A Study on the Concept and Integrability Conditions of Riemann Integration

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Abstract— **Bernhard Riemann integration constitutes one of the foundational concepts of mathematical analysis and plays a central role in the development of modern calculus, functional analysis and applied mathematics. This paper presents a comprehensive study of the theory and applications of Riemann integration, focusing on its conceptual framework, integrability conditions and mathematical significance. The study examines the construction of the Riemann integral through partitions, upper and lower sums and the limiting process that establishes the existence of an integral for bounded functions defined on closed intervals. Particular emphasis is given to the criteria for Riemann integrability, including continuity, monotonicity and the characterization of discontinuous functions.**

By combining theoretical exposition with illustrative examples and applications, this study aims to provide a deeper understanding of Riemann integration and its enduring importance in pure and applied mathematics. The findings underscore the relevance of Riemann integration not only as a classical analytical tool but also as a gateway to advanced topics in real analysis, measure theory, differential equations and mathematical modeling.

Index Terms— **Riemann integration, monotonicity, measure theory, differential equation**

I. INTRODUCTION AND CONCEPT

Scope of Riemann Integration: The scope of Bernhard Riemann integration is fundamental yet extensive within the domain of classical analysis and forms the basis of integral calculus in mathematics. It primarily deals with the integration of bounded real-valued functions defined on closed and compact intervals of the real line. Through the framework of partitions, upper and lower sums, Riemann integration provides a rigorous method for determining the area under curves and serves as a foundational tool for understanding accumulation processes in mathematics.

Within real analysis, its scope extends to continuous functions, monotonic functions and functions with a finite number of discontinuities, all of which are Riemann integrable under standard conditions. It also plays a crucial role in establishing the Fundamental Theorem of Calculus, thereby linking differentiation and integration in a unified theoretical structure. This connection makes it indispensable for the study of differential equations, approximation techniques, and mathematical modeling in physics and engineering.

In applied mathematics, Riemann integration is widely used in computing physical quantities such as displacement, work, mass distribution, probability in elementary settings, and geometric measures like area and volume. It also serves as a preparatory framework for more advanced integration theories, particularly Henri Lebesgue integration, which generalizes its concepts to a broader class of functions and spaces.

Thus, the scope of Riemann integration lies not only in its direct applications but also in its foundational role in developing modern mathematical analysis. It remains a critical stepping stone for students and researchers progressing toward advanced topics such as measure theory, functional analysis, and abstract integration frameworks.

For understanding Riemann integration, we first need some basic definitions and terminology.

Let $a, b \in \mathbb{R}$ with $a < b$. A partition of the interval $[a, b]$ is a set $P = \{x_0, x_1, x_2, \ldots, x_n\}$ of finitely many points of $[a, b]$ such that

$a = x_0 < x_1 < x_2 < \cdots < x_n = b$

In general, let $P = \{x_0, x_1, x_2, \ldots, \}$ be any partition of $[a, b]$, where

$a = x_0 < x_1 < x_2 < \cdots < x_n = b$

We call $[x_{i-1},]$ the i^{th} subinterval of the partition P. Its length $= x_i - x_{i-1}$ is denoted by δ_i or

Δ. $\| P \|$= max $\{\delta_1, \delta_2,,\}$

Given any bounded function f: $[a, b] \to \mathbb{R}$, let us define

(f): = inf $\{f(x) \mid x \in [x_{i-1}, x_i]\}$

and $M_i(f)$: = sup$\{f(x) \mid x \in [x_{i-1}, x_i]\}$, i = 1,2, …, n

Since f is bounded function on $[a, b]$, we define

(f): = inf$\{f(x) \mid x \in [a, b]\}$

and $M(f)$: = sup$\{f(x) \mid x \in [a, b]\}$.

Note that $(f) \leq m(f) \leq M_i(f) \leq M(f)$ for each $i \in$ 1,2, …, n

II. RIEMANN SUM

Given a partition P = c of $[a, b]$ and a bounded function f: $[a, b] \to \mathbb{R}$, we define the lower sum of f w.r.t P to be

(P, f): = $\sum_{i=1}^{n} m_i(_i - x_{i-1})$

and the "upper sum" of f w.r.t P to be

(P, f): = $\sum_{i=1}^{n} M_i(_i - x_{i-1})$

Results: - Let f: $[a, b] \to \mathbb{R}$ be a bounded function.

(i) If P is a partition of $[a, b]$ and P^* is a refinement of P, i.e., $P \subseteq P^*$,

then $L(P, f) \leqslant L(P^*, f)$ & $U(P, f) \geqslant U(P^*, f)$. In short, lower sums increase and the upper sums decrease as the partition becomes finer and finer.

(ii) If P_1, P_2 are any two partitions of $[a, b]$, then $(P_1, f) \leqslant (P_2, f)$.

(iii) Let P be any partition of $[a, b]$. Then

$m(b - a) \leq L(P, f) \leq U(P, f) \leq M(b - a)$

RIEMANN INTEGRABILITY:

Definition: - Let f: $[a, b] \to \mathbb{R}$ be a bounded function and $P = \{x_0, x_1, x_2, \dots, x_n = b\}$ be a partition of $[a, b]$. The lower Riemann Integral of f on $[a, b]$ is defined as Sup $\{(p, f) \mid P$ is a partition of $[a, b]\}$, and is denoted by $\int a\ b\ (x)dx$. i.e.,

$\int_{-a}^{b} \mathrm{f(x)}\ dx$ = Sup $\{(p, f) \mid P$ a partition of $[a, b]\}$ = Sup$_P$ (P, f)

The upper Riemann Integral of f on $[a, b]$ is defined as Inf $\{(p, f) \mid P$ a partition of $[a, b]\}$, and is denoted by $\int_{a}^{\overline{b}} \mathrm{f(x)}\ d$. i.e.,

$\int_{a}^{\overline{b}} \mathrm{f(x)}\ dx$ = Inf $\{(P, f) \mid P$ a partition $\{$[a, b]$\}$ = Inf$_P$ (P, f)

III. DEFINITION

Riemann Integrable- A bounded function f: $[a, b] \to \mathbb{R}$ is said to be "Riemann Integrable" over $[a, b]$ if

$\int_{\overline{a}}^{b} \mathrm{f(x)}\ dx = \int_{a}^{\overline{b}} \mathrm{f(x)}\ dx$,

and it is denoted by $\int_{a}^{b} \mathrm{f(x)}\ dx$

Remarks:

1- If f: $[a, b] \to \mathbb{R}$ is a bounded function

then $\int_{\overline{a}}^{b} \mathrm{f(x)}\ dx \leq \int_{a}^{\overline{b}} \mathrm{f(x)}\ dx$

2- A bounded function f is Riemann integrable on $[a, b]$

$\Leftrightarrow \int_{\overline{a}}^{b} \mathrm{f(x)}\ dx = \int_{a}^{\overline{b}} \mathrm{f(x)}\ dx$

3- If a bounded function f is such that $\int_{\overline{a}}^{b} \mathrm{f(x)}\ dx \neq \int_{a}^{\overline{b}} \mathrm{f(x)}\ dx$, then f is not Riemann integrable on $[a, b]$.

A Necessary and Sufficient condition for Integrability

A bonded function f: $[a, b] \to \mathbb{R}$ is Riemann integrable on [a, b] if and only if for each $\epsilon > 0\ \exists\ a$ partition of P of [a, b] such that $0 \leq (P, f) - (P, f) < \varepsilon$.

Proof: -

Necessary part: Let f be Riemann integroble on [a, b]

$\therefore \int_{\overline{a}}^{b} \mathrm{f(x)}\ dx = \int_{a}^{\overline{b}} \mathrm{f(x)}\ dx = \int_{a}^{b} \mathrm{f(x)}\ dx$… (1)

Let $\epsilon > 0$. By Darboux's theorem $\exists\ \delta > 0$ such that and

$(P, f) < \int_{a}^{\overline{b}} \mathrm{f(x)}\ dx + \epsilon/2$… (2)

$(P, f) > \int_{\overline{a}}^{b} \mathrm{f(x)}\ dx - \epsilon/2$………… (3)

for each P, $\in [a, b]$ with $\| P \| < \epsilon$

From (1) & (2),

We get $(p, f) < \int_{a}^{\overline{b}} \mathrm{f(x)}\ dx + \epsilon/2$

Again, from (1) & (3), we have $\int_{\overline{a}}^{b} \mathrm{f(x)}\ dx < (P, f) + \epsilon\ 2$

$\therefore (r, f) < ((p, f) + \epsilon/2) + \epsilon/2$

$\Rightarrow (P, f) - (P, \mathrm{f}) < \epsilon$

Also, $(P, f) - (P, \mathrm{f}) \geqslant 0$.

$\therefore 0 \leq (P, f) - (P, \mathrm{f}) < \varepsilon$.

SUFFICIENT PART:

Let for each $e > 0\ \exists\ P \in$ [a, b] such that

$0 \leq (P, f) - (P, f) < \varepsilon$

By definition,

$\int_{a}^{\overline{b}} \mathrm{f(x)}\ dx$ = inf $\{(P, f) \mid P \in \wp\ [[a, b]\}$

$\Rightarrow \int_{a}^{\overline{b}} \mathrm{f(x)}\ dx \leqslant U(P, f)$.

By definition

$\int_{\overline{a}}^{b} \mathrm{f(x)}\ \mathrm{dx}$ = sup $\{(P, f) \mid P \in \wp\ [a, b]\}$

$\Rightarrow \int_{\overline{a}}^{b} \mathrm{f(x)}\ \mathrm{dx} \geq (P, f)$

$\Rightarrow -\int_{a}^{b} \mathrm{f(x)}\ dx \leq -L(P, f)$

$\therefore 0 \leq \int_{a}^{\overline{b}} \mathrm{f(x)}\ dx - \int_{\overline{a}}^{b} \mathrm{f(x)}\ \mathrm{dx} \leqslant (P, f) - (p, f) < \varepsilon$

$\therefore$ For each $\epsilon > 0$, we have $0 \le \int_{a}^{\overline{b}} f(x)\, dx - \int_{\overline{a}}^{b} f(x)\, dx < e$

$\Rightarrow \int_{a}^{\overline{b}} f(x)\, dx = \int_{\overline{a}}^{b} f(x)\, dx$.

$\therefore f$ is Riemann integrable on $[a, b]$.

IV. LIMITATIONS OF RIEMANN INTEGRATION

Despite its fundamental importance in classical analysis, Bernhard Riemann integration possesses several limitations that restrict its applicability in advanced mathematical analysis. One of the major limitations is that the Riemann integral can only integrate functions that are bounded on a closed interval. Functions exhibiting extensive discontinuities or highly irregular behavior often fail to satisfy the conditions of Riemann integrability.

Another significant limitation is its inability to effectively handle functions with infinitely many discontinuities, even when such functions may still possess meaningful areas under their curves. For example, certain discontinuous functions that are not Riemann integrable can nevertheless be integrated under the more generalized framework of Henri Lebesgue integration. This demonstrates that the Riemann approach is comparatively restrictive in dealing with complex measurable functions.

Furthermore, Riemann integration relies heavily on interval partitions, making it less suitable for higher-dimensional analysis, abstract spaces and modern probability theory. In advanced branches of mathematics such as measure theory, functional analysis and stochastic processes, the Riemann integral lacks the flexibility and generality required for rigorous treatment of convergence and measurable functions.

The theory also encounters difficulties with limit operations involving sequences of functions. Important convergence theorems, such as the Dominated Convergence Theorem and Monotone Convergence Theorem, cannot be fully developed within the Riemann framework. As a result, mathematicians developed more comprehensive integration theories to overcome these shortcomings.

Therefore, although Riemann integration remains an essential foundation of classical calculus and real analysis, its limitations reveal the necessity for broader and more powerful integration methods in modern mathematical research.

V. CONCLUSION

In conclusion, Bernhard Riemann integration remains one of the most fundamental and influential concepts in mathematical analysis. This study has examined the theoretical foundations of the Riemann integral, including its construction through partitions and sums, the conditions for integrability and its essential properties. The discussion demonstrates how Riemann integration provides a rigorous mathematical framework for measuring areas, analyzing continuous and bounded functions. The research further highlights that although the Riemann integral effectively handles a wide class of functions encountered in classical calculus and applied mathematics, it also possesses certain limitations when dealing with highly discontinuous or irregular functions. These limitations ultimately contributed to the development of more generalized theories such as Henri Lebesgue integration. Nevertheless, the Riemann approach continues to serve as the conceptual and educational foundation for advanced studies in real analysis, functional analysis, differential equations, and mathematical modeling.

Overall, the significance of Riemann integration extends far beyond elementary calculus, as it represents a critical step in the evolution of modern mathematical thought. Its clarity, logical structure, and wide applicability ensure its continued relevance in both theoretical research and practical scientific applications. Therefore, the study of Riemann integration not only strengthens the understanding of classical analysis but also provides an essential bridge to higher mathematical theories and contemporary analytical methods.

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