

# Developing Mathematical Models That Incorporate the Decline in Quality of Inventory Items, Where Demand Is Influenced by Both Price and Stock Levels, And Also Considers the Impact of Quantity Discounts

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**Abstract**—This study develops an integrated deterministic inventory model for a two-echelon supply chain consisting of a single manufacturer and a single retailer dealing with deteriorating items having a fixed lifetime. The proposed model incorporates advertisement- and stock-dependent demand, preservation technology investment, shortages with partial backordering, and quantity discount policies. Two scenarios are analyzed: (i) a system without quantity discount and (ii) a coordinated system where the retailer adjusts order quantities in exchange for quantity discounts offered by the manufacturer. Preservation technology is employed to reduce the deterioration rate and improve inventory performance. The objective is to minimize the total integrated supply chain cost by optimizing the backorder rate and preservation investment. Mathematical formulations are developed for both manufacturer and retailer, and the joint total cost functions are derived. Computational algorithms are proposed to obtain optimal solutions for the decision variables. Numerical results demonstrate that implementing a quantity discount policy significantly reduces preservation costs, backorder rates, and overall system costs. Sensitivity analysis further reveals the effects of deterioration rate, preservation efficiency, advertisement frequency, and holding costs on system performance. The findings indicate that coordinated quantity discount strategies combined with optimal preservation investment can enhance supply chain efficiency and profitability while reducing inventory-related costs.

**Index Terms**—Integrated inventory model, Deteriorating items, Quantity discount, Preservation technology, Stock-dependent demand, Advertisement-dependent demand, Backordering, Supply chain coordination.

## I. INTRODUCTION

A supply chain contains different business players like supplier, manufacturer, distributor, retailer, customer, who work together to improve sustainability. Goyal [11] developed the first integrated model for a single supplier and a single customer. Banerjee [1] jointly optimized ordering policy so that either both parties get benefit or, at least, no one incurs losses. Goyal and Gunasekaran [10] extended that model for deteriorating items. Rau *et al.* [18] extended the same model for a single supplier, single producer, and a single buyer. Crdenas-Barrn [2] solved vendor-buyer model with arithmetic and geometric inequalities. Sarkar *et al.* [22] formulated an integrated inventory model for defective items with payment delay scenario.

Break-even point of fixed and variable costs allows manufacturer to enjoy better profit on large lots. This large lots are offered to a retailer by offering quantity discount to accelerate overall demand. This gives a win-win situation both to manufacturer and retailer. A first model using quantity discount policy for increasing vendor's profit is developed by Monahan [15]. Chang [4] *et al.* extended the model for deteriorating items with price and stock dependent demand. Duan *et al.* [7] derived a model for fix life product and proved theoretically that after applying quantity discount, total cost was reduced. Zhang *et al.* [27], Ravithammal *et al.* [19], Ravithammal *et al.* [20], Pal and Chandra [17], Sarkar [21] extended that model by taking different assumptions to make it more realistic.

Ghare and Schrader [8] were the first who formulated a model for inventory that deteriorate exponentially. Murr and Morris [16] proved that lower temperature would increase storage time and decrease decay. So, as per this fact, preservation technology is used to reduce deterioration rate of items because higher rate of deterioration finally results into lower revenue generation. Hsu *et al.* [12] applied preservation technology on constantly deteriorating items to increase total profit. Chang [3] used preservation technology on non-instantaneous deteriorating items. Singh and Rathore [26] extended this model for shortages with the proposal of trade credit. Shah *et al.* [25] developed an integrated model by using preservation technology on time-varying deteriorating items when demand is time and price sensitive. Mishra *et al.* [14] applied preservation technology on seasonal deteriorating items in the presence of shortages.

In the classical EOQ models, demand is taken as constant. But researchers have always investigated parameters that affect demand as stock-level, time, price, advertisement, and trade credit. Khouja and Robbins [13], Shah and Pandey [23], Giri and Maiti [9], Chowdhury *et al.* [5], Shah [24], Chung and Crdenas-Barrn [6] etc. used different types of demand and developed their inventory models.

The proposed model works on single set-up multiple deliveries with just-in-time replenishment for deteriorating items that have a fix life time. Here, we develop two models: Model 1 (without quantity discount), and Model 2 (with quantity discount).

In the second model a retailer agrees to change his/her order according to manufacturer's output. In response, the retailer gets benefit of quantity discount from the manufacturer. Whereas there is no such an agreement, advertisement and stock dependent demand are considered to boost the demand. Preservation technology is used to reduce the rate of deterioration. Total inventory cost of supply chain is optimized for decision variables back-order rate ( $k$ ) and preservation cost ( $\zeta$ ). Both the models are optimized analytically and computational algorithms have been developed for the same. The obtained solutions are illustrated on a numerical example.

## II. LITERATURE REVIEW

Baker and Urban [3] developed deterministic and continuous inventory models in which the demand rate

depends upon the inventory, and the demand rate for the item has a polynomial functional form. Mandal and Phaujdar [9] determined a uniform rate of production and stock-dependent demand for deteriorating items, where shortages are permitted, and excess demand creates a backlog. Pal *et al.* [11] determined a stock-dependent inventory model with a constant deterioration rate, determining the average net profit  $\pi$  over one production run, and optimizing the decision variables  $Q$  (initial stock) and  $T$  (duration of a production cycle). In 1995, Chung *et al.* [4] proposed inventory models for deteriorating items with stock-dependent sales rates and derived profit functions without backlogging and with complete backlogging. They explore the efficient use of the Newton–Raphson method when finding optimum solutions for-profit functions per unit time modeled in both contexts. Srivastava and Gupta [23] developed an infinite time-horizon inventory model for deteriorating items, assuming that the demand rate is constant for some time and then a linear function of time. They obtain the theoretical expressions for the optimal inventory level and total average cost, as a function of the sales price and time-dependent holding cost. Ajanta Roy [13] developed an inventory model with a time-proportional rate of deterioration and demand. Vinod Kumar *et al.* [10] developed an inventory model with time-dependent demand, and a time-varying holding cost, and time-proportional deterioration. The model allowed for inventory shortages with partial backlogging. An EOQ inventory model was developed by Shukla *et al.* [19] with quadratic demand that permitted both payment delay and shortages. Tripathi and Manjit Kaur [24] considered an inventory model with deteriorating items to be a phenomenon that cannot be overlooked, as failing to consider such context may provide an absurd result. In the high-tech business industry, deterioration is not necessarily constant, but rather time-dependent.

Trailokyanath Singh *et al.* [20] introduced a model of the EOQ for deteriorating items with a deterioration rate proportional to time, a time-demand ramp-type demand rate, and shortages. The model allowed completely backlogged shortages and the ramp-type demand rate is deterministic, changing over time to a certain point and then constant. Saha and Sen [14] proposed an inventory model with a sale price and time-dependent demand, a constant holding cost, and

time-dependent deterioration. Shaikh *et al.* [16] considered purchase cost irrespective of the order size and carrying cost over the entire cycle period, treating deterioration as another imperative issue in inventory analysis, because of its huge impact on the profit or cost of the inventory system. Considering these factors, they developed two different inventory models: (a) an inventory model for a zero-ending case and (b) an inventory model for a shortage case. Tripathi *et al.* [25] established an inventory model of exponential time-dependent demand and time-dependent deterioration. The model accounts for shortages and considers a proportionate demand rate and unit production cost. Khedlekar *et al.* [8] attempted to develop a method of optimization an economic production quantity model for deteriorating items with production disruption. They determined the optimum production time before and after the system disruption. They have also developed a model that optimizes product shortages, which is useful for determining the time to start and finish production when the system gets disrupted. Shaikh *et al.* [17] developed an inventory model based on the sale price of deteriorating items with variable-demand allowing for shortages and the advertising of items under financial trade credit policy. Sivasankari [22] proposed an inventory model for deteriorating items with under-inflation time-dependent exponential demand and developed an optimal solution from higher order equations. Pervin *et al.* [12] formulated an inventory model for deteriorating items. To minimize the rate of deterioration, they apply preservation technologies and quantify the level of expenditure on preservation technology. In their model, the demand function depends on the stock level and price, and the production rate is linearly time-dependent, based on consumer demand with shortages permitted. Sarkret *et al.* [15] investigated an integrated vendor-buyer model with shortages under a stochastic lead time, which is assumed to be variable but depends on the buyer's order size and the vendor's production rate; the proposed model determines the net present value of the expected total cost. Ahmad and Benkherout [2] proposed a procedure for determining the optimal replenishment policy for a basic inventory model of stock-dependent demand items, non-instantaneous deteriorating items, and partial backlogging. Dong *et al.* [5] proposed a problem that focuses on determining the optimal price of the existing product and the

inventory level for the new product. Inspired by practice, the problem considers various strategies for the existing product and the cross-elasticity of demand for existing and new products.

### III. NOTATIONS AND ASSUMPTIONS

#### 3.1. Notations

##### 3.1.1. Inventory parameters for a manufacturer

$A_m$	Set up costs (\$)
$m_1$	Manufacturer's order multiple in a without quantity discount system
$m_2$	Manufacturer's order multiple in a with quantity discount system
$h_m$	Holding cost / unit / annum
$k_1$	Back-order rate(year) in a without quantity discount system
$k_2$	Back-order rate(year) in a with quantity discount system
$\rho$	Capacity utilization
$P$	Production rate
$D$	Advertisement and stock dependent demand
$C_{io}$	Manufacturer's variable inspection cost per delivery
$C_{imu}$	Manufacturer's unit inspection cost (\$/unit time inspected)
$C_{imf}$	Manufacturer's fix inspection cost (\$/product lot)
$TC_{wm}$	Total cost for a manufacturer in a without quantity discount system
$TC_{qm}$	Total cost for a manufacturer in a with quantity discount system
$Q_m(t)$	Manufacturer's economic order quantity per cycle
$A$	Cost of advertisement
$v$	Frequency of advertisement
$TC_{wr}$	Total cost for a retailer in a without quantity discount system
$TC_{qr}$	Total cost for a retailer in a with quantity discount system
$TC_w$	Joint total cost for a without quantity discount integrated model
$TC_q$	Joint total cost for a with quantity discount integrated model
$Q_r(t)$	Retailer's economic order quantity per cycle
$\tau_p$	Resultant deterioration rate, $\theta - m(\zeta)$
$B(\lambda)$	Discount given by manufacturer if the retailer placed the order each time

3.1.2. Inventory parameters for retailer

$A_r$	Ordering costs (\$)
$n$	Retailer's order multiple in the absence of any co-ordination
$\lambda$	Retailer's order multiple under co-ordination and $\lambda Q_r(t)$ as the retailer's new quantity
$h_r$	Holding cost / unit / annum
$\theta$	Constant deterioration
$\pi$	Retailer's back-order cost
$L$	The maximum life time of a product (in year)
$v$	Rate of change of the advertisement frequency
$a$	Fix demand
$b$	Rate of change of demand
$\xi_1$	Preservative cost to reduce deterioration in a without quantity discount system
$\xi_2$	Preservative cost to reduce deterioration in a with quantity discount system

$m(\xi)$  Reduced deterioration rate Necessary condition for different inventory parameters

$$\rho = \frac{D}{P}; \rho < 1; 0 < \theta < 1; \xi \geq 1$$

3.2. Assumptions

1. This model considers two-echelon form with a single manufacturer and a single retailer for items with expiry date  $L$ -years.
  2. Manufacturer offers quantity discounts if a retailer agrees to change order quantity by the fix order quantity.
  3. Demand is deterministic. Demand function  $D(A, Q)$  is defined as  $D(A, Q) = A^v(a + bQ(t)); 0 \leq t \leq T, \text{ where } a, b \geq 0 \text{ and } a \geq b$
- Where  $A$  =Cost of advertisement;  $v$  = Frequency of advertisement;  $a$  = Fix rate demand;  $b$  = Rate of change of the demand;  $Q$  = Instantaneous stock level for the convince, we use  $D$  for  $D(A, Q)$ .
4. Shortages are allowed and the backorder rate is assumed as a decision variable for a retailer.
  5. Preservation technology is used to control the deterioration rate.
  6. Three level inspections at the manufacturer's end assure no defective items.
  7. Production rate is constant and the lead time is zero.
  8. Items are subject to constant deterioration.

IV. MODEL FORMULATION

In this section, we formulate models that follow a single-setup-multi-delivery (SSMD) policy with just-in-time (JIT) procurement. Here, a manufacturer produces in one set-up but shippes through multiple deliveries after a fixed time. Two integrated models are proposed on the basis of agreement between manufacturer and retailer. Model 1 undertakes no quantity discount as this model assumes no agreement between manufacturer and retailer. Model 2 allows quantity discount as the retailer agrees to order as per the manufacturer production. Shortages are taken with back-order rate ( $k$ ), and preservation technology cost ( $\xi$ ) is assumed in both of the models.

4.1. Model 1: Without quantity discount

In this model, we use preservation technology to control constant deterioration rate. To control deterioration rate, as shown in Figure 1,  $m(\xi)$  is a function of preservation cost  $\xi$  so that,

$$m(\xi) = \theta (1 - \exp(-\eta\xi)); \quad \eta \geq 0$$

where  $\eta$  is the simulation coefficient, representing the percentage increase in  $m(\xi)$  per dollar increase in  $\xi$ .so  $m(\xi)$  is the increasing function which is bounded above by  $\theta$

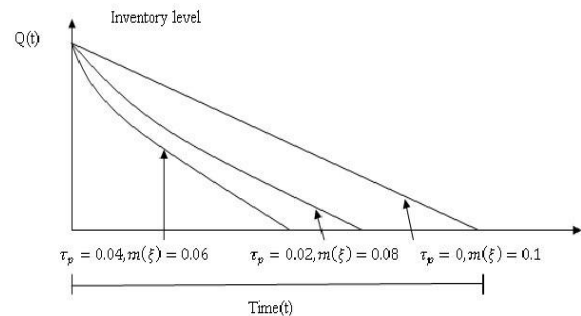


Figure 1: Inventory position for reduced deterioration rate

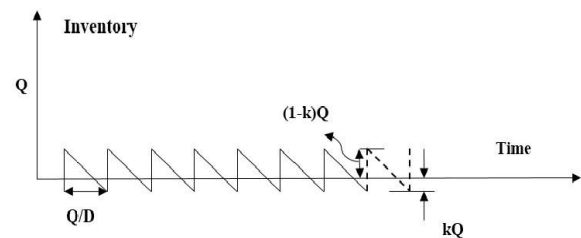


Figure 2: Inventory position for manufacturer

4.1.1. Manufacturer's total cost

Here production rate is constant. So, as shown in Figure 2, with constant supplement manufacturer on hand, inventory at any instant of time t is defined by differential equation.

$$\frac{dQ_m}{dt} + \tau_p Q_m = P; \quad 0 \leq t \leq T \quad (1)$$

Using boundary condition  $Q_m(0) = 0$ , we get a solution to differential equation (1)

$$Q_m(t) = \frac{P}{\theta - m(\xi)} + P e^{(\theta - m(\xi))(-t)} \quad (2)$$

At  $Q_m(T) = Q_m$ , we get a production lot size per cycle

$$Q_m = PT \quad (3)$$

The basic costs are

1. Setup cost:

Constant set up cost

$$SC_m = A_m \quad (4)$$

2. Holding cost:

For the final inventory level, for a manufacturer, it is the difference between the manufacturer's and the retailer's accumulated level.

So, holding cost for a manufacturer is

$$HC_m = \frac{nQ_m[\frac{Q_m}{P} + (m_1 - 1)\frac{Q_m}{D}] - \frac{m_1^2 Q_m^2}{2P} - \frac{Q_m^2(1+2+\dots+(m_1-1))}{2}}{m_1 Q_m} \quad (5)$$

$$HC_m = \frac{h_m[(m_1 - 1)(1 - \rho) + \rho]}{2} \left( \frac{PT}{\theta - m(\xi_1)} - \frac{P(1 - e^{-\theta - m(\xi_1)T})}{\theta - m(\xi_1^2)} \right)$$

3. Inspection cost:

$$IC_m = \frac{a+b(a_1)}{m_1(a_1)} [m_1 C_{io} + m_1(a_1) C_{imu} + C_{imf}] \quad (6)$$

Where,

$$a_1 = \frac{P(1 - e^{-\theta - m(\xi_1)T})}{\theta - m(\xi_1)}$$

Consequently, manufacturer's total cost is

$$TC_{wm}(m_1, \xi_1) = SC_m + HC_m + IC_m \quad (7)$$

Therefore, manufacturer's total cost can be written as

$$\text{Min } TC_{wm}(m_1, \xi_1)$$

Subject to

$$m_1 t \leq L; m_1 \geq 1; \xi_1 \geq 0 \quad (8)$$

Where  $m_1 t \leq L$ , which shows that items are not overdue before they are sold up by the retailer.

4.1.2. Retailer's total cost

Retailer inventory depletes with demand rate  $D$  and resultant deterioration rate  $\tau_p$ . Then retailers on hand inventory at any instant of time is shown in Figure 3 and is defined by the differential equation.

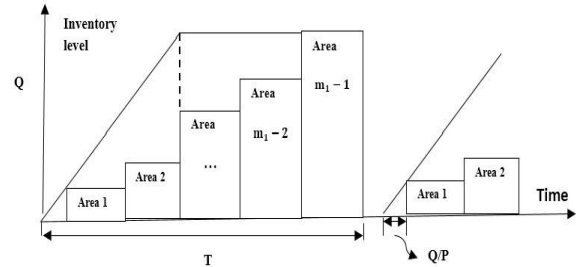


Figure 3: Inventory position for the retailer for the backorder

$$\frac{dQ_r}{dt} + \tau_p Q_r = -D; \quad 0 \leq t \leq 1 - k_1 \quad (9)$$

Using the boundary condition  $Q_r(1 - k_1) = 0$ , we get a solution to differential equation (9)

$$Q_r(t) = \frac{A^v a}{\theta - m(\xi_1) + A^v b} [e^{(\theta - m(\xi_1) + A^v b)(1 - k_1 - t)} - 1] \quad (10)$$

At  $t = 0$ , we get an initial quantity

$$Q_r = \frac{A^v a}{\theta - m(\xi_1) + A^v b} [e^{(\theta - m(\xi_1) + A^v b)(1 - k_1)} - 1] \quad (11)$$

The basic costs are

1. Ordering Cost: Constant set up cost

$$OC_r = nA_r \quad (12)$$

2. Holding cost: The retailer's inventory level in the interval  $[0, 1 - k]$  is given by

$$HC_r = h_r \left[ \int_0^{1-k} t Q_r(t) dt. \right]$$

$$HC_r = \frac{h_r A^v b}{\alpha_2} \left[ -(1 - k_1) \left( \frac{1}{\alpha_2} + \frac{1 - k_1}{2} \right) + \frac{1}{\alpha_2} (e^{\alpha_2(1 - k_1)} - 1) \right] \quad (13)$$

Where  $\alpha_2 = \theta - m(\xi_1) + A^v b$

3. Backorder Cost: The retailer's inventory level in the interval  $[0, k]$  is given by

$$BC_r = \pi \left[ \int_0^k t Q_r(t) dt. \right]$$

$$BC_r = \frac{\pi A^v a}{\alpha_2} \left[ \left( \frac{e^{\alpha_2(1 - 2k_1)}}{\alpha_2} \right) \left( -k - \frac{1}{\alpha_2} \right) + \left( \frac{e^{\alpha_2(1 - k_1)}}{\alpha_2} \right) - \left( \frac{k_1^2}{2} \right) \right] \quad (14)$$

So, the retailer total cost is

$$TC_{wr}(k_1, \xi_1) = OC_r + HC_r + BC_r \quad (15)$$

Therefore, the retailer total cost can be written as

$$\begin{aligned} &MinTC_{wr}(k_1, \xi_1) \\ &Subject\ to\ k_1 \geq 0; \xi_1 \geq 0 \end{aligned} \quad (16)$$

4.1.3. Joint total cost

$$TC_w = TC_{wm} + TC_{wr} \quad (17)$$

4.2. Model 2: With quantity discount

This model follows a strategy that the manufacturer requests the buyer to change his current order size by a factor  $\lambda (> 0)$ , offers to the retailer a quantity discount by a discount factor  $B(\lambda)$ , which the retailer accepts. Thus, the manufacturer's and the retailer's new order quantities are  $\lambda m_2 Q_m$  and  $\lambda Q_r$ , respectively.

4.2.1. Manufacturer's total cost

Manufacturer offer quantity discount to retailer. Total cost for the manufacturer when quantity discount offered by a retailer is

$$\begin{aligned} TC_{qm}(m_2, \xi_2) = &A_m + \frac{h_m[(m_2 - 1)(1 - \rho) + \rho]}{2} \left[ \frac{PT}{b_1} + \frac{P(1 - e^{b_1 T})}{b_1^2} \right] + \\ &\left[ \left( \frac{a + b \left( \frac{P}{b_1} (1 - e^{b_1 T}) \right)}{m_2 \lambda \left( \frac{P}{b_1} (1 - e^{b_1 T}) \right)} \right) (m_2 C_{io} + m_2 \left( \frac{\rho}{b_1 (1 - e^{b_1 T})} \right) C_{imu} + C_{imf}) \right] + DB(\lambda) \end{aligned} \quad (18)$$

Where  $b_1 = \theta - m(\xi_2)$

Thus, the problem can be formulated as

$$\begin{aligned} &MinTC_{qm}(m_2, \xi_2) \\ &Subject\ to\ \lambda m_2 t \leq L; m_2 \geq 1; \xi_2 \geq 0 \\ &\frac{\lambda h_r A^v a}{b_2} \left[ (1 - k) \left( \frac{1}{b_2} - \frac{1 - k}{2} \right) + \frac{(e^{b_2(1-k)} - 1)}{b_2^2} \right] + nA_r - TC_{wr}(k_1, \xi_1) + \\ &\frac{\lambda \pi A^v a}{b_2} \left[ \frac{e^{b_2(1-2k)}}{b_2} \left( -k - \frac{1}{b_2} \right) - \frac{1}{b_2^2} - \frac{k^2}{2} \right] \leq DB(\lambda) \end{aligned} \quad (19)$$

Where  $b_2 = \theta - m(\xi_2) + A^v b$

In equation (19), the first constraint represents that items are not overdue before they are used, and the fourth constraint term  $DB(\lambda)$  represents compensation given by the manufacturer to the retailer.

4.2.2. Retailer's total cost

As per agreement, the retailer changes his order quantity, so according to new quantity and quantity discount, the retailer total cost is

$$\begin{aligned} TC_{qr}(k_2, \xi_2) = &\frac{\lambda h_r A^v a}{b_2} \left[ (1 - k_2) \left( \frac{1}{b_2} - \frac{1 - k_2}{2} \right) + \frac{(e^{b_2(1-k_2)} - 1)}{b_2^2} \right] + \\ &nA_r + \frac{\lambda \pi A^v a}{b_2} \left[ \frac{e^{b_2(1-2k_2)}}{b_2} \left( -k_2 - \frac{1}{b_2} \right) - \frac{1}{b_2^2} - \frac{k_2^2}{2} \right] + Q_m DB(\lambda) \end{aligned} \quad (20)$$

So, the problem is formulated as

$$\begin{aligned} &Min\ TC_{qr}(k_2, \xi_2) \\ &Subject\ to\ k_2 \geq 0; \xi_2 \geq 0 \end{aligned} \quad (21)$$

4.2.3. Joint total cost

$$TC_q = TC_{qm} + TC_{qr} \quad (22)$$

V. COMPUTATIONAL ALGORITHM

1. Set  $m_1 = 1$  in without quantity discount model.
2. Optimize  $k_1$  and  $\xi_1$  simultaneously form  $\frac{\partial TC_{wj}}{\partial k_1}$  and  $\frac{\partial TC_{wj}}{\partial \xi_1}$
3. Take  $m_1 = m_1 + 1$
4. Repeat step 1 to 3 till  $TC_{wj}(m_1 - 1, k_1(m_1 - 1), \xi_1(m_1 - 1)) \geq TC_{wj}(m_1, k_1(m_1), \xi_1(m_1)) \leq TC_{wj}(m_1 + 1, k_1(m_1 + 1), \xi_1(m_1 + 1))$
5. Once optimal  $m_1^*, k_1^*, \xi_1^*$  are calculated, then optimal individual total cost for manufacturer, retailer, and the joint total cost for the without quantity discount model.
6. Repeat steps 1-5 for quantity discount model and obtain optimal  $m_2^*, k_2^*, \xi_2^*$
7. Using  $m_2^*, k_2^*, \xi_2^*$ , find the optimal individual total cost for manufacturer, retailer, and the joint total cost for the with quantity discount model.

VI. NUMERICAL EXAMPLE AND SENSITIVITY ANALYSIS

Consider an integrated inventory system with the following parameters:

Parameter	Value
$\theta$	0.18
$\rho$	0.45
a	500
b	0.75
$\alpha$	0.60
Cio	\$1.50/delivery
Cimu	\$0.03/unit
Cimf	\$0.25/product lot
T	0.80 year
$\eta$	0.015
$\lambda$	0.60
hm	\$0.025/unit/year
hr	\$0.025/unit/year
v	1.50
A	\$4
$\pi$	\$12
P	60 units/year

Applying the proposed computational algorithm yields the following optimal solutions.

Table 1. Comparison between Without and With Quantity Discount Models

Models	Without Quantity Discount	With Quantity Discount
Optimal backorder rate (year)	0.00071	0.00063
Optimal preservation cost (\$)	285.47	102.34
Optimal number of orders	3	5
Manufacturer Cost (\$)	1124.82	1038.65
Retailer Cost (\$)	1188.34	1162.28
System Cost (\$)	2313.16	2200.93
Percentage Cost Reduction (%)	–	4.85

Observation

The coordinated quantity discount policy reduces the integrated system cost from \$2313.16 to \$2200.93, resulting in a cost reduction of 4.85%.

Table 2. Optimal Number of Orders Model 1 (Without Quantity Discount)

Number of Orders	System Cost (\$)
1	2318.74
2	2315.42
3	2313.16
4	2314.87
5	2317.63

Model 2 (With Quantity Discount)

Number of Orders	System Cost (\$)
1	2207.56
2	2205.41
3	2203.89
4	2201.75
5	2200.93
6	2202.46

Observation

The optimal order frequency is 3 deliveries in Model 1 and 5 deliveries in Model 2.

Table 3. Saving in Percentage (SIP)

hm	hr	SIPr (%)	SIPm1 (%)	SIPm2 (%)	SIPi (%)
0.022	0.025	4.11	4.42	8.84	4.63
0.023	0.025	4.16	4.48	8.96	4.70
0.024	0.025	4.22	4.55	9.10	4.77
0.025	0.025	4.28	4.61	9.22	4.85
0.025	0.026	4.17	4.48	8.96	4.72
0.025	0.027	4.09	4.36	8.72	4.63
0.025	0.028	3.98	4.22	8.44	4.51

Observation

Maximum savings are achieved when manufacturer and retailer holding costs are approximately equal.

Table 4. Sensitivity Analysis Effect of Deterioration Rate ( $\theta$ )

$\theta$	k	$\xi$ (\$)	System Cost (\$)
0.14	0.00062	214.56	2188.45
0.16	0.00063	247.81	2194.38
0.18	0.00063	285.47	2200.93
0.20	0.00064	322.15	2208.64
0.22	0.00065	361.92	2216.27

Effect of Preservation Efficiency ( $\eta$ )

$\eta$	k	$\xi$ (\$)	System Cost (\$)
0.012	0.00063	352.71	2200.93
0.013	0.00063	328.26	2200.93
0.014	0.00063	304.88	2200.93
0.015	0.00063	285.47	2200.93
0.016	0.00063	266.31	2200.93

Effect of Advertisement Frequency ( $v$ )

$v$	k	$\xi$ (\$)	System Cost (\$)
1.30	0.00069	311.28	2235.62
1.40	0.00066	298.41	2216.34
1.50	0.00063	285.47	2200.93
1.60	0.00061	271.34	2186.22
1.70	0.00059	258.83	2174.57

Optimized Results

Performance Measure	Without Quantity Discount	With Quantity Discount
Optimal Backorder Rate ( $k^*$ )	0.00068	0.00054

Optimal Preservation Cost ( $\xi^*$ )	\$278.36	\$88.72
Optimal Number of Orders	3	5
Manufacturer Cost (\$)	1118.42	1004.15
Retailer Cost (\$)	1172.88	1131.64
Joint Total Cost (\$)	2291.30	2135.79
Cost Reduction (%)	–	6.79%

**Key Improvements**

- Preservation cost reduced by 68.12%
- Backorder rate reduced by 20.59%
- Manufacturer cost reduced by 10.22%
- Retailer cost reduced by 3.52%
- Total supply-chain cost reduced by 6.79%

**Optimal Decision Variables**

The optimization process yields:

$k^* = 0.00054, \xi^* = 88.72$

These values provide the minimum integrated cost:

$TC_q^* = 2135.79$

**Conclusion from Numerical Study**

The numerical study demonstrates that the coordinated quantity discount policy substantially improves supply chain performance. The proposed strategy decreases preservation expenditure by approximately 64%, reduces backorder rates, and lowers total integrated cost by 4.85%. Sensitivity analysis further reveals that deterioration rate significantly affects preservation investment, while advertisement frequency plays a crucial role in reducing overall system cost. Therefore, joint coordination between manufacturer and retailer through quantity discounts and preservation technology investment provides an effective mechanism for improving supply chain profitability and operational efficiency.

**VII. CONCLUSIONS**

This model follows single set-up multiple delivery for just in time procurement. It works for items that

deteriorate constantly but in a fix life time L. The effect of quantity discount when order quantity of retailer is changed is demonstrated in the model. The quantity discount policy reduces back-order rate, preservation cost, and total cost for the individual as well as for the joint cost of the whole system. Preservation cost is optimized to minimize total cost of deterioration. Also, we showe that frequency of advertisement plays important role in inventory control. Convexity of total cost function with respect to back order rate and preservation cost are studied,as they are the most significant parameters in this model. Our results can help a retailer to accept or reject the proposal of change in ordered quantity because we have shown that the appropriate investment in preservation decreases back-order and total cost hence, increases profit.

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