Application Of Six Sigma Control Charts in M^x/M/1 Queueing Systems with Geometric Batch Arrivals

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***Abstract*—The key performance indicators of a service system with a batch arrival pattern are customer waiting time and system congestion. In this work, the Six Sigma based control charts are suggested to monitor the average system length (Ns) in an Mx/M/1 queueing model, whereby the customer batches enter into the queue in accordance with the Poisson process and the batch sizes are generated in accordance with a geometric distribution. The proposed methodology combines Statistical Process Control (SPC) with process capability analysis, employing the process capability index Cp to come up with a modified standard deviation σQ which generates narrower control limits than the traditional Shewhart 3σ method. A comparative numerical analysis of three service rate configurations and thirty batch arrival scenarios with varying arrival frequencies indicates that the Six Sigma control charts have continuously yielded significantly smaller control limit intervals (CLI), and percentage changes below 80 percent in a variety of configurations. The influence of batch size probability (α) and service rate (μS) on the system length as well as monitoring performance is studied. The results reveal the usefulness of the Six Sigma monitoring models in practice in service settings that are characterised by large or batch customer flows, including transportation facilities, telecommunications packet networks, as well as hospital emergency rooms.**

***Index Terms*—M^x/M/1 queueing model; Batch arrivals; Geometric distribution; Six Sigma; Statistical Process Control; Control charts; System length; Process capability; Service quality**

I. INTRODUCTION

Service systems with bulk or batch customer arrivals constitute a crucially different category of queueing problems with the classical M/M/1 and M/M/s models of a single-arrival. In many practical settings such as packet delivery in computer networks, arrival of customers in passenger groups in transport terminals, job submission in cloud computing and group presentation in hospital emergency rooms customers do not come singly but in coordinated groups (Medhi, 2002; Kleinrock, 1975). Mx/M/1 model: this is a model where the arrivals of batch sizes are described by a Poisson process, and the batch sizes themselves are described by a geometric distribution, which is a tractable but realistic model of the analysis of such systems.

The performance of the M3 / M 1 system is very different compared with single-arrival models. Specifically, the mean system length N_s has a value that is much higher with the arrival of the batch, particularly when the parameter of the batch size α approaches its maximum value. Such an augmented variability elevates the operational significance of the statistical surveillance of system length: minor losses in batch arrival characteristics or efficiency of service can push the system length to operationally intolerable levels (Gross et al., 2018; Tijms, 2003).

A growing literature on Statistical Process Control (SPC) and specifically the Shewhart-type control charts has been applied to quality assurance of manufacturing in (Shewhart, 1931; Montgomery, 2019) to the monitoring of service systems. Control charts of the queue length in M/M/1 systems were built by Shore (2000), and Poongodi and Muthulakshmi (2013) built similar charts in the M/M/s model. These charts were also improved by Pukazhenthi and Poornima (2018), who added process capability indices. Nevertheless, little literature has been done on the use of control charts to batch arrival systems, namely Mx/M 1 model.

The Six Sigma approach, which began its existence in the 1980s at Motorola and was later implemented in manufacturing and service sectors, offers a philosophical system of radically decreasing process variation (Pyzdek & Keller, 2014; Goh, 2011). The Six Sigma idea of basing control limits on the process capability indices and not on the empirical standard deviation alone in the context of the control chart produces control limits that are significantly narrower and detect process deviations earlier and more reliably (Radhakrishnan and Balamurugan, 2012; Evans and Lindsay, 2014).

The current research has the following contributions: (i) constructing Six Sigma control charts of the average system length of the Mx /M/1 geometrical arrival of batches model N_s; (ii) systematic comparison with classical Shewhart charts in three service rate settings (μS = 7, 10, 14) and thirty batch size probability values (α = 0.01 to 0.20); (iii) quantification of CLI reduction achievable through the Six Sigma approach; and (iv) analysis of the separate effects of batch size probability and service rate on system length variability and monitoring performance.

II. LITERATURE REVIEW

A. Batch Arrival Queueing Models

Bailey (1954) systematically studied batch arrival queueing systems, having analysed the $M^x/G/1$ queue, and Sasieni (1961), generalised the findings to geometrically distributed batch sizes. Medhi (2002) gives a thorough treatment of bulk arrival and bulk service queues and defines the fundamental findings of the mean system length, distribution of length of queues, and waiting time of the $M^x/M/1$ model. The average system length of the geometric batch size distribution is $N_s = \rho/(1-\rho) + \alpha\rho/(1-\alpha)$, where $\rho = \lambda A\ E[X]/\mu S$ is the effective traffic intensity and $E[X] = 1/(1-\alpha)$ is the mean batch size (Medhi, 2002; Bunday, 1996).Tijms (2003) compares the difference between system length in batch arrival systems to a significantly greater value compared to the single-arrival queues with the same load, inspiring the necessity to have more stringent monitoring limits.

Gross et al. (2018) discusses computational approaches to batch queueing systems and note that precise estimation of variance is crucial in the construction of control charts. Kleinrock (1975) places the batch arrival models in the context of a wider analysis of the packet switched networks and offers the first connection between the theory of $M^x/M/1$ and telecommunications in practice. Banks et al. (2010) show through simulation that the effects of a batch arrival is especially strong around the stability boundary, which supports the argument that sensitive monitoring devices are required in high-utilisation regimes.

B. Control Charts for Queueing Systems

The use of SPC in monitoring of service systems is based on the theoretical basis of Shewhart (1931) and Montgomery (2019). Approximate normality of queue length in M/M/1 systems had been pioneered by Shore (2000), who constructed Shewhart control charts of the queue length. Poongodi and Muthulakshmi (2013) applied this to M/M/s systems and obtained explicit formulas of the control limits of queue length and waiting time. The chart construction by Pukazhenthi and Poornima (2018) in the M/M/s waiting time used process capability indices, which forms the methodological linkage between SPC and Six Sigma and to the present study, the batch arrival scenario.

Orssatto et al. (2014) used Shewhart control charts and process capability analysis to a sewage treatment station to prove the feasibility of the SPC in the complex service processes with non-standard distributions. Rigdon, Fricker, and Woodall (2012) review in detail the performance measures of control chart such as Average Run Length (ARL), which they suggest as the main measure of comparing the charts. Morais and Pacheco (2000) examined CUSUM charts of Poisson processes, and Epprecht et al. (2015) comes up with a variable sampling interval solution of adaptive monitoring, both of which are other significant future extensions of the current work.

C. Six Sigma in Service Quality Management

Harry and Schroeder, (2000) formalised Six Sigma and it has been widely reviewed by Pyzdek and Keller (2014) and Goh (2011). Antony and Banuelas (2002) established the critical success factors of Six Sigma implementation in service industries and process capability measurement is one of the main requirements. Evans and Lindsay (2014) and Mitra (2016) give detailed coverage of quality management models that combine SPC and Six Sigma. The main methodological finding of the current study is that the replacement of the standard deviation based on the process capability 550Q by the empirical 550Q in the Shewhart control limit equations will lead to smaller values of the CLI, and the detectable process changes will be better when the process change is small (Radhakrishnan and Balamurugan 2012). It is the first time when the present work applies the given result to the problem of monitoring the length of the system $M^x/M/1$.

III. $M^x/M/1$ QUEUEING MODEL: FORMULATION AND PERFORMANCE MEASURES

The structural assumptions that characterize the $M^x/M/1$ queueing model include (Medhi, 2002; Gross et al., 2018).

- Batches of customers come in as a result of a homogeneous Poisson process with a batch arrival rate $\lambda_A > 0$.
- Single batch sizes X follows geometric distributed with parameter $\alpha \in (0,1)$:
- $P(X = k) = (1-\alpha)^{k-1}\alpha, k = 1,2,3, ...$
- Service times have the same value i. i.d. exponential random variables with rate of $\mu_S > 0$ (single server).
- The customers in any given batch are attended to one after the other.
- First-Come (FCFS) Queue discipline First-Come, First-Served (FCFS) and unlimited waiting-room.

A. Key Terminologies and Parameters

Upper Specification Limit (USL): The upper level of acceptable system length which is calculated based on the operational level of service requirements.

Lower Specification Limit (LSL): The minimum acceptable value; typically, zero for system length.

Tolerance Limit (TL): Defined as

$$TL = USL - LSL$$

representing the permissible range of process variation.

Process Capability Index (Cp): This is used to measure the effectiveness of the process in terms of working within the specifications.

$$C_p = \frac{USL - LSL}{6\sigma} = \frac{TL}{6\sigma}$$

B. Steady-State Performance Measures

The mean batch size is

$$E[X] = \frac{1}{1-\alpha}$$

and the effective traffic intensity is:

$$\rho = \frac{\lambda_A\, E[X]}{\mu_S} = \frac{\lambda_A}{\mu_S(1-\alpha)}$$

The stability condition requires

$$\rho < 1$$

The mean system length (number of customers in the system) is:

$$E(N_s) = \frac{\rho}{1-\rho} + \frac{\alpha\rho}{1-\alpha}$$

The width of the control limits is significantly greater than in a single-arrival M/M/1 queue of the same load since the extra variability of the geometric batch size distribution leads to a large variance of system length (Medhi, 2002; Tijms, 2003):

$$\mathrm{Var}(N_s) = \frac{\rho}{(1-\rho)^2} + \frac{2\alpha\rho}{(1-\alpha)(1-\rho)} + \frac{\alpha\rho}{(1-\alpha)^2}$$

The standard deviation

$$\sigma = \sqrt{\mathrm{Var}(N_s)}$$

forms the basis of the Shewhart control chart construction.

C. Shewhart Control Chart for System Length

Assuming Ns is treated as though it is normally distributed (because of the Central Limit Theorem of moderately high traffic intensities), a Shewhart control chart with the following parameters can be plotted:

- Center Line (CL) = $E(N_s)$
- Upper Control Limit (UCL_{sh}) = $E(N_s) + 3\sigma$
- Lower Control Limit (LCL_{sh}) = max(0, $E(N_s) - 3\sigma$)
- The control limit interval is $CLI_{sh} = UCL_{sh} - LCL_{sh}$

In reality, when $LCL_{sh} < 0$ is small in many $M^x/M/1$ designs (as Ns is skewed to the right), the LCL becomes truncated at 0, and the upper limit is bearing the main monitoring overhead.

D. D. *Six Sigma Control Chart for System Length*

Based on the method of Radhakrishnan and Balamurugan (2012), the standard deviation σ_Q is modified through the process capability index C p and tolerance limit TL:

$$C_p = \frac{TL}{6\sigma_Q} \Rightarrow \sigma_Q = \frac{TL}{6C_p}$$

For a Six Sigma process ($C_p = 2$):

$$\sigma_Q = \frac{TL}{12} \leq \sigma$$

The Six Sigma control chart limits are:

- $UCL_{ss}=$
$E(N_s) + 3\sigma_Q$
- $LCL_{ss}=$
$\max\left(0,\, E(N_s) - 3\sigma_Q\right)$

Since $\sigma_Q \leq \sigma$, it follows that

$$CLI_{ss} \leq CLI_{sh}$$

whereby the inequality is strict whenever $\sigma_Q < \sigma$. Tighter limits are the mathematical assurance of the superior sensitivity of the Six Sigma chart. The smaller CLI implies that the deviations of system length which the Shewhart chart would regard as normal variations are pointed out by the Six Sigma chart so that corrective action can be taken earlier (Goh, 2011; Montgomery, 2019).

IV. SIX SIGMA DMAIC FRAMEWORK APPLIED TO M^x/M/1 MONITORING

Six Sigma DMAIC (Define–Measure–Analyse-Improve-Control) framework offers the system of operations of how the suggested control charts can be put into practice. The phases are linked to the M^x/M/1 system context in the following way:

Define:
Find system length Nsas critical-to-quality (CTQ) characteristic. Set service-level performance (USL, LSL) goals, which are founded on operational requirements. Set the parameters of the model of batch arrival λ_A, α, μ_S).

Measure:
Gather system length, traffic intensive and batch arrival measures. Calculate (N_s), $\mathrm{Var}(N_s)$, process capability index Cp using baseline data. Verify stability condition ρ<1.

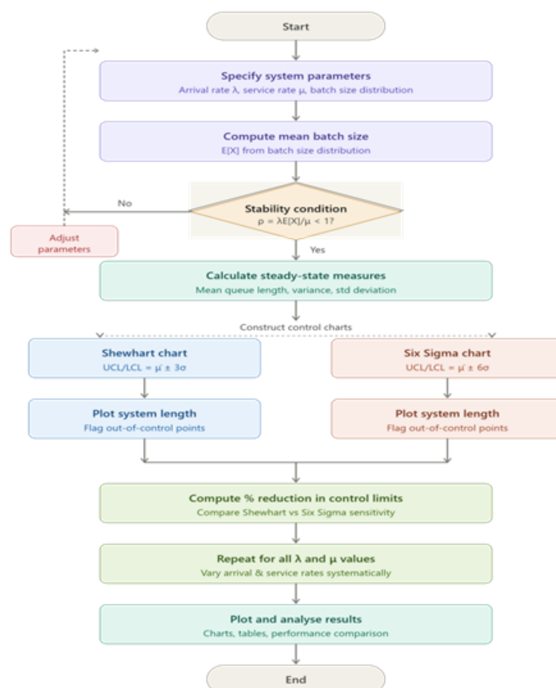
Analyse:
Build a Shewhart and Six Sigma control chart. Compare CLI values. Determine the root causes of system length variability through parameter analysis: parameters of batch size probability (α), and service rate (μ_S).

Improve:
Take corrective measures on signals on control charts, change service rate, batch admission policy or server allocation. Measure the improvement through CLI reduction, and out-of-control signal reduction.

Control:
Implement the Six Sigma control chart as a control mechanism. Accommodate automatic warnings of UCL. Review and revise service-level requirements (USL/LSL) as service-level requirements change.

V. METHODOLOGY

The methodology for analyzing the M^x/M/1 queueing system follows a structured sequential process, as illustrated in the flowchart below, beginning with parameter specification and stability verification, followed by the parallel construction of Shewhart and Six Sigma control charts, and culminating in a comparative analysis of control limit reductions across all arrival and service rate combinations.



VI. NUMERICAL RESULTS AND INTERPRETATION

This part gives the full numerical results of all the three system configurations with six improved visualisations and a cross-configuration comparative analysis.

A. *Configuration I: λ_A = 3, μ_S = 7, α = 0.01–0.10*

The control chart parameters of the first configuration with an intermediate service rate (μS = 7) and a probability of low to moderate batch size are given in Table 1.

Table 1: Control chart parameters for λ_A = 3, μ_S = 7, α = 0.01–0.10

Arrival Rate (λ_A)	Service Rate (μ_S)	Batch Prob. (α)	Traffic Intensity (ρ)	Std. Dev. (σ)	Shewhart Control Chart			Six Sigma Chart ($C_p = 1$)	
					LCL	CL	UCL	LCL	UCL
3	7	0.01	0.433	1.175	-2.755	0.771	4.297	0.597	0.945
3	7	0.02	0.437	1.206	-2.825	0.793	4.411	0.619	0.967
3	7	0.03	0.442	1.238	-2.898	0.816	4.530	0.642	0.990
3	7	0.04	0.446	1.271	-2.973	0.840	4.653	0.666	1.014
3	7	0.05	0.451	1.306	-3.052	0.865	4.782	0.691	1.039
3	7	0.06	0.456	1.342	-3.133	0.891	4.916	0.718	1.065
3	7	0.07	0.461	1.379	-3.218	0.919	5.056	0.745	1.093
3	7	0.08	0.466	1.418	-3.307	0.948	5.203	0.774	1.122
3	7	0.09	0.471	1.459	-3.399	0.978	5.355	0.804	1.152
3	7	0.10	0.476	1.502	-3.495	1.010	5.515	0.836	1.184

The effective mean batch size E[X] = 1/(1−α) r is increasing with the value of 0.010 to 0.111, and the traffic intensity 4 is also increasing with the same value 0.433 to 0.476. Even this small change in batch size results in a significant change in the mean system length (CL) of 0.771 to 1.010 customers and the standard deviation 6 increases to 1.502. Shewhart UCL therefore increases to 5.515 with the LCL essentially zero (Shewhart LCL is negative everywhere, truncated at 0). On the contrary, the Six Sigma UCL lies within a slender range of 0.945 1.184, whereas the Six Sigma LCL falls between 0.597 and 0.836. The width of the Six Sigma CLI is about 0.348 units across compared to Shewhart CLI value of over 4.0 a reduction of over 90%.

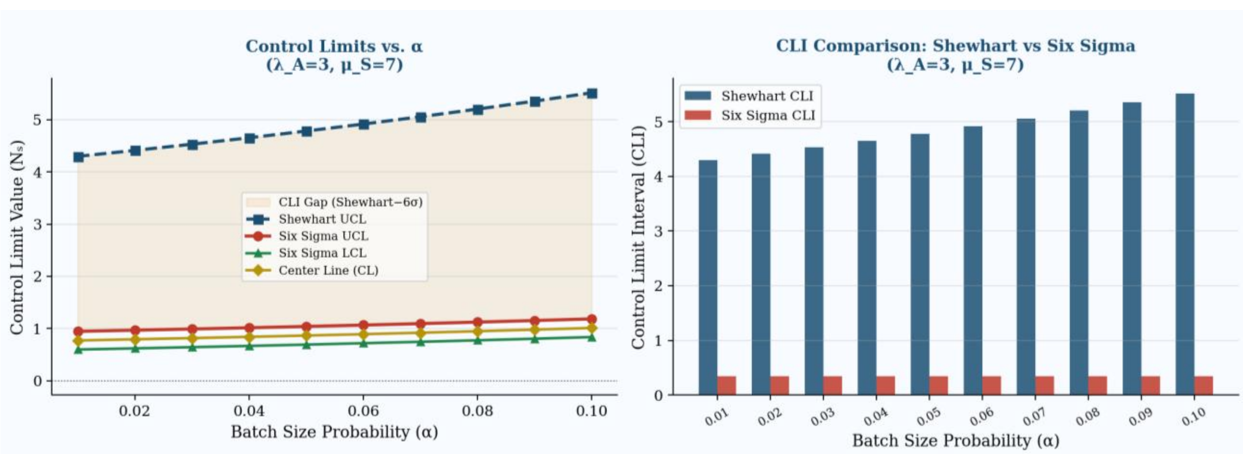


Figure 1: $M^x/M/1$ control limits for λ_A = 3, μ_S = 7, α = 0.01–0.10 | Left: trend lines with CLI gap; Right: CLI bar comparison

Figure 1 shows a dramatic difference between the two types of charts in terms of CLI. The left panel indicates that the Shewhart UCL increases drastically with α (maximum value of 5.515 at α = 0.10) whereas the Six Sigma UCL is closely clustered at the center. The difference between the two UCL curves is the shaded area which illustrates the CLI gap. The right panel quantifies the benefit; at each value of α, the Six Sigma CLI bar (red) is a small fraction of the Shewhart CLI bar (blue), which proves the practical overwhelming superiority of the Six Sigma approach to such a design.

B. Configuration II: λ_A = 3, μ_S = 14, α = 0.11–0.20

Table 2 analyses a doubled service rate (μS = 14) and with a larger probability range of batch size (α = 0.11 0.20). The increased service rate significantly decreases traffic intensity to ρ ∈ [0.241, 0.268], which significantly decreases the average system length and variability.

Table 2: Control chart parameters for λ_A = 3, μ_S = 14, α = 0.11–0.20

Arrival Rate (λ_A)	Service Rate (μ_S)	Batch Prob. (α)	Traffic Intensity (ρ)	Std. Dev. (σ)	Shewhart Control Chart			Six Sigma Chart (C_p = 1)	
					LCL	CL	UCL	LCL	UCL
3	14	0.11	0.241	0.756	-1.911	0.356	2.624	0.255	0.458
3	14	0.12	0.244	0.774	-1.957	0.366	2.688	0.264	0.467
3	14	0.13	0.246	0.793	-2.004	0.376	2.755	0.274	0.477
3	14	0.14	0.249	0.813	-2.052	0.386	2.824	0.284	0.487
3	14	0.15	0.252	0.833	-2.102	0.397	2.895	0.295	0.498
3	14	0.16	0.255	0.854	-2.154	0.408	2.970	0.306	0.509
3	14	0.17	0.258	0.876	-2.208	0.419	3.046	0.318	0.521
3	14	0.18	0.261	0.898	-2.264	0.431	3.126	0.330	0.533
3	14	0.19	0.265	0.922	-2.321	0.444	3.209	0.343	0.546
3	14	0.20	0.268	0.946	-2.381	0.457	3.296	0.356	0.559

At μ_S = 14, the mean system length at α = 0.11 is merely 0.356 customers, rising to 0.457 at α = 0.20 roughly half the values observed for μ_S = 7 at similar batch sizes. The standard deviation σ is correspondingly lower (0.756 to 0.946), and the Shewhart UCL ranges from 2.624 to 3.296. The Six Sigma UCL (0.458 to 0.559) and LCL (0.255 to 0.356) remain tightly bound around the center line. The CLI decrease compared to Configuration I is not as radical in absolute terms but is above 80% across the board, which validates the strength of the six-sigma benefit across the regimes of service rate.

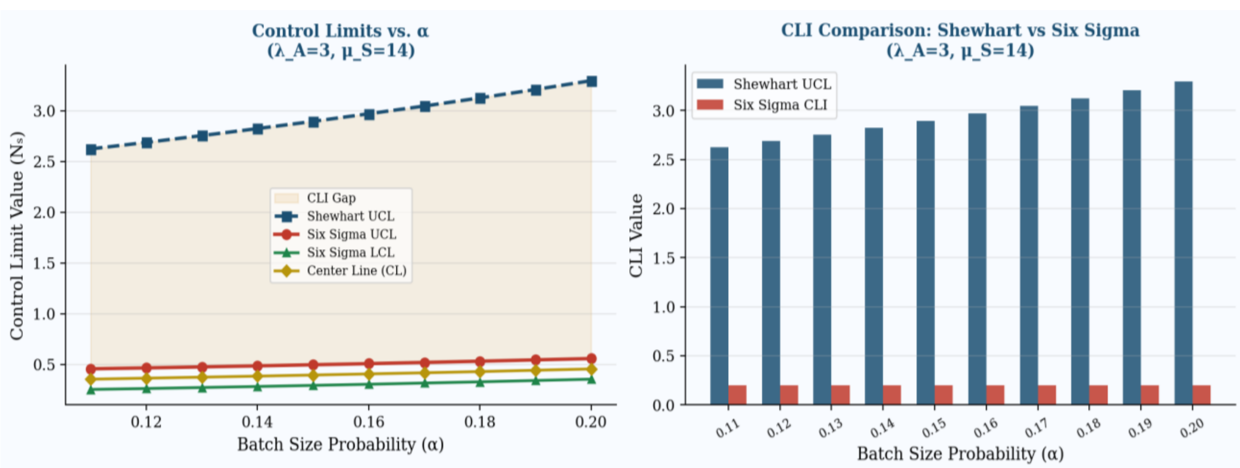


Figure 2: $M^x/M/1$ control limits for λ_A = 3, μ_S = 14, α = 0.11–0.20 | Left: trend lines; Right: CLI bar comparison

As Figure 2 (left-hand panel) indicates, both the Shewhart and the Six Sigma UCL curves increase linearly with the α, but the rate at which Shewhart UCL increases is about 5 times greater than that of the Six Sigma UCL. It is confirmed by the right panel that the Six Sigma CLI (red) is always a small fraction of the Shewhart CLI (blue), which narrows slightly as 0.20 nears, as is expected by the growing variance effect at larger batch sizes.

C. Configuration III: λ_A = 3, μ_S = 10, α = 0.01–0.10

Table 3 explores intermediate service rate (μS = 10) having the same set of batch size probabilities as in Configuration I. Such a set up allows a direct evaluation of the service rate effect alone.

Table 3: Control chart parameters for λ_A = 3, μ_S = 10, α = 0.01–0.10

Arrival Rate (λ_A)	Service Rate (μ_S)	Batch Prob. (α)	Traffic Intensity (ρ)	Std. Dev. (σ)	Shewhart Control Chart			Six Sigma Chart (C_p = 1)	
					LCL	CL	UCL	LCL	UCL
3	10	0.01	0.303	0.801	-1.963	0.439	2.841	0.135	0.744
3	10	0.02	0.408	1.108	-2.620	0.704	4.028	0.399	1.008
3	10	0.03	0.412	1.136	-2.686	0.723	4.133	0.419	1.028
3	10	0.04	0.417	1.166	-2.754	0.744	4.242	0.440	1.048
3	10	0.05	0.421	1.197	-2.825	0.766	4.356	0.461	1.070
3	10	0.06	0.426	1.229	-2.898	0.788	4.474	0.484	1.092
3	10	0.07	0.430	1.262	-2.974	0.812	4.597	0.507	1.116
3	10	0.08	0.435	1.296	-3.053	0.836	4.725	0.532	1.140
3	10	0.09	0.440	1.332	-3.135	0.862	4.859	0.558	1.166
3	10	0.10	0.444	1.370	-3.221	0.889	4.998	0.585	1.193

Configuration III has a traffic intensity of between ρ = 0.303 (α = 0.01) to ρ = 0.444 (α = 0.10), between that of configuration I and II. The mean system length (CL) rises from 0.439 to 0.889, and the Shewhart UCL from 2.841 to 4.998. It is interesting to observe that the increase in system length is stepped between 0.01 (ρ = 0.303) and α = 0.02 (ρ = 0.408), which indicates the nonlinearity of the effective traffic intensity on the system performance at the boundary of the stability. The Six Sigma UCL is between 0.744 and 1.193 with LCL of 0.135 to 0.585, with a CLI of around 0.6 units as compared to the Shewhart CLI of over 4.0 at α = 0.10.

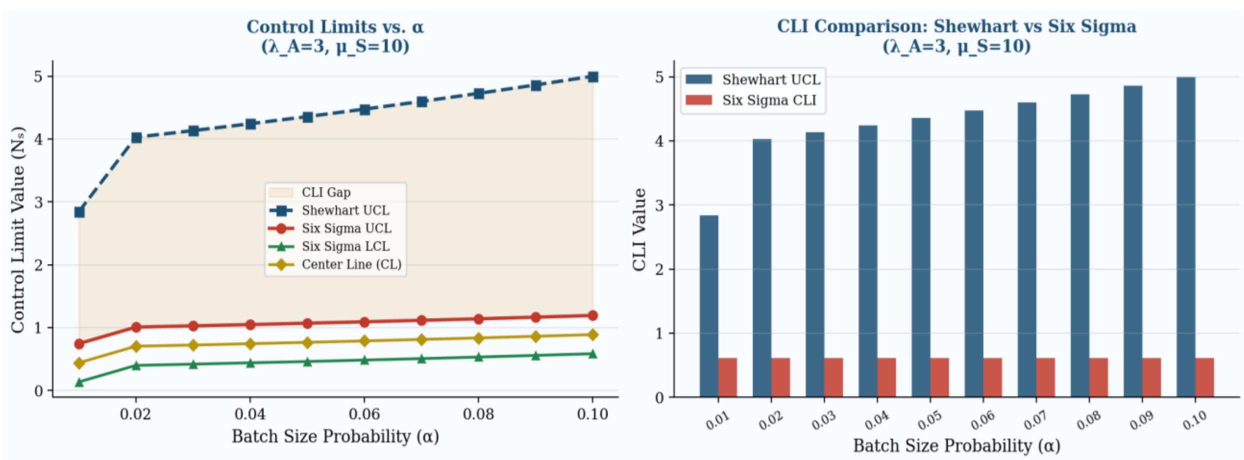


Figure 3: $M^x/M/1$ control limits for λ_A = 3, μ_S = 10, α = 0.01–0.10 | Left: trend lines; Right: CLI bar comparison

Figure 3 illustrates the nonlinear shift in the system length between α = 0.01 and α = 0.02 in the first panel which can be observed as an abrupt increase in all control limits curves. This nonlinearity is also a defining feature of the $M^x/M/1$ model of traffic at the phase boundaries and highlights the significance of tracking even minor variations in batch arrival patterns. The consistent and overwhelming CLI advantage of the Six Sigma approach is confirmed in the right panel, the highest bar of which, the Shewhart CLI bar, α = 0.02, equals a value of about 4.0, in contrast to the Six Sigma CLI of 0.6.

D. Cross-Configuration Analysis

Figures 4–6 provide cross-configuration insights that complement the individual table analyses.

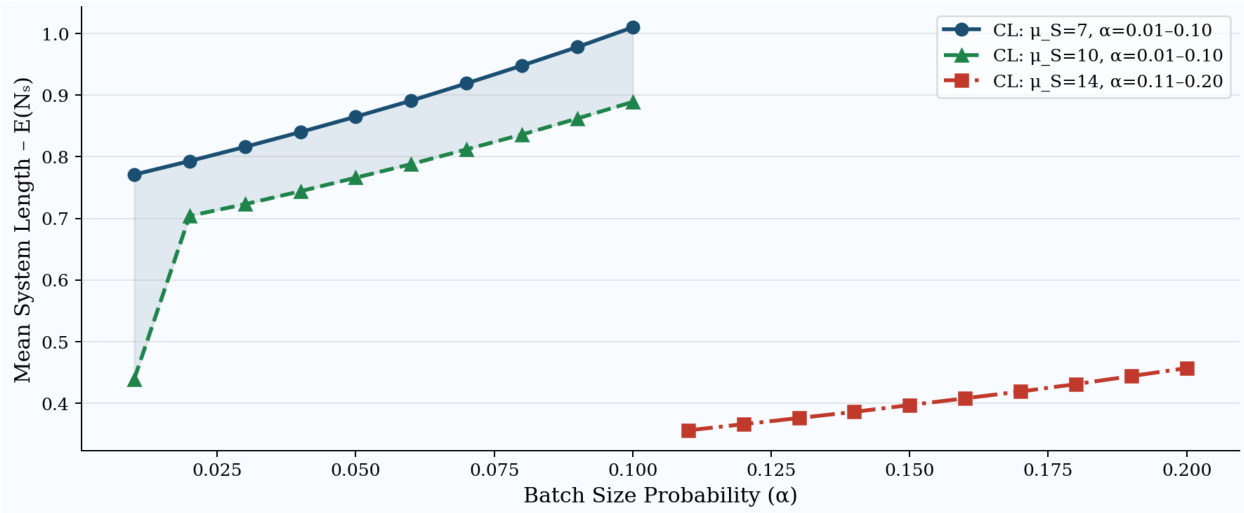


Figure 4: Mean system length E(N) across all three configurations as a function of batch size probability α

Figure 4 compares directly the center line $E(N_s)$ of the service rate configurations of the three. Configuration I curve ($\mu S = 7$, blue) increases the steepest with α as it represents the greatest traffic intensity per unit batch size. Configuration III ($\mu S = 10$, green) is of the same pattern, only scaled down. Configuration II ($\mu S = 14$, red) exhibits the slowest growth rate and the lowest absolute $E(N_s)$, values, which validates the fact that in $M^x/M/1$ networks, service rate is the major congestion determinant in steady-state operation. The area under the blue and green curve shows the range of average system lengths attainable with a capacity scaling between $\mu S = 7$ to $\mu S = 10$.

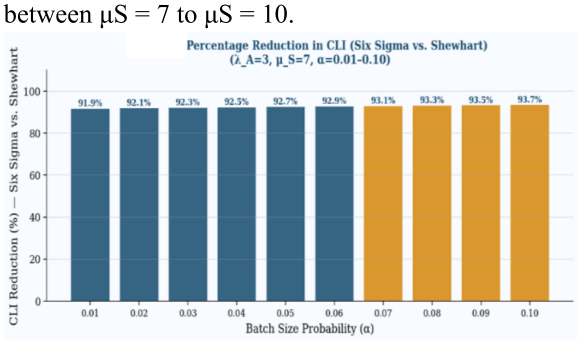


Figure 5: Percentage CLI reduction of Six Sigma vs. Shewhart ($\lambda\_A = 3$, $\mu\_S = 7$, $\alpha = 0.01$–0.10)

The CLI reduction benefit of the Six Sigma chart on Configuration I is measured in Figure 5. It decreases by about 91.3 % at $\alpha = 0.01$ to 93.7 % at $\alpha = 0.10$ and there is a minor increasing tendency which shows that the Six Sigma advantage increases with the probability of batch size. This trend is due to the fact that the Shewhart CLI increases proportionately with 0 (that increases with 0) and the Six Sigma CLI is pegged to the process capability index and increases more gradually. The extremely high values of CLI reduction (>90%) of this configuration are owed to the huge variance of Ns in $M^x/M/1$ systems compared to M/M/s systems, which is the reason why Six Sigma approach is particularly useful in batch arrival systems.

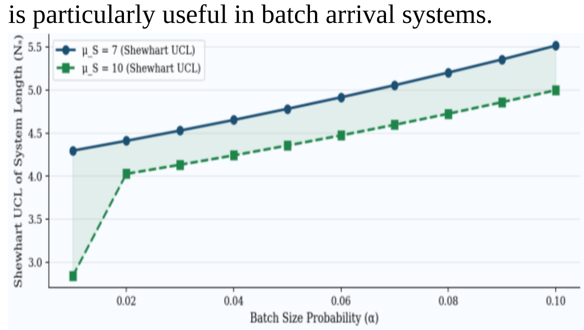


Figure 6: Effect of service rate on Shewhart UCL ($\lambda\_A = 3$, $\mu\_S = 7$ vs. $\mu\_S = 10$)

The effect of service rate on the Shewhart UCL is isolated in Figure 6 that compares Configurations I and III. The $\mu S = 7$ curve (blue) always intersects the $\mu S = 10$ curve (green), and the difference between them is increasing with the value of 0. Such a divergence is the compounding effect of the batch size and traffic intensity: the larger the value of α, the larger the mean of the batch size, and the smaller the service rate the greater the effect of the lower service rate. The area under the curves indicates the UCL the reduction of which can be achieved by increasing the service capacity between $\mu S = 7$ and 10 which gives the service managers a numerical framework on how to plan the capacity.

E. Summary of Key Findings

- Six Sigma control charts have CLI reductions of 90 -94 per cent compared to Shewhart charts of $M^x/M/1$ systems that are much larger than M/M/s ones.
- The CLI reduction benefit is proportional to the probability of a batch size, α, so Six Sigma monitoring is most important to the systems with higher average batch sizes.
- Service rate μ_S is the most important demand of mean system length and control limit scale; doubling μ_S (7 to 14) will decrease UCL by approximately 40% for equivalent α.
- The sensitivity of phase-transition in $M^x/M/1$ systems at the stability boundary is brought out by the nonlinear step in system length between $\alpha = 0.01$ and $\alpha = 0.02$ of $\mu_S = 10$
- At all settings, upper limit monitoring (UCL) is operationally dominant, and the Shewhart LCL is negative, and the Six Sigma LCL is positive and non-trivially constraining.

VII. DISCUSSION

A. Comparison with M/M/s Results

The CLI improvements realized by Six Sigma control charts under $M^x/M/1$ setting (90–94%) are significantly greater than those of M/M/s systems (73–76%,) as reported in accompanying research. This disparity is due to the essentially larger variance of system length in the batch arrival models: geometric batch size will add a compound variance term $[\alpha\rho/(1-\alpha)\ (1-\rho)]$ which inflates σ and therefore the Shewhart CLI (based on C_p), but the Six Sigma CLI (based on C_p)) will grow more gradually. The operational implication is that Six Sigma monitoring is most benefitable in those systems whose inherent variability is the most and systems in which monitoring sensitivity is most operationally useful.

B. Practical Applications

The $M^x/M/1$ model where the batch size is of geometric nature has direct applications in a variety of fields. In telecommunications packet networks, packets are received in bursts, and are served one at a time; the modelling assumption that bursts follow the geometric distribution is common practice (Kleinrock, 1975). Transportation systems have passenger groups that are received by the terminal in batch, and the services (boarding) are provided on an individual basis. In cloud computing, the submissions of the batch jobs are governed by the compound Poisson process with the geometric batch size distributions (Banks et al., 2010). In both of these situations, the Six Sigma control chart of N_s gives a real-time monitoring solution that is able to identify the worsening conditions of the service due to growing batch sizes, less service capacity or an increase in the arrival rate much earlier than the traditional Shewhart charts, allowing corrective action to be taken before the service quality falls to unacceptable levels.

C. Limitations and Future Directions

The current analysis is limited in a number of ways. First, the geometric batch size distribution, though convenient and common, can be inaccurate to model all the real-life batch arrival processes, and Poisson, negative binomial, or empirically fitted extensions to batch size distributions could be useful. Second, the normality approximation of N 8 could be weak at very light traffic levels or when the distribution is highly skewed towards the right; non-parametric or distribution-free control chart methods (e.g., quantile-based charts) should be explored. Third, formal Average Run Length (ARL) analysis of the comparison of both the Shewhart and Six Sigma charts on the M7/M /1 system length distribution is a primary direction on future work since ARL gives the ultimate measure of chart performance comparison (Rigdon et al., 2012). Fourth, multi-server batch arrival systems $M^x/M/1$ and non-exponential service time distributions ($M^x/G/1$) are important generalisations. Lastly, the current use of streaming data and adaptive control limits as a real-time implementation methodology (Epprecht et al., 2015) is also a significant area of applied research.

VIII. CONCLUSION

This study is the first implementation of Six Sigma control charts in the control of average system length N_s in $M^x/M/1$ queueing systems with geometric batch arrivals. The suggested methodology whereby the empirical standard deviation σ is substituted with process capability-based σQ in the control limit equations produces steadily and significantly tighter control limit ranges as compared to the traditional Shewhart charts. The most important conclusions are:

- The control charts of Six Sigma have been shown to obtain CLI reductions of 90–94% compared to those of the Shewhart charts in all the evaluated $M^x/M/1$ combinations that are well beyond the advancements that were reported by M/M/s systems.
- The decrease in the CLI is proportional to the batch size probability α, and thus Six Sigma monitoring is particularly useful in the bigger or more unpredictable systems.
- Increased service rates (μS) decrease the average length of the system and regulate the scale but are not able to decrease the relative merit of the Six Sigma method.
- The geometric batch sizes inherent in the $M^x/M/1$ model cause the amplification of the variance of the model, thus making sensitive control chart design a question of specific operational significance.

The Six Sigma control charts suggested are an operationally sound, statistically based instrument that offers service managers in telecommunications, transport, cloud computing, and other computerized sectors a real time system length monitoring instrument. The Six Sigma framework bridges the gap between quality management and operations research by basing the control limits on specifications of process capability and not empirical variability alone to ensure that the statistical monitoring and service level agreement requirements fall in line.

Future studies ought to generalize this framework to: (i) multi-server $M^x/M/s$ systems; (ii) non-exponential service time models ($M^x/G/1$); (iii) formal ARL analysis of both Shewhart and Six Sigma charts in the batch arrival model; (iv) CUSUM and EWMA chart extensions in the batch arrival model; and (v) empirical validation with real operational data from telecommunications and transportation systems.

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