systems, and machine learning algorithms—particularly spectral clustering techniques—would bridge theoretical research with practical applications. Finally, hybrid extensions combining Pythagorean fuzzy graphs with soft set theory or neutrosophic structures may open new avenues for modeling higher levels of uncertainty. Overall, future research should aim to develop a unified, computationally efficient, and application-oriented spectral framework for Pythagorean fuzzy graph theory. These features may be studied through other extension of Pythagorean fuzzy graph theory like circular PFS concepts. This would provide a deeper research environment and a better result.

IV. RESEARCH GAP

Despite the growing body of literature on Pythagorean fuzzy graphs, the study of their spectral characterization remains relatively fragmented and underdeveloped. Most existing works primarily focus on defining structural properties, graph operations, degree concepts, and basic energy measures, while comprehensive spectral frameworks comparable to classical graph theory are still lacking. In particular, rigorous theoretical connections between adjacency and Laplacian spectra and fundamental graph invariants—such as connectivity, robustness, and structural stability—have not been fully established in the Pythagorean fuzzy setting. Moreover, the impact of structural transformations, including graph products, contractions, and dynamic modifications, on eigen values and spectral radius has received limited systematic investigation. Computational aspects, such as efficient eigen value algorithms for large-scale Pythagorean fuzzy matrices, also remain largely unexplored. Furthermore, there is insufficient comparative analysis between Pythagorean fuzzy graphs and other advanced fuzzy graph models, which restricts a deeper understanding of their relative advantages. Therefore, a unified, rigorous, and application-oriented spectral theory for Pythagorean fuzzy graphs is still needed, highlighting a clear and significant research gap in the current literature.

Problem Statement

Although Pythagorean fuzzy graphs provide a more flexible framework for modeling uncertainty compared to classical and intuitionistic fuzzy graphs, their spectral theory has not yet been systematically developed. Existing studies largely concentrate on defining basic structural properties, graph operations, and energy measures, but a unified mathematical framework connecting spectral parameters with structural transformations is still absent. In particular, the influence of operations such as union, product, complement, contraction, and subdivision on eigen values, spectral radius, and Laplacian characteristics has not been thoroughly analysed. Moreover, the lack of efficient computational techniques for evaluating large-scale Pythagorean fuzzy matrices limits practical applicability in real-world network systems. There is also no comprehensive synthesis of recent advancements that integrates spectral analysis with structural behaviour under the Pythagorean fuzzy environment. Therefore, the central problem addressed in this review is the need to systematically examine and consolidate the theoretical developments in spectral characterization and structural transformations of Pythagorean fuzzy graphs, while identifying limitations and potential directions for future advancement.

V CONCLUSION

The study of spectral characterization and structural transformations of Pythagorean fuzzy graphs represents a significant and evolving area within fuzzy graph theory. By extending classical spectral concepts such as adjacency and Laplacian eigen values into the Pythagorean fuzzy environment, researchers have begun to uncover deeper insights into the structural behaviour, stability, and connectivity of uncertain networks. Existing contributions have established foundational definitions, explored graph operations, and introduced energy-based measures; however, a comprehensive and unified spectral framework is still emerging. Structural transformations, including graph products, complements, and homomorphisms, further highlight the dynamic nature of Pythagorean fuzzy graph models and their applicability in complex decision-making and network systems. Despite notable progress, several theoretical and computational challenges remain unresolved, offering ample opportunities for future research. Overall, the continued integration of spectral analysis with structural theory is expected to enhance both the mathematical depth and practical relevance of Pythagorean fuzzy graph research in modeling uncertainty across diverse scientific domains.

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