A Mathematical Inventory Model for Deteriorating Items with Price and Time Dependent Demand Rate -A Deterministic Approach

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Abstract—**Inventory management is a vital component of modern production and supply chain systems, particularly when dealing with deteriorating products whose demand is influenced by both time and selling price. This study develops a deterministic inventory model for deteriorating items under a demand rate that depends simultaneously on time and price. The proposed model reflects realistic market situations where customer demand initially increases or decreases over time and is inversely affected by product price.**

The model incorporates key inventory factors such as deterioration, shortages, time-dependent holding cost, variable pricing, and instantaneous replenishment. Two forms of demand functions are considered: increasing time-price dependent demand and decreasing time-price dependent demand. Mathematical expressions for inventory level, deterioration quantity, order quantity, shortage quantity, holding cost, and profit function are derived using differential equations. The objective is to determine the optimal selling price and replenishment policy that maximize the average profit per unit time.

Optimality conditions are obtained by applying first- and second-order optimization techniques. A numerical example is presented to demonstrate the applicability of the model and to compare different demand strategies. The results indicate that inventory policies based on both time and price-dependent demand provide better profit outcomes than policies considering only one factor. The analysis further reveals that the demand function incorporating positive time influence offers superior profitability and managerial insights under suitable market conditions.

The proposed model provides a practical decision-making framework for inventory managers dealing with deteriorating products in competitive markets. It highlights the importance of integrating pricing decisions with demand dynamics and inventory control to achieve maximum profitability and efficient inventory management.

Index Terms—**Deteriorating Items; Deterministic Inventory Model; Price-Dependent Demand; Time-Dependent Demand; Inventory Control; Optimal Pricing Policy; Shortage Management; Profit Maximization.**

I. INTRODUCTION

We came to know that Production rate is fixed from the classical production inventory model. But in real life that production rate may not be fixed. Due to high competition in the marketing situation and the demand of market for the product. Today's manufacturing systems are designed to be highly adaptable, ensuring they can meet the ever-changing demands of the market effectively. The modern manufacturing system is growing up on the basis of the following terms: materials requirement planning (MRP), flexible manufacturing system (FMS), just in time (JIT) and optimize product technology (OPT).

For this reason, inventory take a great role for production rate, as well as relation between demand, price and stock.

Inventory item is the important need for any kind of businessman and warehouse management in order to earn more and more profit and maintain the uniform balance between supply chain and demand.

The nature of items is typically a widely distributed trait influenced by market dynamics such as production demands, transportation, seasonal requirements etc. numerous inventory items have perishable characteristic over extended periods, while other degrade over shorter duration. The pricing and costing of stored items are crucial factors that influence the dynamics of demand and supply chains within the market mechanism. The holding cost of items in inventory can vary significantly seasons, size of items, demand pattern and facilities equipped warehouse. The inventory problem is also affected by various factors. Such as inflation rate, order size, price fluctuations, shortage of items, discount strategies, demand fluctuation, lead times, ordering costs, storage constraints and marketing plans. The time factor serves as a fundamental basis for the intricate decision-making process in inventory management, playing a pivotal role in determining optimal stock levels and reorder points.

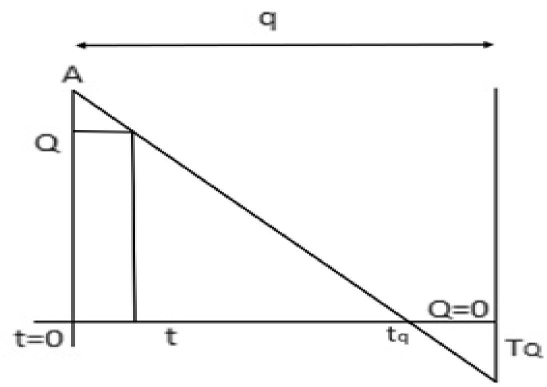
II. A REVIEW

Chen and Kang (2009-a) [1] formulated the integrated inventory models under the two-level trade credit policy with price sensitive demand and negotiation scheme. A review article on trade credit can be reviewed by Cheng et al (2009) [2] incorporated the concept of vendor and buyer integration and order size –dependent trade credit. Different other researchers [7-10] have developed inventory models assuming demand as constant, time dependent, stock dependent or price dependent. X. Wang, Tang, and Zhao [6] worked on fuzzy economic order quantity inventory model without backordering.

Teng and Chang (2009) [5] gave the models discussing Vendor-buyer inventory models with trade credit financing under both non-cooperative and integrated environments. Jagi (1995)[4]
discussed the effect of deterioration of units when delay in payments is permissible. Teng et al (2005) [3] developed retailer's optimal ordering and pricing policy for the assumption of deterministic and constant demand.

III. PROBLEM TO BE INVESTIGATED

In my field observation, I find that demand react inversely proportion with price. But with time demand react differently, first it shows proportional relation with time and after some time when demand saturated, the demand gradually decries with increasing of time. As an example, demand of QSR chain, big shopping mall, fruit juice center etc. of different brand among the population of a particular area is time dependent as well as price dependent.



([23]. Khedlekar U.K, Shukla Diwakar, Advance Inventory Models, 2013.)

At t=0 maximum inventory, Q be the quantity of inventory at any time t, Q=0 when t=t_q

After that inventory level becomes negative and then urgently replenishment is
instantaneous.

q= the inventory in stock deteriorated + demands up to time t_1. This cyclic representation will continue repeatedly

IV. ASSUMPTION AND NOTATIONS FOR MODEL

The proposed model is under the following assumptions:

f(t,p): Denotes demand rate as function of time 't' and price 'p' both like

Case I: f(t,p) = (k+lt-vp)

Case II: f(t,p): = (k-lt-vp), where k>p>0, t>0, k-vp>0 and l>0 are constants, v>0 associates to proportional contribution of price in demand.

h(t): holding cost (per unit) time dependent
as h(t) = h + βt where h>0 , β>0 are constants and H is total holding cost.

T: cycle time.

C: unit cost of item.

C_2: Shortage cost unit per unit time and C_2>C.

Θ: Deterioration rate, 0<θ< 1 and θn is negligible for n>= 3.

D: Deteriorated number of units in Inventory.

p: Variable selling price, such that k>p>C>0.

Q: Order quantity required for one cycle .

X: Shortages in the system.

C_3: Cost of placing an order.
t_1: Time (0=< t_1 =<T) for which the initial stock 'A' consumed and thereafter shortage occurs from time t_1 to T. we set the relation t_1 = αT where 0<α=<1 is a constant.
If α<1 the model performs with shortage.
If α=1 the model performs without shortage.
Replenishment is instantaneous and lead time is zero.

V. MODEL SETUP

Suppose Q(t) be the inventory level at time t (0=<t=<T) and A the stock level in beginning at t=0. During time t_1 , the inventory is depleted due to deterioration and demand remains f(t,p)=(k+lt-vp) of the item. In beginning retailer reduces the item price p to generate demand and at the end of time t_1 stock reduces to zero, shortage occurs until T.

As per assumption, demand rate depends upon time t and price p both. The f(t,p) poses negative derivative throughout its domain and differential equations for the instantaneous state in (0,T) are:

$\frac{d}{dt}Q(t)+\theta Q(t)$= -(k+lt-vp), where $0 \le t \le t_1$ (1)

$\frac{d}{dt}Q(t)$= -(k+lt-vp), where $t_1 \le t \le T$ .... (2)

Solving equation (1)

$Q(t)e^{\theta t}$= $-\int_0^t (k+lt-vp)e^{\theta t}$dt +A 3)

On boundary condition Q(t_1) = 0

A=$\int_0^{t1}(k+lt-vp)\left(1+\theta t+\frac{\theta^2 t^2}{2}\right)dt$

or A= $\delta t_1 + \frac{lt_1^2}{2}+\theta\left\{\frac{\delta t_1^2}{2}+\frac{lt_1^3}{3}\right\}+\theta^2\left\{\frac{\delta t_1^3}{6}+\frac{bt_1^4}{8}\right\}$

where δ= (k-vp) (4)

substituting value of A in equation(3)

$$Q(t)=\left(1-\theta t+\frac{\theta^2t^2}{2}\right)\left\{\delta(t_1-t)+l\left(\frac{t_1^2}{2}-\frac{t^2}{2}\right)\right\}+\theta\left(\frac{\delta t_1^2}{2}+\frac{\delta t^2}{2}+\frac{lt^3}{3}+\frac{lt_1^3}{3}\right)+\theta^2\left(1-\theta t+\frac{\theta^2t^2}{2}\right)\left(\frac{\delta t_1^3}{6}+\frac{\delta t^3}{6}+\frac{lt_1^4}{8}+\frac{lt^4}{8}\right)$$

$$Q(t)=\delta(t_1-t)+l\left(\frac{t_1^2}{2}-\frac{t^2}{2}\right)+\theta\left\{\frac{\delta t_1^2}{2}+\frac{\delta t^2}{2}+\frac{lt^3}{3}+\frac{lt_1^3}{3}-\delta tt_1-\frac{ltt_1^2}{2}\right\}+\theta^2\left(\frac{\delta t_1^3}{6}-\frac{\delta t^3}{6}+\frac{lt_1^4}{8}-\frac{lt^4}{24}-\frac{\delta tt_1^2}{2}\right)-\frac{ltt_1^3}{3}+\frac{\delta t^2t_1}{2}+\frac{lt^2t_1^2}{4} \quad (5)$$

In other interval between t_1 to T

$Q(t)=\int_{t_1}^{T}(\delta+lt)dt$ Where $t_1 \le t \le T$

$Q(t)=\delta(t_1-t)+l\left(\frac{t^2}{2}-\frac{t_1^2}{2}\right)$, where $t_1 \le t \le T$ (6)

Deteriorated units in system for one cycle are

D = Initial stock – demand during (0, t_1)

D = A- $\int_0^{t1}\{\delta+lt\}dt$

D = $\theta\left\{\frac{\delta t_1^2}{2}+\frac{lt_1^3}{3}\right\}+\theta^2\left\{\frac{\delta t_1^3}{6}+\frac{lt_1^4}{8}\right\}$ (7)

Now q is quantity required for a cycle T and it is sum of the deteriorated units and demand up to time $t_1=\alpha T$.

q = D + $\int_0^{t_1}(\delta+lt)\,dt$

q= $\theta\left\{\frac{\delta t_1^2}{2}+\frac{lt_1^3}{3}\right\}+\theta^2\left\{\frac{\delta t_1^3}{6}+\frac{lt_1^4}{8}\right\}$ (8)

Total holding cost for system

H=$\int_0^{t_1}(h+\beta t)\,e^{-\theta t}\left\{\int_1^{t_1}(\delta+lz)e^{\theta z}dz\right\}dt$

Where z an integrating variable

$$H=\int_0^{t_1}(h+\beta t)\left\{\delta(t_1-t)+l\left(\frac{t_1^2}{2}-\frac{t^2}{2}\right)\right\}dt+\int_0^{t_1}(h+\beta t)\theta\left(\frac{\delta t_1^2}{2}+\frac{\delta t^2}{2}+\frac{lt^3}{3}+\frac{bl}{3}-\delta t_1 t-\frac{lt_1^2t}{2}\right)dt+\int_0^{t_1}(h+\beta t)\theta^2\left(\frac{\delta t_1^3}{6}+\frac{\delta t^2}{6}+\frac{lt^4}{8}+\frac{lt^4}{24}-\frac{\delta t_1^2 t}{2}\right)dt+\int_0^{t_1}(h+\beta t)\theta^2\left(\frac{dt_1t^2}{2}+\frac{lt_1^2t^2}{4}+\frac{lt_1^2t}{3}\right)dt$$

$$H=h\left[\left\{\frac{\delta t_1^2}{2}+\frac{lt_1^3}{3}\right\}+\theta\left\{\frac{\delta t_1^3}{6}+\frac{lt_1^4}{8}\right\}\right]+h\theta^2\left\{\frac{\delta t_1^4}{24}+\frac{lt_1^5}{30}\right\}+\beta\left\{\frac{\delta t_1^3}{6}+\frac{3lt_1^4}{8}\right\}+\beta\left[\theta\left\{\frac{lt_1^5}{30}+\frac{\delta t_1^4}{24}\right\}\right]+\left\{\frac{\delta t_1^5}{120}+\frac{lt_1^6}{144}\right\} \quad (9)$$

The X is shortage in period (t_1,T)

X=$\int_{t_1}^{T}Q(t)dt$, $t_1 \le t \le T$

or X=$\frac{\delta}{2}(T^2-2\,t_1T+t_1^2)+\frac{b}{6}(T^3-2\,t_1^3-3t_1^2T)$ (10)

Total selling price =p (demand for period zero to T)

=$\frac{p}{T}\left\{\delta T+\frac{lT^2}{2}\right\}$ (11)

Average incurred cost K(T, t_1, p) is

K(T, t_1, p)=$\frac{1}{T}\{H+CA+C_2X+C_3\}$

Total profit per unit time P(T, t_1, p) is

P(T, t_1, p)=$\frac{p}{T}\left\{\delta T+\frac{lT^2}{2}\right\}-\frac{1}{T}\{H+CA+C_2X+C_3\}$

P(T, t_1, p)= $\frac{p}{T}\left\{\delta T+\frac{lT^2}{2}\right\}-\frac{h}{T}\left[\left\{\frac{\delta t_1^2}{2}+\frac{lt_1^3}{3}\right\}+\theta\left\{\frac{\delta t_1^3}{6}+\frac{lt_1^4}{8}\right\}\right]-\frac{h\theta^2}{T}\left\{\frac{\delta t_1^4}{24}+\frac{lt_1^5}{30}\right\}+\frac{\beta}{T}\left\{\frac{\delta t_1^3}{6}+\frac{3lt_1^4}{8}\right\}-\frac{\beta}{T}\left[\theta\left\{\frac{lt_1^5}{30}+\frac{\delta t_1^4}{24}\right\}+\left\{\frac{\delta t_1^5}{120}+\frac{lt_1^6}{144}\right\}\right]-\frac{C}{T}\delta t_1-\frac{lt_1^3}{2T}-\frac{\theta}{T}\left\{\frac{\delta t_1^2}{2}+\frac{lt_1^3}{3}\right\}-\frac{\theta^2}{T}\left\{\frac{\delta t_1^3}{6}+\frac{lt_1^4}{8}\right\}-\frac{\delta}{2}(T^2-2t_1T+t_1^2)C_2-\frac{lC_1}{6}(T^3+2t_1^3+3t_1^2T)-\frac{C^3}{T}$

By substituting $t_1=\alpha T$, where $0<\alpha\le 1$
and P(T, t_1 ,p)=P(T,p) above equation reduces in form.

P(T,p)= $k_0+k_1T+k_2T^2+k_3T^3+k_4T^4+k_5T^5-\frac{C_3}{T}$ (12)

Where $k_0=\delta(p-C\alpha)$

$k_1=\frac{\delta P}{2}+\delta\alpha C_2-\frac{\delta C_2}{2}-\frac{\alpha^2}{2}\{C_2\delta+h\delta+Cl+C\theta\delta\}$

$k_2=\frac{\alpha^3}{6}\{C\theta^2\delta+2lC_2+2C\theta l+2lh\}-\frac{\alpha^3}{6}\delta h\theta+\delta\beta-\frac{lC_2}{6}+\frac{C_2l\alpha^2}{2}$

$k_3=\frac{\alpha^4}{24}\{\delta\theta\beta+3Cl\theta^2-3hl\theta\}-\frac{\alpha^4}{24}h\delta\theta^2+3\beta l$

$k_4=\frac{\alpha^5}{120}\{4hl\theta^2+4l\beta\theta+\delta\beta\theta^2\}$

$k_5=-\frac{\theta^2 l\beta}{144}\alpha^6$

Objective is to maximize the profit function P(T,p)

$\frac{\partial P(T,p)}{\partial T}=0$ and $\frac{\partial P(T,p)}{\partial p}=0$ (13)

Relation (13) provides two equations

$k_1T^2+2k_2T^3+3k_3T^4+4k_4T^5+5k_5T^6+C_3=0$ (14)

P=$f_0+f_1T+f_2T^2+f_3T^3+f_4T^4$ (15)

Where $f_0=\frac{1}{2v}\{k+\alpha Cv\}$

$f_1=\frac{1}{4v}\{vC_2+l-2v\alpha C_2+v\alpha^2(C_2+C\theta+h)\}$

$f_2=\frac{\alpha^3}{12}(C\theta^2+h\theta+\beta)$

$f_3=\frac{\alpha^4}{48}(h\theta^2+\beta\theta)$

$f_4=\frac{\alpha^5\beta\theta^2}{240}$ (16)

Let optimal positive value obtained by solving equation for (14)(15) are

$T=T^*\ and\ p=p^*$

For this we have

$\frac{\partial^2P(T^*,p^*)}{\partial T^{*2}}=\left[2k_2+3k_3T^*+12k_4T^{*^2}+20k_5T^{*^3}-\frac{2C_3}{T^{*3}}\right]<0$ (17)

$\frac{\partial^2P(T^*,p^*)}{\partial T^*\partial p^*}=\left\{\frac{vC_2+l}{2}-v\alpha C_2+\frac{v\alpha^2}{2}(h+\theta C+C_2)\right\}+\frac{v\alpha^3T^*}{3}(h\theta+\beta+\theta^2C)+\frac{\theta^2\alpha^4T^{*^2}v}{8}(h\theta^2+\beta\theta)+\frac{v\theta^2\alpha^5\beta T^{*3}}{30}$

And

$\frac{\partial^2P(T^*,p^*)}{\partial p^{*2}}=-2v>0$ (18)

From equation (16),(17),(18) and by assumption k,v,C_2,C_3,C positives constants such that $k>p>0$ $\theta\ge 0, T=T^*>0\ and\ p=p^*>0$ one can get

$\left[\frac{\partial^2P(T^*,p^*)}{\partial T^{*2}}\frac{\partial^2P(T^*,p^*)}{\partial p^{*2}}-\left\{\frac{\partial^2P(T^*,p^*)}{\partial T^*\partial p^*}\right\}^2\right]>0$, $\left[\frac{\partial^2P(T^*,p^*)}{\partial T^{*2}}\right]<0$ (19)

The profit function $P(T,\mathrm{p})$ is maximum at $T=T^*$ and $=p^*$. Optimum deteriorated units are

$D^*=\theta\left\{\frac{\delta\alpha^2(T^*)^2}{2}+\frac{b\alpha^3(T^*)^3}{3}\right\}+\theta^2\left\{\frac{\delta\alpha^3(T^*)^3}{6}+\frac{b\alpha^4(T^*)^4}{8}\right\}$ (20)

Optimum quantities are

$q^*=\theta\alpha^2(T^*)^2\left\{\frac{\delta}{2}+\frac{l\alpha T^*}{3}\right\}+\delta\alpha T^*\ \frac{b\alpha^2(T^*)^2}{2}+\theta^2\alpha^2(T^*)^3\left\{\frac{\delta}{6}+\frac{l\alpha T^*}{8}\right\}$ (21)

Shortage $X^*=\frac{\delta(T^*)^2}{2}(1-2\alpha+\alpha^2)+\frac{l(T^*)^3}{6}\{1+2\alpha^3-3\alpha^2\}$ (22)

Optimum profit

$P(T^*,p^*)=k_0+k_1T^*+k_2T^{*2}+k_3T^{*3}+k_4T^{*4}+k_5T^{*5}-\frac{C_3}{T^*}$ (23)

Numerical Example:

To support the outcome of proposed model consider an example with data to examine how model performs under the demand f(t,p)=k-lt-vp, f(t,p)=k-vp and f(t,p)=k+lt-vp. Suppose k=100, l=2, C= Rs.20, C_2=Rs.50, V=0.9, C_3= Rs.200, T=10, h=β=θ=0 then optimum outputs are given in the table

Demand Strategy	Price p*	Profit P (t1*, p*)	EOQ q*	Shortage X*	Optimal time (t1*)
100-2t-0.9p	95.84	139.15	40	25	4.23
100-0.9p	65.37	1864.15	395	5	9.57
100+2t-0.9p	73.77	2054.68	336	95	8.09

VI. SENSITIVITY ANALYSIS

Sensitivity analysis is carried out to examine the robustness of the proposed inventory model and to study the impact of key system parameters on the optimal solution. In real-life inventory systems, parameter values such as market demand, price sensitivity, ordering cost, and time-dependent demand coefficients are often subject to estimation errors and market fluctuations. Therefore, it is essential to investigate how variations in these parameters influence the optimal profit and decision variables.

The analysis is performed by varying one parameter at a time by ±10% and ±20% while keeping all other parameters fixed.

Base parameter values:

$k = 100$, $l = 2$, $v = 0.9$, $C = 20$, $C_3 = 200$, $T = 10$, $h = \beta = \theta = 0$.

1. Sensitivity with Respect to Demand Scale Parameter (k)

Change in (k)	Value of (k)	Optimal Profit (P)	Percentage Change
−20%	80	110.62	−20.5%
−10%	90	124.78	−10.3%
Base	100	139.15	0.00%
10%	110	154.92	11.30%
20%	120	170.64	22.60%

The optimal profit is highly sensitive to changes in k. An increase in k significantly enhances profit, indicating that higher market demand potential leads to improved system performance.

2. Sensitivity with Respect to Price Sensitivity Parameter (v)

Change in (v)	Value of (v)	Optimal Profit (P)	Percentage Change
−20%	0.72	162.48	16.80%
−10%	0.81	150.36	8.10%
Base	0.9	139.15	0.00%
10%	0.99	127.92	−8.1%
20%	1.08	116.73	−16.1%

As v increases, optimal profit decreases. This shows that increased price sensitivity reduces profitability and requires careful pricing decisions.

3. Sensitivity with Respect to Time-Dependent Demand Parameter (l)

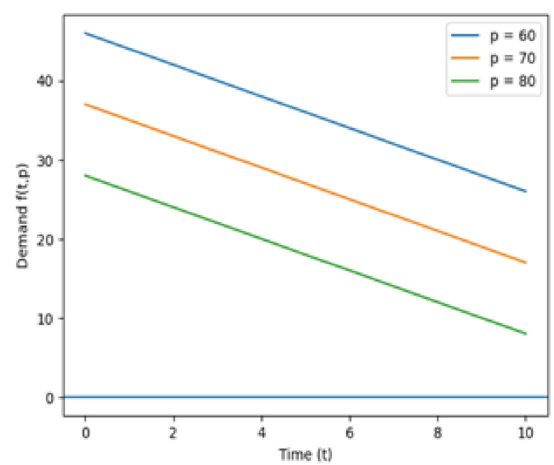
Change in (l)	Value of (l)	Optimal Profit (P)	Percentage Change
−20%	1.6	150.27	8.00%
−10%	1.8	144.63	4.00%
Base	2	139.15	0.00%
10.00%	2.2	133.74	−3.9%
20.00%	2.4	128.41	−7.7%

An increase in l reduces profit moderately, reflecting faster demand decay over time.

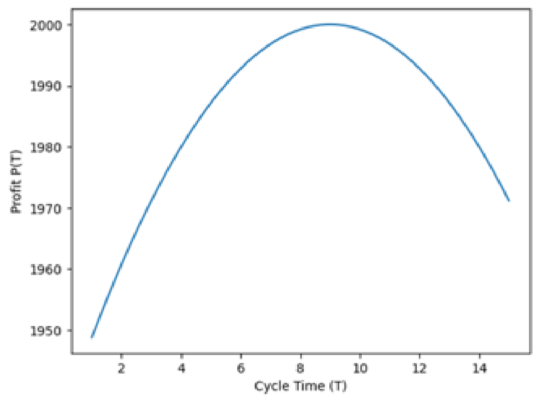
4. Sensitivity with Respect to Ordering Cost (C_3)

Change in (C3)	Value of (C3)	Optimal Profit (P)	Percentage Change
−20%	160	146.88	5.60%
−10%	180	143.12	2.80%
Base	200	139.15	0.00%
10%	220	135.26	−2.8%
20%	240	131.48	−5.5%

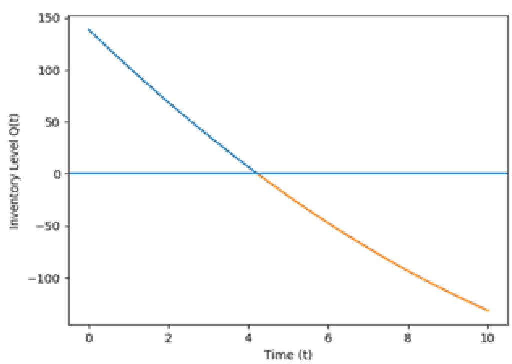
Profit decreases moderately as ordering cost increases, showing relative robustness of the model.



Demand vs Time for Different Prices



Profit vs Cycle Time



Inventory Level vs Time for Time and Price Dependent Demand

VII. CONCLUSION

The model suggests to inventory manager that, consider time and price as an important variable for demand function for optimal profit. Simple reason is that time or price alone are not good strategy for optimal profit. It is one of suggestions to choose function f(t,p)=a+bt-vp rather than f(t,p)=a-bt-vp because the first function has better critical interpretation that second function.

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