Structural Analysis of Continuous Monotone Star Decompositions in Graphs

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***Abstract*—Let $G = (V, E)$ be a connected simple graph with p vertices and q edges. Suppose $G_1, G_2, \dots, G_n$ are pairwise edge-disjoint subgraphs of G such that $E(G) = E(G_1) \cup E(G_2) \cup \dots \cup E(G_n)$.**
Then the collection $(G_1, G_2, \dots, G_n)$ is called a decomposition of G. A star decomposition refers to a decomposition in which each subgraph is a star. In this paper, we establish that certain classes of star graphs admit a continuous monotonic star decomposition.

***Index Terms*—Decomposition, Wheel, Monotonic**

DEFINITION 1.1
A graph G consists of a finite non empty set V = V(G) of p points together with a prescribed set X of q unordered pairs of distinct points of V. Each pair x = {u,v} of points in X is a line of G and x is said to join u and v.

DEFINITION 1.2
If x = {u,v} is a line of a graph G. We write x = uv and say that u and v are adjacent points; point u and line x are incident with each other, as are v and x. If two distinct lines x and y are incident with a common point, then they are adjacent lines. A graph with p points and q lines is called a (p,q) graph.

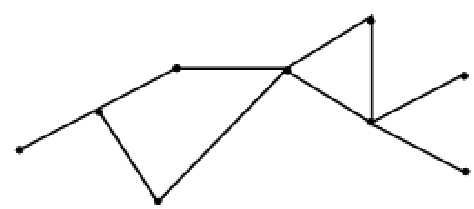
DEFINITION 1.3
A graph G is labeled when the p points are distinguished from one another by names such as v_1, v_2,.., v_p

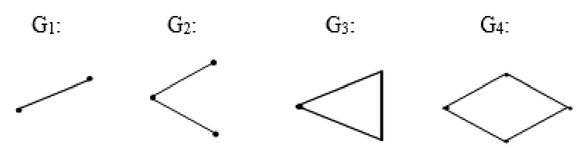
DEFINITION 1.4
Let G_1 and G_2 be two graphs with disjoint point sets V_1 and V_2 and line sets X_1 and X_2 respectively. Their union $G = G_1 \cup G_2$ has, as expected, $V = V_1 \cup V_2$ and $X = X_1 \cup X_2$. Their Join defined by is denoted $G_1 + G_2$ and consists of $G_1 \cup G_2$ and all lines joining V_1 with V_2. For $n \geq 4$, the wheel W_n is defined to be the graph $C_{n-1} + K_1$.

DEFINITION 1.5
Let G = (V, E) be a connected simple graph of order p and size q. If G_1, G_2,…., G_n are edge disjoint subgraphs of G such that $E(G)=E(G_1) \cup E(G_2) \cup \dots \cup E(G_n)$, then $(G_1, G_2, \dots, G_n)$ is said to be a decomposition of G.

DEFINITION 1.6
A decomposition $(G_1, G_2, \dots, G_n)$ of G is said to be a continuous monotonic decomposition (CMD) if each G_i is connected and $|E(G_i)|=i$ for each i = 1,2 …, n.

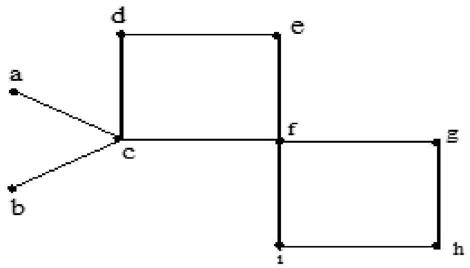
Example:
For the graph G in fig 2.1 (G_1, G_2, G_3, G_4) is a continuous monotonic decomposition.
G:





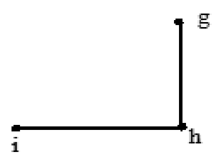
DEFINITION 1.7
A continuous monotonic decomposition in which each G_i is a star is said to be a continuous monotonic star decomposition (CMSD).

Example:
For the graph G in fig 2.2, (S_1,S_2,S_3,S_4) is a continuous monotonic star decomposition of the graph G.
G:

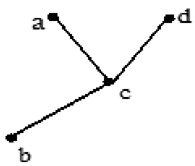


S_1:

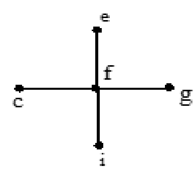
S_2:



S_3:



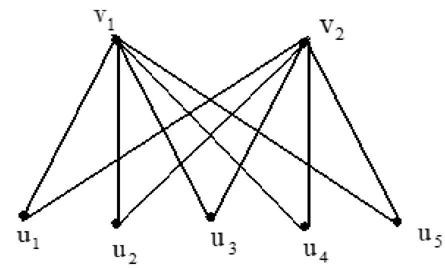
S_4:

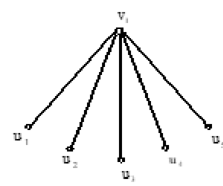


Theorem 1.1
1. $K_{n,2n+1}$ admits CMSD for all $n \geq 1$
2. $K_{n+1,2n+1}$ admits CMSD for all $n \geq 1$.

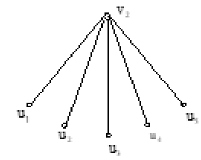
Proof:
Let $V = \{ v_1,v_2\ldots\ldots, v_n\}$ and $U = \{ u_1,u_2\ldots., u_{2n+1}\}$ be the bipartition of $K_{n,2n+1}$.
Let T_i denote the star $K_{1,2n+1}$ centered at v_i.
Then T_1 can be decomposed into stars S_1 and S_{2n}; T_2 can be decomposed into two stars S_2 and S_{2n-1}.
Continuing this process, T_n can be decomposed in to stars S_n and S_{n+1}.
Thus $K_{n,2n+1}$ is decomposed into 2n stars of sizes 1,2 , 2n.
Similary $K_{n+1,2n+1}$, is decomposed into 2n+1 stars of sizes 1,2, …, 2n +1.

Example
The decomposition of $K_{2,5}$ into continuous monotonic stars is illustrated in $K_{2,5}$



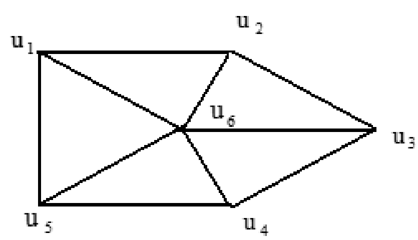


$K_{1,5} = S_1 \bigcup S_4$

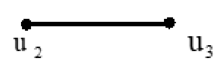


$K_{1,5} = S_2 \bigcup S_3$

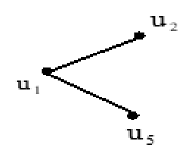
Example
A decomposition of W_6 into continuous monotonic stars is illustrated in below figure W6:



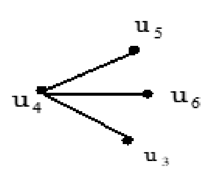
S_1:



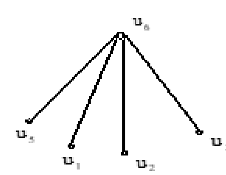
S_2:



S_3:



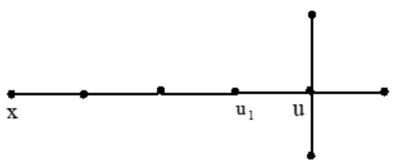
S_4:



Proof:
Let S_1, S_2,.., S_n be the CMSD of T.
We claim that T – W is a path.
Suppose T-W is not a path.

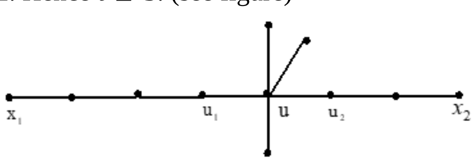
Then there exists at least three pendant vertices x_1, x_2 and x_3 such that d (u, x_i) >2, for all
i = 1, 2, 3 Clearly no internal vertex of u - x_i path can be a center of star of size $\geq$ 3.
Hence none of the pendant edges incident with x_i can be fitted in any star of size $\geq$ 3, which is a contradiction to the hypothesis. Hence T-W is a path, say P_t.
We claim that t $\leq$ 3.
We consider two cases.

Case (i):
u is the origin of P_t.
Since T is a spider tree, all the internal vertices and terminus of P_t are of degree 2 in T.
Let x be the unique pendant vertex adjacent to the terminus of P_t and u_1 be the vertex adjacent to u in P_t (See figure)



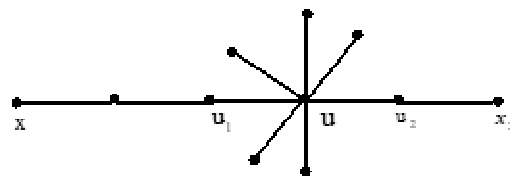
Then no vertex (except origin) of P_t can be a center of a star of size$\geq$3. Hence u_i–x path of T must be decomposed into non- isomorphic stars of sizes one and two so that d(u,x) $\leq$ 4. Thus t $\leq$ 3.

Case (ii):
u is not the origin of P_t.
There exists exactly two pendant vertices x_1 and x_2 which are not adjacent to u in T.
Let u_1 be a vertex adjacent to u in the x_1 –u section of P_t and u_2 be a vertex adjacent to u in the x_2 –u section of P_t
Since u_1-x_1 and u_2-x_2 sections must be decomposed into S_1 and S_2, we must have d(u,x_1)$\leq$3 and d(u,x_2) $\leq$ 2. Hence t $\leq$ 3. (see figure)

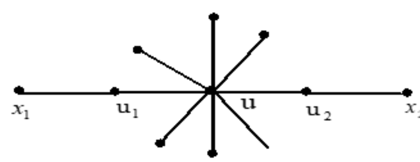


Conversely,
let T-W be a path of length $\leq$ 3.
We consider three cases.

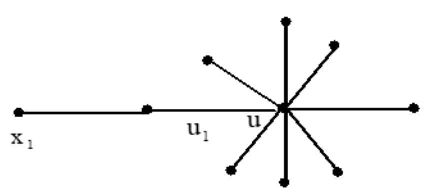
Case (i)
Let T-W = P_3
Let u_1 be the vertex adjacent to u in P_3.
If u is the origin of P_3, then the u_1-x path can be decomposed into S_1 and S_2 and remaining part of tree is a star which can be decomposed into S_3,S_4, ……, S_n
If u is not the origin of P_3, let u_1 and u_2 be the vertices adjacent to u in P_3 and d(u_1, x_1) =2 and d(u_2,x_2) =1.
Then S_1 = u_2 x_2 and S_2= u_1 – x_1 path in T and the remaining part of T is a star



Case (ii)
T – W = P_2.
If u is the origin of P_2, let u_1 be adjacent to u in P_2.
Then the u-x path in T is decomposed into S_1 and S_2 and the remaining part is a star
If u is not the origin of P_2, let u_1 and u_2 be two vertices adjacent to u in P_2 and let x_1 and x_2 be the unique vertices adjacent to u_1 and u_2 respectively.
Then S_1 = u_1 x_1 and S_2 = u-x_2 path in T and the remaining part is a star



Case (iii)
T –W = P_1
Let u_1 be the vertex adjacent to u in P_1 and x_1 be the unique vertex adjacent to u_1.
Then S_1=$u_1$$x_1$ and the remaining part is a star



Hence the theorem.

CONCLUSION

In this paper, we investigate the concept of continuous monotone decomposition of graphs and examine its various extensions, with particular emphasis on continuous monotone star decompositions. Furthermore, we analyze the decomposition of bipartite graphs into star subgraphs. As a direction for future research, we propose to extend these decomposition techniques to additional classes of graphs that have not yet been explored.

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