The Sir Epidemic Mathematical Model for The Transmission, Prevention and Treatment of Dipterid Disease

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***Abstract*— Dipterid disease is a vaccine- preventable illness, where an outbreak will not happen if a significant portion of the population is immune. Current research indicates that vaccinated people with insufficient immunity coverage might face the possibility of subclinical diphtheria infection. In this study, we create an SIR mathematical model for diphtheria transmission employing simplifying assumptions, to identify parameters contributing to both endemic and non-endemic conditions.**

The model operates under the assumption that infected individuals can naturally recover without medical intervention. We aim to assess and analyze the dynamics of diphtheria transmission, despite its lack of realism and inability to eradicate the disease in practice. We utilize the basic Reproduction Number to examine the endemic equilibrium. We then explore the possibility of enhancing the model by incorporating additional complex variables and parameters.

***Index Terms*—Diphtheria Infection, Explore, Subclinical.**

I. INTRODUCTION

Diphtheria is a sudden contagious illness of the respiratory system sparked by bacteria called corynebacterium diphtheriae, with symptoms varying from mild to severe. In severe instances, it can lead to paralysis and even death. Despite the existence of highly efficient childhood vaccines, diphtheria transmission still remains a public health threat in numerous nations due to the potential mortality rate of 5 – 10%. This rate may escalate to over 20% in children under 5 years old and adults over 40 years old. In Thailand, diphtheria remain a significant contributor to children mortality and morbidity as evidenced by reports from 2917 - 2019. During this period, diphtheria cases saw a gradual increase, with an incidence rate ranging from 0 12 to 0.26 cases per 100,000 population. Particularly noteworthy is the considerable proportion of healthy adolescents affected by these outbreaks, showing a decline in protective antibody levels from previous diphtheria vaccinations over time.

This disease becomes epidemic in United States of America (U.S.) in 1921 with 206,000cases. But there are no such problems today due to improved conditions of living (S. Kanchanarat,2022). The complications from Diphtheria, such as airway blockage and heart muscle infection, often results in death. It spreads through droplets from infected persons or carriers.

Mathematical models in infectious disease epidemiology have been crafted to provide understanding into the transmission dynamics of diseases that may offer permanent immunity. Earn et al., (2022), introduced the classic susceptible-infectious-removed (SEIR) model to enhance comprehension of the intricate dynamical shifts in measles transmission. Kermach and McKendrick, (1927) were the first to utilize the SIR models to mathematically handle epidemic diseases where the population is sub-grouped into various classes such as the SIR epidemic model.

Kharapov and Loginova, (2019) utilized the modified SIR model for analyzing the transmission dynamics of COVID-19 pandemic in China. It has been a common knowledge that preventive vaccines induce immunity against some of the diseases mentioned above. There was a recent outbreak of Diphtheria in Indonesia which was confirmed by Indonesia's Ministry of Health. There were 95 regions and cities from 20 provinces that have reported cases from January to November 2017, and 622 cases recorded, 32 of these were fatal cases (Balieux F. et al, (2018). Diphtheria

has caused thousands of deaths and remain endemic in underdeveloped countries of the world. Those who survived from the disease suffer from paralysis of the muscles and permanent damage to the heart and kidneys. The very sensitive age to this disease is the children whose ages are one to ten years.

We model the Diphtheria disease utilizing the SIR Compartmental model in which the population is stratified into three groups: Susceptible compartment (S), Infected compartment (I), and Recovered compartment (R). The study focuses on the evaluation, analysis and interpretation of the Basic Reproduction Number of the SIR mathematical model of diphtheria disease.

1.1 Transmission Dynamics of Diphtheria

Diphtheria bacteria lives in the mouth, throat and nose of an infected person and so it can be passed to others by sneezing or coughing.

Another source of diphtheria bacteria transmission occurs from the skin, sores or through articles soiled with discharge from sores of infected persons.

If the disease is cutaneous, it can be spread by coming into contact with the wounds or lesion of an infected person or agent.

It can also be transmitted through sexual activity if any of the partner has diphtheria bacteria. Smooching can transmit diphtheria bacteria since it involves close contact and anything involving close contact with an infected person may attract infection.

1.2 Symptoms of Diphtheria

The common diphtheria symptoms include the following:

i. Fatigue or weakness
ii. Throat pain
iii. Fever
iv. Swollen neck glands
v. Difficulty swallowing
vi. Breathing problems due to tissues obstructing the nose and throat.
vii. Kidney, nerve or heart problems (if the bacteria enter the bloodstream).

Four identifiable diphtherias are (1) classical respiratory diphtheria (2) laryngeal diphtheria (3) nasal diphtheria, and (4) cutaneous diphtheria (skin lesion).10th may, 2023 (Ministry of Health, Republic of Indonesia, https/www.kemkesgo.id).

II. METHODS

In this research, we employ the analytical method of the Basic Reproduction Number and the SIR mathematical model. We utilize the simplified assumptions within the model equations to clarify the intricate workings. It therefore becomes simpler to identify the procedures to solving a mathematical model. Through researching relevant materials, we acquire a broad overview of diphtheria mathematical models, SIR epidemic models and systems of differential equations, equilibrium points, eigen-values and eigen-vectors. Ultimately, we examine the fundamental Reproduction number. Stability assessment is conducted to identify the key parameters that significantly influence the basic reproduction number. The study delves into the results of the model, exploring the local asymptotical stability analysis and investigating the presence of endemic diphtheria equilibria.

III. MODEL FORMULATION AND PROCESSES

In the model analysis of the spread of diphtheria bacteria, we sub-grouped the human population into three compartments of: (1) Susceptible compartment (S): those individuals who are uninfected and vulnerable to infection, (2) Infected compartment (I): which are those persons who are infected by the disease, and (3) Recovered compartment (R): These are the individuals who have recovered and acquired permanent immunity to the disease.

3.1 Model Assumptions

When modeling the transmission of an extremely contagious and infectious illness, we must confine ourselves to specific constraints. In this consideration, we assume WOLOG that:

i. The population is small.
ii. The birth and death rates are the same so that the total population is a constant.
iii. There is no immigration and emigration process so that the population is closed.
iv. No immune to the disease.
v. Transmission of Diphtheria is through direct contact with the patients.
vi. Only one disease (diphtheria bacteria) spreads in the population.

vii. Each child is born vulnerable due to passive immunity or material antibodies which are not fully effective due to their limited duration.
viii. People who are infected can either naturally recover from the illness or succumb to it.

The list of model variables is as shown in the table 3. 1 below:

Variables	Description	Terms
N = N(t)	Sum of population at time t.	N(t) ≥ 0
S = S(t)	Total individuals who are susceptible to infectious disease at time t.	S(t) ≥ 0
I = I(t)	Total persons who are infected with the disease at time t.	I(t) ≥ 0
R = R(t)	Sum of persons who have recovered from the disease at time t.	R(t)≥ 0

Table3. 1: List of model variables.

List of model parameters is as shown below in table 3.2.

Parameters	Description	Term
μ	Birth and death rates	μ > 0
β	Transmission rate between susceptible persons and infected persons.	β > 0
ϒ	Recovery rate	ϒ > 0

Figure 3.2: List of model parameters.

From the preceding, the modelflow-chart of disease transmission of diphtheria is depicted in figure3.3 below:

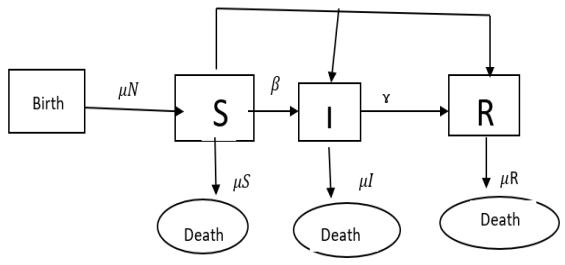

Figure 3.3: model flow- chart of disease transmission of diphtheria.

The corresponding SIR epidemic model system (1) is given by the following deterministic non-linear differential equations

$$f1: \frac{dS}{dt} = \mu N - \beta SI - \mu S \quad (3.1)$$

$$f2: \frac{dI}{dt} = \beta SI - (\mu + \gamma)I \quad (3.2)$$

$$f3: \frac{dR}{dt} = \gamma I - \mu R \quad (3.3)$$

SIR epidemic Model system (1).

IV. ANALYSIS OF THE SIR EPIDEMIC MODEL

Since the total population N is a constant, we can simplify the system (1) by computing proportions for each compartment. The respective proportions for the various persons in each compartment are the following:

Proportion for Susceptible compartment = $s = \frac{S}{N}$, proportion for Infectious compartment = $i = \frac{I}{N}$, and proportion for Recovered = $r = \frac{R}{N}$.(4.1).

From which we have that $s + i + r = \frac{S}{N} + \frac{I}{N} + \frac{R}{N} = \frac{S+I+R}{N}$.

But $N = S + I + R$, therefore, $s + I + r = \frac{N}{N} = 1$. (4.2).

So that the model equations become

f1: $\frac{dS}{dt} = \mu - \beta SI - \mu S$.(4.3).

f2: $\frac{dI}{dt} = \beta SI - (\mu + \gamma)I$.(4.4).

f3: $\frac{dR}{dt} = \gamma I - \mu R$.(4.5).

4.1 Disease- Free Equilibrium (Dfe) Of Diphtheria Disease

Disease- free equilibrium (DFE) refers to a situation where there is no disease in the population.In this case I=0 (absence of disease att =0), and R=0,

Therefore $\frac{dS}{dt} = \mu - \mu S$, and $\frac{dS}{dt} = 0$ implies that $\mu - \mu S = 0$ or $\mu S = \mu \Longrightarrow S = 1$. (4.6).

Hence the free dipterid Equilibrium point gives

$E^O = (S, I^O, R^O) = (1, 0, 0)$ (4.7).

4.2 The Endemic Disease Equilibrium (Ede)

This is a state where there is disease in the system.In this consideration, we equate all the derivatives of the model equations to zero, that is,

$\frac{dS}{dt} = 0, \frac{dI}{dt} = 0, \frac{dR}{dt} = 0$, and solve the resulting equations for the Endemic Dipterid Equilibrium point $E^* = (S^*, I^*, R^*)$.

From equation (3.7), $\frac{dI}{dt} = 0 \Longrightarrow \beta S^* I^* - (\mu + \gamma)I^* = 0$,

$\therefore \beta S^* I^* = (\mu + \gamma)I^*, \Longrightarrow \beta S^* = \mu + \gamma$

And$S^* = \frac{\mu+\gamma}{\beta}$,Adding equations (3.6) and (3.7) yields

$\frac{dS}{dt}+\frac{dI}{dt} = \mu - \mu S^* - \beta S^* I^* + \beta S^* I^*$ -- $(\mu + \gamma)I^*$

$= \mu - \mu S^* - \mu I^* - \gamma I^*$.

$= \mu(1-S^*) - ((\mu + \gamma)I^*$and so,$\frac{dS}{dt}=\frac{dI}{dt} = 0 \Longrightarrow (\mu + \gamma)I^* = \mu(1-S^*)$

Therefore,$I^* = \frac{-\mu(1-S^*)}{\mu+\gamma}$, and

$= \frac{-\mu(\mu+\gamma-\beta)}{\beta(\mu+\gamma)}$. (Substituting the value of S^*).

Similarly, from$\frac{dR}{dt} = 0$, we find $\mu R^* = \gamma I^* \Longrightarrow R^* = \frac{\gamma I^*}{\mu}$,

$\Longrightarrow R^* = \frac{-\gamma(\mu+\gamma-\beta)}{\beta(\mu+\gamma)}$. (Substituting the value of I^*).

Hence the Endemic Dipterid Equilibrium point is $E^* = (S^*, I^*, R^*)$.

That is, $E^* = (\frac{\mu+\gamma}{\beta}, \frac{-\mu(\mu+\gamma-\beta)}{\beta(\mu+\gamma)}, \frac{-\gamma(\mu+\gamma-\beta)}{\beta(\mu+\gamma)})$ (4.8).

4.3 Stability About the Disease – Free Equilibrium (Dfe)

Theorem 4.1: The disease-free equilibrium of the model is locally asymptotically stable if $R_O < 1$ and unstable if$R_0 > 1$.

PROOF: To prove this theorem, we linearize the system of model system (1) by computing its Jacobian Matrix JE^0. This is done at the disease-free equilibrium point by taking the partial derivatives of each equation in the system for the state variables. Thus, we have

$$J = \begin{pmatrix} \frac{\partial f1}{\partial S} & \frac{\partial f1}{\partial I} & \frac{\partial f1}{\partial R} \\ \frac{\partial f2}{\partial S} & \frac{\partial f2}{\partial I} & \frac{\partial f2}{\partial R} \\ \frac{\partial f3}{\partial S} & \frac{\partial f3}{\partial I} & \frac{\partial f3}{\partial R} \end{pmatrix} = \begin{pmatrix} -\beta I - \mu & -\beta S & 0 \\ \beta I & \beta S - (\mu+\gamma) & 0 \\ 0 & \gamma & -\mu \end{pmatrix} \quad (4.9).$$

Substituting the critical point S = 1, I = 0, R = 0, we find

$$J_{DFE} = \begin{pmatrix} -\mu & -\beta & 0 \\ 0 & \beta - (\mu+\gamma) & 0 \\ 0 & \gamma & -\mu \end{pmatrix} \quad (4.10).$$

We obtain the eigen-values of the Jacobian matrix by taking the determinant of J_{DFE} i.e.

$$|J_{DFE} - \lambda| = 0, \text{ that is, } \begin{vmatrix} -\mu-\lambda & -\beta & 0 \\ \beta I & \beta - (\mu+\gamma) - \lambda & 0 \\ 0 & \gamma & -\mu-\lambda \end{vmatrix} = 0$$

From which we obtain the corresponding eigen-values as

$$\left.\begin{aligned} \lambda_{1=-\mu,}\lambda_2 &= -\frac{1}{2}I\beta + \frac{1}{2}\beta S - \frac{1}{2}\gamma - \mu - \frac{1}{2}\sqrt{(\beta^2 I^2 - 2\beta^2 IS - 2\beta\gamma I - 2\beta\gamma S + \gamma^2)}, \text{ and} \\ \lambda_3 &= -\frac{1}{2}I\beta + \frac{1}{2}\beta S - \frac{1}{2}\gamma - \mu - \frac{1}{2}\sqrt{(\beta^2 I^2 - 2\beta^2 IS + \beta^2 S^2 - 2\beta\gamma I - 2\beta\gamma S + \gamma^2)}. \end{aligned}\right\} \quad (4.12).$$

According to the Routh- Hurwitz criterion, the necessary and sufficient condition for the stability of any system is that all the factors of the characteristic polynomial of a system must be negative. Since the eigen-values of the system (1) are all negative and conditions for the Routh-Hurwitz criteria are met, therefore the disease is asymptotically stable. This proves the above theorem.

4.4 Computation of the Basic Reproduction Number (R_0)

The Basic Reproduction Number (R_0) is defined as the average number of individuals an infected person can transmit the disease to throughout his life time.From equation (4.11), when we have $|J - \lambda I| = 0$, we use the factor

$\lambda_2 = \beta S^0 - (\mu + \gamma)$, for the eigen-value λ_2.

$= (\mu + \gamma)[\frac{\beta S^0}{\mu+\gamma} - 1]$,

$= (\mu + \gamma)[R_0 - 1]$, where$R_0 = \frac{\beta S^0}{\mu+\gamma}$.

Thus, we have the basic reproduction number $R_O = \frac{\beta S^0}{\mu+\gamma}$.

But $S^o = 1$ implies that $R_O = \frac{\beta}{\mu+\gamma}$. (4.13).

Therefore $R_O < 1$ occurs when$\beta < (\mu + \gamma)$ and$R_0 > 1$ occurs when $\beta > (\mu + \gamma)$. By utilizing this result, we make $R_O < 1$, when the death due to diphtheria factor μ does not increase and the rate of healing γ can increase. Hence, the key parameters that can significantly influence the basic Reproduction number

R_0 are β and ɤ and so, the model nature is dependent on the two factorsβ and ɤ.

4.5. Conclusion

The SIR epidemic model for the spread of diphtheria without treatment is evaluated and assessed by analytic method by utilizing the basic reproduction number operator and the fact that the Ordinary Differential Equations of the model satisfy the Routh-Hurwitz criterion for local stability. The Dipterid disease will not spread if $R_O < 1$ otherwise the disease will remains endemic in the system if $R_0 > 1$, that is, if $\frac{dI}{dt}$stays non-negative (i.e, when $\frac{dI}{dt} > 0$) and the infection I decreases. The key parameters that can be controlled to maintain $R_O < 1$ (or that increase or decrease the values of R_0) are β which is the transmission rate of diphtheria between susceptible humans and infected humans and the recovery rate ɤ. Note, we have assumed in this research that Diphtheria can cure naturally. Further research is needed to ascertain the comparison made with natural cure rate by utilizing vaccines or anti-virals.

4.6. Recommendation

Antibiotics, such as, penicillin or erythromycin, which help to kill bacteria in the body by clearing up infections is highly recommended as the best treatment for Diphtheria disease.

Antibiotics lessen the duration that someone with Diphtheria is contagious. If a doctor suspects Diphtheria, he or she will demand or request a medication that counteracts the Diphtheria toxin in the body (antitoxin). The best prevention against Diphtheria is immunization and getting a booster vaccination,

All contacts should also receive a course of antibiotics. Practicing strict hygiene by washing hands frequently before handling, preparing or eating food if one is caring for someone with diphtheria is also encouraged. Currently, there is no proof that Diphtheria bacteria are transmitted through semen or virginal fluids. But wounds or sores can be found in the male organ or virginal of females. Additional research is needed to determine if diphtheria bacteria are transmitted sexually.

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