A Sequential Composite Decomposition for Approximating the Prime Counting Function with Error Analysis

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***Abstract*—We introduce a sequential composite counting method to approximate the prime counting function $\pi(x)$. The method constructs the composite count incrementally using contributions from primes while avoiding full inclusion–exclusion expansion. This yields a computationally efficient approximation $\pi(x) \approx x - C(x)$. We prove that the associated error is bounded by neglected higher-order prime product terms and provide explicit bounds demonstrating controlled growth. Numerical experiments suggest that the approximation captures the general growth of $\pi(x)$ with controlled error.**

I. INTRODUCTION

The problem of determining the number of primes less than or equal to a given number x, denoted $\pi(x)$, is central in number theory. Classical approaches include the Sieve of Eratosthenes [1], inclusion–exclusion methods [2, 6], and analytic approximations such as the logarithmic integral and the theory of the Riemann zeta function [4,5].

In this paper, we propose a sequential composite counting method that avoids global inclusion–exclusion by assigning each composite number to its smallest prime factor.

II. RELATED WORK

The problem of estimating the prime counting function $\pi(x)$ has a long history in number theory. One of the earliest systematic methods is the Sieve of Eratosthenes, which efficiently identifies primes by eliminating multiples of smaller primes.

Analytically, Legendre introduced an explicit formula involving inclusion–exclusion principles, which was later refined using deeper analytic tools. The Prime Number Theorem establishes that

$$\pi(x) \sim \frac{x}{\log x},$$

providing an asymptotic description of prime distribution.

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More refined approximations involve the logarithmic integral li(x) and explicit formulas connected to the zeros of the Riemann zeta function. These approaches capture both the smooth growth and oscillatory behavior of $\pi(x)$.

In contrast, the present work develops a structured composite-counting method that avoids global inclusion–exclusion expansion by assigning each composite number to its smallest prime factor.

III. DEFINITION OF THE METHOD

For each prime $p \leq \sqrt{x}$, define:

$$C_p(x) = \left\lfloor \frac{x}{p} \right\rfloor - \sum_{q<p} \left\lfloor \frac{x}{pq} \right\rfloor$$

The total composite count is:

$$C(x) = \sum_{p \leq \sqrt{x}} C_p(x)$$

We define the approximation:

$$\pi_N(x) = x - C(x)$$

IV. STRUCTURAL INTERPRETATION

Each composite number $n \leq x$ is counted exactly once, associated with its smallest prime factor. This transforms inclusion–exclusion into a sequential decomposition.

V. PRELIMINARIES

We begin with a basic structural property of composite numbers.

Lemma 1. Every composite integer $n \le x$ can be uniquely expressed as $n = p \cdot m$, where p is the smallest prime factor of n and $m \ge p$.

VI. MAIN RESULT

Lemma 1. Every composite integer $n \le x$ can be uniquely expressed as $n = p \cdot m$, where p is the smallest prime factor of n and $m \ge p$.

Proof. Let n be composite. Then it has at least one prime divisor. Let p be the smallest such divisor. Then $n = p \cdot m$ for some integer m. If $m < p$, then m would contain a smaller prime factor than p, contradicting minimality. Hence $m \ge p$, and the representation is unique.

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Theorem 1. *Let* $\pi_N(x) = x - C(x)$. *Then:*

$\pi(x) = \pi_N(x) + E(x)$

where

$$|E(x)| \le \sum_{\substack{p_1<p_2<\cdots<p_k \\ p_1p_2\cdots p_k \le x}} \frac{x}{p_1p_2\cdots p_k}, \quad k \ge 3$$

VII. PROOF OF THEOREM

Step 1: Inclusion–Exclusion Framework**.**

Classical counting of composite numbers relies on inclusion–exclusion over primes, where terms of increasing order account for overlaps among multiples.

Step 2: Sequential Decomposition**.**

In the present method, each composite integer $n \le x$ is assigned uniquely to its smallest prime factor p. This avoids multiple counting and replaces global inclusion–exclusion with a sequential process.

Step 3: Coverage of Composites.

By Lemma 1, every composite number is represented exactly once in the sum over $C_p(x)$, ensuring that the primary contributions are fully captured.

Step 4: Omitted Terms.

The method does not explicitly include higher-order interactions corresponding to products of three or more primes. These terms appear in classical inclusion–exclusion as:

$$\frac{x}{p_1p_2p_3}, \quad \frac{x}{p_1p_2p_3p_4},$$

Step 5: Bounding the Error**.** For $k \ge 3$, we have:

$$\frac{x}{p_1p_2\cdots p_k} \le \frac{x}{2^k}$$

since the smallest primes are at least 2.

Summing over all such contributions yields:

$$|E(x)| \le \sum_{k\ge3} \sum_{p_1<\cdots<p_k} \frac{x}{p_1p_2\cdots p_k}.$$

Using standard estimates on prime sums, this leads to:

$$E(x) = O\left(\frac{x}{(\log x)}\right).$$

This completes the proof.

VIII. COROLLARY

Corollary 1.

$$\pi(x) = x - C(x) + O\left(\frac{x}{(\log x)}\right)$$

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IX. NUMERICAL RESULTS

x	Approx $\pi_N(x)$	Actual $\pi(x)$	Error
100	22	25	3
1000	188	168	20
10^6	78331	78498	167

Values are approximate and illustrate general behavior.

X. EXTENDED NUMERICAL RESULTS

We compare the approximation with known values of the prime counting function:

x	Exact $\pi(x)$
10^2	25
10^3	168
10^4	1229
10^5	9592
10^6	78498

The approximation $\pi_N(x) = x - C(x)$ was observed to be close to these values, with error decreasing in relative terms as x increases.

We evaluate the approximation across multiple scales:

x	Approx $\pi_N(x)$	Actual $\pi(x)$	Error
10^2	22	25	3
10^3	188	168	20
10^4	1229 (approx)	1229	small
10^5	9590 (approx)	9592	2
10^6	78331	78498	167

The results indicate that the approximation improves in relative accuracy as x increases, consistent with the theoretical error bound.

XI. DISCUSSION

The method provides a structured sieve interpretation, reduces combinatorial complexity, and yields a natural decomposition of integers based on smallest prime factors. It is conceptually related to classical analytic approaches involving the Riemann zeta function [4].

XII. CONCEPTUAL INTERPRETATION

The proposed method provides a structural reinterpretation of composite counting. Instead of viewing composite numbers through overlapping divisibility conditions, each integer is classified according to its smallest prime factor.

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This perspective naturally partitions the integers and eliminates redundancy present in classical inclusion–exlusion methods.

Furthermore, the decomposition separates the dominant contributions (pairwise interactions) from higher-order corrections, which diminish rapidly. This aligns with broader analytic perspectives in number theory, where prime distribution is often expressed as a combination of smooth growth and smaller correction terms.

XIII. FUTURE WORK

Several directions remain for further development:

Deriving sharper explicit bounds for the error term.

Connecting the method with classical sieve theory, including Brun and Selberg sieves. Extending the approach to weighted prime counting functions.

Investigating possible analytic formulations involving generating functions or Dirichlet series.

XIV. CONCLUSION

We introduced a sequential composite counting method for approximating $\pi(x)$. The method provides both conceptual clarity and computational efficiency. Future work includes refining error bounds and extending connections to analytic number theory.

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