Odd Fibonacci Co-Prime Labeling of Twig Graph, Olive Tree and Caterpillars

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***Abstract*—The notion of Prime labeling was introduced by Roger Entringer and was discussed in a paper by Tout. Fibonacci Prime labeling of graphs was introduced by Sekar and Chandrakala. Diverse results on Fibonacci Prime labeling can be viewed in Gallian's Dynamic Survey of graph labelings. A new labeling method termed as Odd Fibonacci Co-Prime Labeling was introduced by J Jeba Jesintha and J Sathyapriya. Let G be a simple undirected graph with p vertices and q edges. An injective function $f: V(G) \rightarrow \{F_1, F_3, F_5, F_7, \ldots\}$, where each F_i is an odd indexed Fibonacci number such that adjacent vertices are Co-Prime is called an Odd Fibonacci Co-Prime Labeling. A graph which admits an Odd Fibonacci Co-Prime Labeling is called an Odd Fibonacci Co-Prime Graph. In this paper the odd Fibonacci Co-Prime labeling of few trees such as twig, olive tree and caterpillars are proved.**

***Index Terms*—Fibonacci Prime labeling, Odd Fibonacci Co-Prime labeling, twig graph, olive tree, caterpillar.**

I. INTRODUCTION

Graph labeling was first introduced by Alexander Rosa in 1967 [1]. The notion of prime labeling was later proposed by Roger Entringer, and further developed by Tout, Dabboucy and Howalla in 1982 [9]. A graph is said to admit a prime labeling if its vertices can be assigned distinct positive integers such that the labels of every pair of adjacent vertices are relatively prime. Dertsky [3] proved that the cycle C_n on n vertices possesses a prime labelling. Lee [7] established that a wheel graph is prime if and only if n is even. In 2018, Chandrakala and Sekar [8] introduced Fibonacci prime labeling and showed that several cycle-related graphs satisfy this property. A comprehensive survey of graph labeling including both prime and Fibonacci prime labelling can be found in Gallian [4].

The Odd Fibonacci Co-Prime labeling was introduced by J Jeba Jesintha and J Sathyapriya [5]. Let G be a simple undirected graph with p vertices and q edges and an injective function $f: V(G) \rightarrow \{F_1, F_3, F_5, F_7, \ldots\}$ where each F_i is an odd indexed Fibonacci number such that adjacent vertices are co-prime. A graph which admits an odd Fibonacci coprime labeling is called an odd Fibonacci co-prime graph. Odd Fibonacci co-prime labeling has potential applications in network security, coding theory and cryptographic key generation due to the mathematical uniqueness of Fibonacci primes. In this paper the odd Fibonacci co-prime labeling of twig, olive tree and caterpillars are proved.

II. PRELIMINARY DEFINITONS

In this section we give few definitions.

Definition 2.1 [8] The Fibonacci sequence is a series of numbers starting with 0 and 1, where each succeeding number is the sum of two preceding numbers. Hence the Fibonacci sequence is 0, 1, 1, 2, 3, 5, 8,13, 21 and so on. It can be obtained by the recursive formula $F_n = F_{n-1} + F_{n-2}$, for $n \geq 2$, with initial values $F_0 = 0$ and $F_1 = 1$.

Definition 2.2 [3] The prime labeling of a graph G is an injective function f: $V(G) \rightarrow \{1, 2, \ldots, |V(G)|\}$ such that for every pair of adjacent vertices u and v, gcd(f(u), f(v)) = 1. A graph which admits a prime labeling is called a prime graph.

Definition 2.3 [8] A Fibonacci prime labeling of a graph G = (V, E) with n vertices is an injective function f: $V(G) \rightarrow \{F_1, F_2, \ldots, F_{n+1}\}$ where F_n is the nth Fibonacci number, such that greatest common divisor of adjacent vertices is 1.

Definition 2.4 [6] An Odd Fibonacci Co-Prime labeling of a graph G with p vertices and q edges is an injective function $f : V(G) \rightarrow \{F_1, F_3, F_5, F_7, F_9 \ldots\}$ where each F_i is an odd indexed Fibonacci number such that adjacent vertices are coprime. A graph which admits an odd Fibonacci co-prime labeling is called an Odd Fibonacci Co-Prime graph.

Definition 2.5 [2] A twig graph is a graph obtained from a path of n vertices P_n by attaching exactly two pendant edges to each internal vertex of P_n.

Definition 2.6 [11] An Olive tree is a rooted tree consisting of k branches, where the i^{th} branch is a path of length i.

Definition 2.7 [10] A caterpillar is a tree, where deleting all the one degree vertices, leaves a path.

Definition 2.8 [4] A comb graph, denoted as $P_n \odot K_1$, is formed by taking a path graph P_n with n vertices and attaching a single pendant edge to each vertex of the path.

III. MAIN RESULTS

In this section, we prove the twig graph, olive tree, caterpillar are odd Fibonacci co-prime graphs.

Theorem 3.1. Twig graph admits odd Fibonacci co-prime labeling.

Proof: Let G be the twig graph. The vertices of twig graph are shown in Figure 1.

Define the vertex labels are as follows:

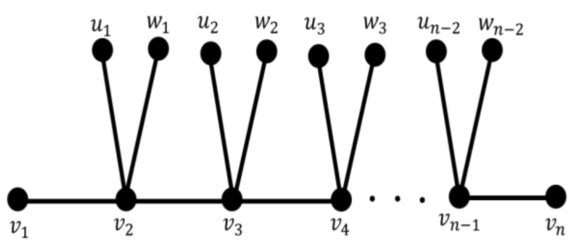

Figure 1. Twig graph

$f(v_i) = F_{6i-5}$; $1 \le i \le n-1$

$f(v_n) = F_{6n-9}$

$f(u_j) = F_{6j-3}$; $1 \le j \le n-2$

$f(w_j) = F_{6j-1}$; $1 \le j \le n-2$

It is observed that

$$\gcd(f(v_i), f(v_{i+1})) = \gcd(F_{6i-5}, F_{6i+1}) = F(_{\gcd(6i-5, 6i+1)}) = 1$$

$$\gcd\left(f(v_i), f(u_j)\right) = \gcd(F_{6i-5}, F_{6j-3}) = 1$$

$$\gcd\left(f(v_i), f(w_j)\right) = \gcd(F_{6i-5}, F_{6j-1}) = 1$$

Hence twig graph admits odd Fibonacci co-prime labeling. An illustration for Theorem 3.1 is given in Figure 2.

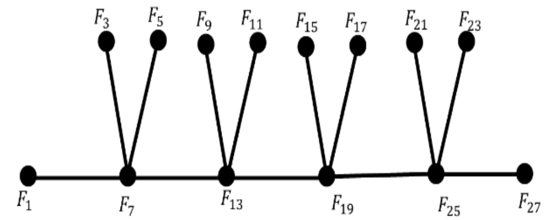

Figure 2. Odd Fibonacci Co-prime Twig graph with $n = 6$

Theorem 3.2. Olive tree admits odd Fibonacci co-prime labeling.

Proof: Let O_n be the Olive tree. The vertices of Olive tree are described as follows: The root vertex of O_n is denoted as v_0. The i^{th} branch attached at the root v_0 consists of vertices $v_{1,1}, v_{i,2}, \ldots v_{i,i}$ from top to bottom. The Olive tree O_n is shown in Figure 3.

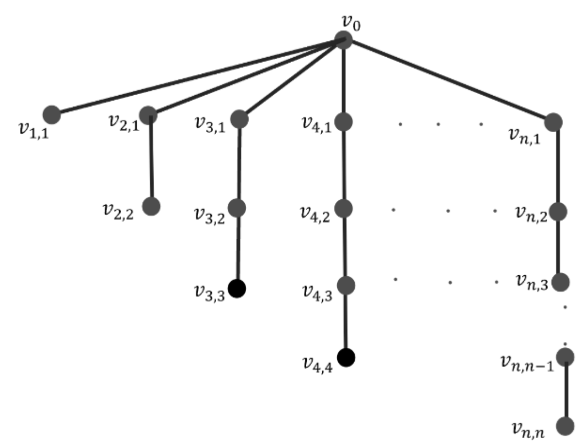

Figure 3. Olive tree O_n

Define the vertex labels are as follows:

$f(v_0) = F_1$

$f(v_{i,j}) = F_{i(i-1)+2j+1}$;

$1 \le i \le n$, $1 \le j \le n$, $j \le i$

It is observed that

$$\gcd\left(f(v_0), f(v_{i,1})\right) = \gcd(F_1, F_{i(i-1)+3}) = 1$$

$$\gcd\left(f(v_{i,j}), f(v_{i,j+1})\right) = \gcd(F_{i(i-1)+2j+1}, F_{i(i-1)+2j+3}) = 1$$

Hence, Olive tree admits odd Fibonacci co-prime labeling. An illustration for Theorem 3.2 is given in Fig

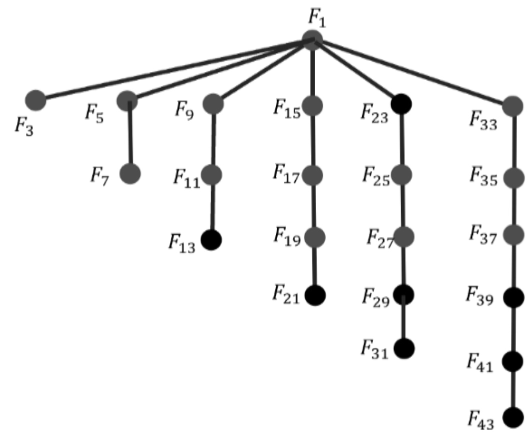

Figure 4. Odd Fibonacci Co-prime Olive Tree O_6

Theorem 3.3 The caterpillar C(m, n) admits odd Fibonacci Co-Prime labeling

Proof**:** Let G be a caterpillar tree with m vertices on the central path and $n_1, n_2, \ldots, n_m$ leaves attached to each of these m central vertices where $n_1 + n_2 + \ldots + n_m = n$ as shown in Figure 5

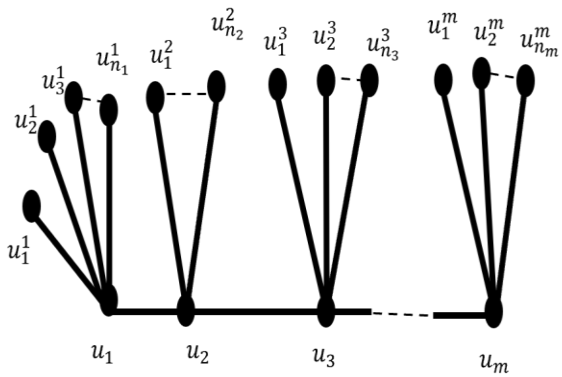

Figure 5. Caterpillar tree C(m, n).

Define the vertex labels as follows:

$f(u_i) = F_{2i-1},\ 1 \leq i \leq m$

$f(u_j^i) = F_p$,

where p′s are distinct primes > 2m-1, $1 \leq j \leq n_i$

It is observed that

$\gcd(f(u_i), f((u_j^i)) = \gcd(F_{2i-1}, F_p) =$

$F_{\gcd(2i-1,p)} = F_1 = 1,\ 1 \leq i \leq m,\ 1 \leq j \leq n_i$

Hence caterpillars admit odd Fibonacci co-prime labeling.

An illustration for Theorem 3.3 is given in Figure 6.

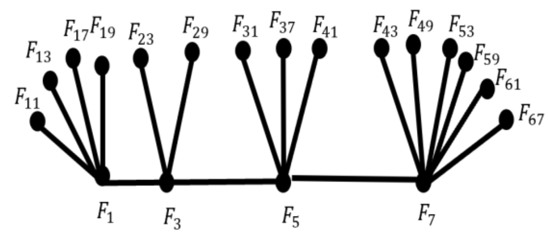

Figure 6. Odd Fibonacci co-prime Caterpillar C(4,15)

Corollary: Every comb graph is Fibonacci co-prime. This is obtained when each $n_i = 1$ in C(m, n).

An illustration of Fibonacci co-prime comb graph is given in Figure 7.

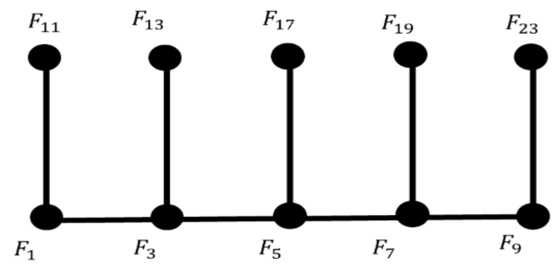

Figure 7. Odd Fibonacci co-prime comb graph C(5,5)

IV. CONCLUSION

In this paper the odd Fibonacci co-prime labeling of few trees were discussed. Analogous work can be carried for other families and in the context of different types of graph labeling techniques.

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