Lyapunov-Based Stability Certification for Data-Driven Control with Explicit Error Margins

Rajeev Muthu[1], Sekhar Babu Gosala[2], Jyothi Unduru[4], Mounika Mallipudi[5]
[1,2,3,4,5]*Department of Mathematics, Government College (Autonomous), Rajahmundry, Andhra Pradesh, India, PIN: 533105*

Abstract— **This paper develops a theorem-driven certification framework for data-driven control under model uncertainty. The central result establishes that if a learned closed-loop proxy admits a computable quadratic Lyapunov certificate with a strict decrease margin on a prescribed compact domain, and if the mismatch between the true and proxy closed-loop maps satisfies a certified relative error bound below an explicit threshold, then the true closed loop is exponentially stable with a computable decay rate. The admissible error margin is given in closed form in terms of the Lyapunov matrix, the decrease constant and the certification radius.**

The analysis then converts certified residual bounds from data equations into certified relative closed-loop error bounds, thereby exposing the role of data conditioning in a directly checkable way. Building on the primary theorem, the paper derives explicit robustness margins for switched and networked implementations, including computable average dwell-time thresholds. Event-triggered implementations are treated through explicit dissipation inequalities, yielding certified decay rates together with a strictly positive inter-event lower bound that excludes Zeno behavior.

The paper is fully deterministic and margin-based. It avoids qualitative small-gain arguments, simulation-only validation and unspecified smoothness assumptions. Worked examples compute all constants symbolically from stated quantities and illustrate both the certification mechanism and the precise points at which the program fails when verifiability is lost.

Index Terms— **data-driven control, Lyapunov certification, robust stability margins, switched systems, event-triggered control MSC 2020— 93D05, 93C25, 93D20, 93C57, 93C85**

I. INTRODUCTION

Proposition 1 (Governing theorem idea: data-driven Lyapunov under identified model error). Let $x^+ = f(x)$ be a discrete-time system on a set $\mathcal{X} \subset \mathbb{R}^n$ and let $\hat{f}: \mathcal{X} \to \mathcal{X}$ be a learned model. Assume that there exist constants $\alpha_1, \alpha_2, c > 0$, a rate $\rho \in (0,1)$, and a function $V: \mathcal{X} \to \mathbb{R}_{\geq 0}$ such that for all $x \in \mathcal{X}$,
$\alpha_1 \|x\|^2 \leq V(x) \leq \alpha_2 \|x\|^2$,
$V\left(\hat{f}(x)\right) - V(x) \leq -c\|x\|^2$. Assume moreover that the modeling error is bounded in a verifiable way: $\|f(x) - \hat{f}(x)\| \leq \varepsilon \|x\|$ for all $x \in \mathcal{X}$, and that V is locally Lipschitz on $\mathcal{X}$ with an explicit modulus L_V on $\mathcal{X}$. If $\varepsilon \leq \varepsilon_\star := \frac{c}{L_V}$, then the true system $x^+ = f(x)$ is exponentially stable on $\mathcal{X}$, with a computable rate depending only on $(\alpha_1, \alpha_2, c, L_V)$.

This proposition is the spine of the paper. It will be proved as Theorem 11 after the verifiability program is made explicit.

Implementation in this paper. In what follows, the certificate V is taken to be quadratic ($V(x) = x^\top Px$) so that the perturbation of V can be bounded by an explicit quadratic term on $\mathcal{X}$; see Theorem 11 and Proposition 13.

The argument is short. The difficulty is to make [eq:intro:V-decrease-hat] and [eq:intro:model-error] checkable from data and from the learned controller/model, without inserting non-verifiable smoothness or "small-gain" assumptions as decoration. The recent revival of data-driven control provides many design routes, but the certification step is often treated informally (learn → simulate → claim).

What Is Missing in Typical Arguments

A common proof pattern is the following. One estimates a one-step prediction error for $\hat{f}$ from a finite dataset. One observes that trajectories of $\hat{f}$ converge in simulation. One asserts that the true closed-loop inherits this stability when the prediction error is

"small". This does not yield a certificate. It confuses a pointwise regression quantity with a dynamical robustness margin. The separation is explicit in robust control: stability is a property of the closed-loop operator, not of the open-loop fit.
The present program replaces the informal step by one inequality. It is [eq:intro:eps-star]. All subsequent results exist to make its constants computable, and to explain what must be excluded when they are not.

Main Contributions
The results are numbered now, and proved later.

1. (Theorem 11) A quantitative robustness theorem of the form stated in Proposition 1: a Lyapunov decrease for $\hat{f}$ plus an identified error bound implies exponential stability for f, with an explicit allowable error margin $\varepsilon_\star$. The proof keeps constants visible.
2. (Section 3.3) A Verifiable hypotheses checklist for Theorem 11: the reader is told exactly what must be computed (bounds for V, a decrease margin c, and a dataset-derived bound for ε) and what is forbidden when it is not checkable. The checklist is constructed to match the rank-type and noise-consistency conditions used in data-driven methods.
3. (Theorem 26) A computable identification-error margin $\varepsilon_\star = \varepsilon_\star(\alpha_1, \alpha_2, c, L_V)$ in a norm suited for certification. The statement is deliberately not asymptotic. It is a finite, checkable inequality.
4. (Theorem 34 and Corollaries 36, 40 and 42) Corollaries for networked and event-triggered implementations: the spine yields explicit sampling and triggering margins once the modeling error induced by sampling/holding is bounded in the same norm. The derivations use standard constructions, but the constants are rederived to remain compatible with the verifiability discipline.

A specific failure mode in common arguments
The failure is concrete.
Let $\hat{f}$ be stable and suppose one has a bound

$$\|f(x) - \hat{f}(x)\| \leq \delta$$

on a dataset region. This does not imply that f is stable. The reason is structural. A uniform additive bound is not scaled to the state. Near the origin it dominates the contraction of $\hat{f}$ and can generate an invariant annulus. A simulation of $\hat{f}$ does not detect this, because it does not propagate the mismatch.
This leads us to a dilemma. Either one strengthens the hypothesis to a relative bound like [eq:intro:model-error], or one introduces an ISS-type gain with an explicit ultimate bound, which is a different claim.
The present framework avoids the dilemma by fixing the norm in which the modeling error is measured, and by forcing the Lyapunov inequality to carry an explicit margin c. Then a single perturbation estimate yields [eq:intro:eps-star]. No qualitative "small enough" remains. An explicit numerical margin remains.

Roadmap
Section 2 fixes notation and admissible norms. Section 3 proves Theorem 11 and the margin theorem Theorem 26, with verifiable hypotheses stated as checklists. Section [sec:extensions] derives corollaries for sampled, networked, and event-triggered implementations. Section 5 gives worked examples with explicit constants. Section 6 states what was proved and what was excluded.

Preliminaries
Notation and standing conventions
Fix a dimension $n \in \mathbb{N}$. Vectors are column vectors in $\mathbb{R}^n$. For a matrix M, $\|M\|$ denotes the induced operator norm. For $r > 0$ and $x \in \mathbb{R}^n$, write

$$\mathbb{B}(x,r) := \{z \in \mathbb{R}^n : \|z - x\| \leq r\}, \qquad \mathbb{B}(r) := \mathbb{B}(0,r).$$

Throughout, $\mathcal{X} \subset \mathbb{R}^n$ is a nonempty compact set containing 0. All statements are restricted to $\mathcal{X}$ unless said otherwise.

Definition 2 (Local Lipschitz modulus on $\mathcal{X}$). Let $V: \mathcal{X} \to \mathbb{R}$. A number $L_V \geq 0$ is called a Lipschitz modulus of V on $\mathcal{X}$ if $|V(x) - V(z)| \leq L_V\|x - z\| \quad$ for all $x, z \in \mathcal{X}$. When V is differentiable on an open neighborhood of $\mathcal{X}$, one may take $L_V := \sup_{x\in\mathcal{X}} \|\nabla V(x)\|$.

Definition 3 (Exponential stability on $\mathcal{X}$). Let $x^+ = f(x)$ with $f: \mathcal{X} \to \mathcal{X}$. The origin is exponentially stable on $\mathcal{X}$ if there exist constants $M \geq 1$ and $\lambda \in (0,1)$ such that for every $x_0 \in \mathcal{X}$ the solution is well-defined in $\mathcal{X}$ and satisfies $\|x_k\| \leq M\lambda^k\|x_0\| \quad$ for all $k \in \mathbb{N}$.
System model and the learned proxy

We work with a discrete-time closed-loop map

$$x^+ = f(x), \qquad f\colon \mathcal{X} \to \mathcal{X},$$

and a learned proxy

$$\boldsymbol{x}^+ = \hat{\boldsymbol{f}}(\boldsymbol{x}), \qquad \hat{\boldsymbol{f}}\colon \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{X}}.$$

No differentiability is assumed unless stated. The analysis is Lyapunov-based, hence metric rather than spectral.

Definition 4 (Relative modeling error on $\boldsymbol{\mathcal{X}}$). For $\boldsymbol{\varepsilon} \geq \mathbf{0}$ we say that [eq:prelim:hat] approximates [eq:prelim:true] with relative error $\boldsymbol{\varepsilon}$ on $\boldsymbol{\mathcal{X}}$ if $\|\boldsymbol{f}(\boldsymbol{x}) - \hat{\boldsymbol{f}}(\boldsymbol{x})\| \leq \boldsymbol{\varepsilon}\|\boldsymbol{x}\|$ for all $\boldsymbol{x} \in \boldsymbol{\mathcal{X}}.$

The choice [eq:prelim:relerr] is deliberate. An additive bound $\|\boldsymbol{f}(\boldsymbol{x}) - \hat{\boldsymbol{f}}(\boldsymbol{x})\| \leq \boldsymbol{\delta}$ has no scale near $\mathbf{0}$ and is not suited to certifying exponential stability. This is the failure mode stated in Section 1.3. For quantitative robustness one must keep track of gains and margins, as is standard in robust stability theory.

Lyapunov inequalities used in the spine

The spine is the passage from a strict decrease for $\hat{\boldsymbol{f}}$ to a strict decrease for $\boldsymbol{f}$ under an explicit perturbation margin.

Assumption 5 (Quadratic Lyapunov certificate for the proxy). Fix a compact set $\boldsymbol{\mathcal{X}} \subset \mathbb{R}^{\boldsymbol{n}}$ with $\mathbf{0} \in \boldsymbol{\mathcal{X}}$ and choose $\boldsymbol{R} > \mathbf{0}$ such that $\boldsymbol{\mathcal{X}} \subseteq \mathbb{B}(\boldsymbol{R})$. There exists a symmetric matrix $\boldsymbol{P} \in \mathbb{R}^{\boldsymbol{n}\times\boldsymbol{n}}$ with $\boldsymbol{P} \succ \mathbf{0}$ and a constant $\boldsymbol{c} > \mathbf{0}$ such that the quadratic function $\boldsymbol{V}(\boldsymbol{x}) := \boldsymbol{x}^\top \boldsymbol{P}\boldsymbol{x}$ satisfies, for all $\boldsymbol{x} \in \boldsymbol{\mathcal{X}}$,

$$\boldsymbol{\alpha_1}\|\boldsymbol{x}\|^2 \leq \boldsymbol{V}(\boldsymbol{x}) \leq \boldsymbol{\alpha_2}\|\boldsymbol{x}\|^2,$$
$$\boldsymbol{V}\left(\hat{\boldsymbol{f}}(\boldsymbol{x})\right) - \boldsymbol{V}(\boldsymbol{x}) \leq -\boldsymbol{c}\|\boldsymbol{x}\|^2.$$

Here $\boldsymbol{\alpha_1} := \boldsymbol{\lambda_{min}}(\boldsymbol{P})$ and $\boldsymbol{\alpha_2} := \boldsymbol{\lambda_{max}}(\boldsymbol{P})$.

Remark 6. Assumption 5 is classical in spirit. The novelty is that $\boldsymbol{c}$ and $\boldsymbol{L_V}$ must be computable on $\boldsymbol{\mathcal{X}}$. The analysis in Section 3 refuses to hide them under asymptotic notation.

Verifiable hypotheses: what can be checked from data

We isolate the data-dependent part as a separate hypothesis. The paper later gives checkable sufficient conditions in concrete settings.

Assumption 7 (A verifiable bound for the relative error). A number $\boldsymbol{\varepsilon} \geq \mathbf{0}$ is available such that [eq:prelim:relerr] holds on $\boldsymbol{\mathcal{X}}$. The method producing $\boldsymbol{\varepsilon}$ must output a certificate, not only a fitted model.

Remark 8. In linear settings, such certificates can be derived from regression residual bounds under bounded-noise assumptions, and can be sharpened using persistency-of-excitation conditions. In trajectory-based data-driven control, rank conditions and consistency constraints play an analogous role. The present paper uses these ideas only as sources of numerical bounds entering [eq:prelim:relerr]. No claim in the main theorems depends on an uncheckable generalization.

A basic comparison principle used later

The passage from a strict quadratic decrease to exponential stability is standard. We record it to fix constants.

Lemma 9 (Quadratic Lyapunov decrease implies exponential stability). Assume [eq:prelim:true]. Suppose there exist $\boldsymbol{\alpha_1}, \boldsymbol{\alpha_2}, \tilde{\boldsymbol{c}} > \mathbf{0}$ and $\boldsymbol{V}\colon \boldsymbol{\mathcal{X}} \to \mathbb{R}_{\geq \mathbf{0}}$ such that $\boldsymbol{\alpha_1}\|\boldsymbol{x}\|^2 \leq \boldsymbol{V}(\boldsymbol{x}) \leq \boldsymbol{\alpha_2}\|\boldsymbol{x}\|^2$, $\boldsymbol{V}(\boldsymbol{f}(\boldsymbol{x})) - \boldsymbol{V}(\boldsymbol{x}) \leq -\tilde{\boldsymbol{c}}\|\boldsymbol{x}\|^2$ for all $\boldsymbol{x} \in \boldsymbol{\mathcal{X}}$. Then the origin is exponentially stable on $\boldsymbol{\mathcal{X}}$ in the sense of Definition 3, with explicit constants $\|\boldsymbol{x_k}\| \leq \sqrt{\frac{\alpha_2}{\alpha_1}}\left(\mathbf{1} - \frac{\tilde{c}}{\alpha_2}\right)^{\boldsymbol{k}/\mathbf{2}}\|\boldsymbol{x_0}\|.$

Proof. From the decrease inequality and $\boldsymbol{V}(\boldsymbol{x}) \leq \boldsymbol{\alpha_2}\|\boldsymbol{x}\|^2$,

$$\boldsymbol{V}(\boldsymbol{x}^+) \leq \boldsymbol{V}(\boldsymbol{x}) - \tilde{\mathrm{c}}\|\boldsymbol{x}\|^2 \leq \left(\mathbf{1} - \frac{\tilde{\boldsymbol{c}}}{\boldsymbol{\alpha_2}}\right)\boldsymbol{V}(\boldsymbol{x}).$$

Iterating yields $\boldsymbol{V}(\boldsymbol{x_k}) \leq (\mathbf{1} - \tilde{\boldsymbol{c}}/\boldsymbol{\alpha_2})^{\boldsymbol{k}}\boldsymbol{V}(\boldsymbol{x_0})$. Using $\boldsymbol{\alpha_1}\|\boldsymbol{x_k}\|^2 \leq \boldsymbol{V}(\boldsymbol{x_k})$ and $\boldsymbol{V}(\boldsymbol{x_0}) \leq \boldsymbol{\alpha_2}\|\boldsymbol{x_0}\|^2$ gives the claim. □

Remark 10. An ISS route would yield an ultimate bound under additive disturbances. The present paper does not pursue that line. It pursues exponential stability under a relative error margin, because this produces a single checkable inequality of the form $\boldsymbol{\varepsilon} \leq \boldsymbol{\varepsilon_\star}$.Finite-sample radii.

We use standard concentration-based radii for least-squares or ridge estimators; only the explicit constants matter.

Main results

Primary theorem and proof

We now prove the governing statement announced in Proposition 1. The proof is short. The work is in making the constants checkable, which is handled in Section 3.3.

Theorem 11 (Quadratic Lyapunov robustness under relative model error). Consider the true closed loop [eq:prelim:true] and the proxy [eq:prelim:hat]. Assume Assumption 5 and Assumption 7. Let $R > 0$ satisfy $\mathcal{X} \subseteq \mathbb{B}(R)$ and let $V(x) = x^\top P x$ be the quadratic certificate from Assumption 5, with $P \succ 0$ and proxy decrease margin $c > 0$. If $\varepsilon < \varepsilon_\star := \frac{c}{2\lambda_{max}(P)\,R}$, then the origin is exponentially stable for the true system [eq:prelim:true] on $\mathcal{X}$. More precisely, defining the robust decrease margin $\tilde{c} := c - 2\lambda_{max}(P)\,R\,\varepsilon > 0$, the Lyapunov function V satisfies, for all $x \in \mathcal{X}$, $V(f(x)) - V(x) \le -\tilde{c}\,\|x\|^2$. Consequently, the exponential bound of Lemma 9 holds with $\tilde{c}$ in place of c, namely $\|x_k\| \le \sqrt{\frac{\alpha_2}{\alpha_1}}\left(1 - \frac{\tilde{c}}{\alpha_2}\right)^{k/2}\|x_0\| \qquad (k \in \mathbb{N})$.

Proof. Fix $x \in \mathcal{X}$. By Assumption 5 we have the strict proxy decrease [eq:prelim:Vdec]. By Proposition 13 (quadratic perturbation bound) and the relative error certificate [eq:prelim:relerr],

$$\left|V(f(x)) - V\left(\hat{f}(x)\right)\right| \le 2\lambda_{max}(P)\,R\,\varepsilon\,\|x\|^2.$$

Therefore

$$\begin{aligned} V(f(x)) - V(x) &\le \left(V\left(\hat{f}(x)\right) - V(x)\right) \\ &\quad + \left|V(f(x)) - V\left(\hat{f}(x)\right)\right| \\ &\le -c\|x\|^2 + 2\lambda_{max}(P)\,R\,\varepsilon\,\|x\|^2, \end{aligned}$$

which is [eq:main:true-dec] with $\tilde{c}$ as in [eq:main:c-tilde]. Finally, Lemma 9 yields [eq:main:exp-rate]. □

Remark 12. The proof shows a constraint. A Lipschitz modulus for V produces a linear perturbation term. To retain a quadratic decrease suitable for Lemma 9 with explicit constants, we work with certificates where the perturbation of V is itself controlled by a quadratic expression. Quadratic Lyapunov functions provide precisely this. This is standard in robust control, but it is usually hidden behind qualitative small-gain language.

A computable modulus for quadratic certificates

We now make the previous point explicit. This is the regime used in all worked examples.

Proposition 13 (Quadratic V yields a checkable perturbation bound). Let $\mathcal{X} \subset \mathbb{R}^n$ be compact and assume $\mathcal{X} \subseteq \mathbb{B}(R)$ for some $R > 0$. Let $P \in \mathbb{R}^{n\times n}$ be symmetric positive definite and define $V(x) := x^\top P x$. Then [eq:prelim:Vbounds] holds with $\alpha_1 = \lambda_{min}(P)$, $\alpha_2 = \lambda_{max}(P)$. Moreover, for all $x, z \in \mathcal{X}$, $|V(x) - V(z)| \le 2\lambda_{max}(P)\,R\,\|x - z\|$. If, in addition, $f, \hat{f}: \mathcal{X} \to \mathcal{X}$ satisfy [eq:prelim:relerr], then for all $x \in \mathcal{X}$, $\left|V(f(x)) - V\left(\hat{f}(x)\right)\right| \le 2\lambda_{max}(P)\,R\,\varepsilon\|x\|$. If $\mathcal{X}$ is chosen so that $\|x\| \le R$ for all $x \in \mathcal{X}$, then [eq:main:V-pert-quad] strengthens to $\left|V(f(x)) - V\left(\hat{f}(x)\right)\right| \le 2\lambda_{max}(P)\,R^2\,\varepsilon\,\frac{\|x\|^2}{R} = 2\lambda_{max}(P)\,R\,\varepsilon\|x\|^2$. Thus the perturbation of V is quadratically dominated on $\mathcal{X}$ with an explicit constant.

Proof. The bounds with $\lambda_{min}(P)$ and $\lambda_{max}(P)$ are standard. For [eq:main:quad-Lip], write

$$V(x) - V(z) = (x - z)^\top P(x + z),$$

hence

$$\begin{aligned} |V(x) - V(z)| &\le \|x - z\|\,\|P\|\,\|x + z\| \\ &\le \lambda_{max}(P)\,\|x - z\|\,(\|x\| + \|z\|) \\ &\le 2\lambda_{max}(P)\,R\,\|x - z\|. \end{aligned}$$

For [eq:main:V-pert-quad] apply [eq:main:quad-Lip] to $x = f(x)$ and $z = \hat{f}(x)$ and use [eq:prelim:relerr]. The last strengthening uses $\|x\| \le R$ on $\mathcal{X}$. □

Remark 14. Quadratic certificates with explicit eigenvalue bounds are the natural objects for LMI-based synthesis and verification. This includes robust stabilization and data-driven LMI formulas. The point here is not the existence of LMIs. It is the explicit conversion of their feasible solution into a numerical margin $\varepsilon_\star$.

Verifiable hypotheses for the primary theorem

We now state, without ambiguity, what the reader must check to apply Theorem 11 in the quadratic regime. The list is short.

1. Domain. Choose a compact $\mathcal{X} \subset \mathbb{R}^n$ with $0 \in \mathcal{X}$, and compute a radius R such that $\mathcal{X} \subseteq \mathbb{B}(R)$.
2. Certificate. Exhibit $P \succ 0$ and set $V(x) = x^\top P x$. Compute $\alpha_1 = \lambda_{min}(P)$ and $\alpha_2 = \lambda_{max}(P)$.
3. Proxy decrease margin. Verify, on $\mathcal{X}$, the strict inequality

$$V\left(\hat{f}(x)\right) - V(x) \le -c\|x\|^2$$

- with an explicit $c > 0$. For linear $\hat{f}(x) = \hat{A}x$ this reduces to the matrix inequality

$$\hat{A}^\top P \hat{A} - P \preccurlyeq -cI,$$

- which is checkable by eigenvalues.

4. Perturbation modulus. Compute the modulus $L_V^{(2)} := 2\lambda_{max}(P)R$ from Proposition 13. This yields an explicit bound for $\left|V(f(x)) - V\left(\hat{f}(x)\right)\right|$ in terms of $\|f(x) - \hat{f}(x)\|$.
5. Relative modeling error bound. Produce a certified bound ε such that
$$\|f(x) - \hat{f}(x)\| \le \varepsilon\|x\| \qquad (x \in \mathcal{X}).$$

- In linear data-driven control this can be derived from residual bounds under bounded disturbances and rank conditions. In trajectory-based frameworks one may extract such bounds from consistency constraints and persistency of excitation assumptions.

6. Margin check. Compute the allowable error
$$\varepsilon_\star = \frac{c}{L_V^{(2)}} = \frac{c}{2\lambda_{max}(P)R},$$

- and verify $\varepsilon < \varepsilon_\star$. Then Theorem 11 yields exponential stability with rate [eq:main:exp-rate].

Remark 15. If one cannot produce a certified ε, the program stops. A simulation error is not a certificate. If one insists on an additive bound $\|f(x) - \hat{f}(x)\| \le \delta$, the conclusion changes to an ultimate bound (ISS-type) and must be stated as such. The present paper excludes that route.

A new lemma: from dataset residuals to a certified closed-loop relative error

Theorems such as Theorem 11 require a bound of the form
$$\|f(x) - \hat{f}(x)\| \le \varepsilon\|x\| \qquad (x \in \mathcal{X}),$$
with an explicit ε. Many papers replace this by a regression statement. That replacement is incorrect. A regression residual is not a dynamical robustness margin.

We now give a lemma that converts a certified data residual bound into a closed-loop relative error bound. It is not a trivial variant of a standard inequality because it performs a logically necessary reduction: it ties the object "noise in the data equation" to the object "relative error in the closed-loop map," in the norm demanded by Theorem 11. No such step is provided by the usual "learn → simulate → claim" pattern.

Assumption 16 (Data equation with certified residual bound). Consider an unknown discrete-time LTI system $x_{t+1} = Ax_t + Bu_t + w_t$, where $A \in \mathbb{R}^{n\times n}$ and $B \in \mathbb{R}^{n\times m}$ are unknown. Fix a horizon $T \ge 1$ and form the data matrices $X := [x_0 \quad x_1 \quad \cdots \quad x_{T-1}] \in \mathbb{R}^{n\times T}$, $U := [u_0 \quad u_1 \quad \cdots \quad u_{T-1}] \in \mathbb{R}^{m\times T}$, $X^+ := [x_1 \quad x_2 \quad \cdots \quad x_T] \in \mathbb{R}^{n\times T}$, $W := [w_0 \quad w_1 \quad \cdots \quad w_{T-1}] \in \mathbb{R}^{n\times T}$. Assume the exact data equation $X^+ = \Theta Z + W$, $\quad \Theta := [A \quad B]$, $\quad Z := \begin{bmatrix} X \\ U \end{bmatrix}$. Assume that the disturbance matrix admits a certified operator-norm bound $\|W\| \le \bar{W}$, where $\bar{W} \ge 0$ is obtained from sensor guarantees or from a validated noise model.

Definition 17 (Least-squares proxy consistent with the data). Assume Z has full row rank. Define the proxy parameter $\hat{\Theta} := X^+ Z^\dagger$, where $Z^\dagger$ is the Moore–Penrose pseudoinverse.

Lemma 18 (Residual-to-relative-error conversion for a fixed feedback gain). Assume Assumption 16 and Definition 17, and fix a feedback gain $K \in \mathbb{R}^{m\times n}$. Define the true and proxy closed-loop maps $f(x) := (A + BK)x$, $\quad \hat{f}(x) := \left(\hat{A} + \hat{B}K\right)x$, where $\hat{\Theta} = \left[\hat{A}\ \hat{B}\right]$. Then for every $x \in \mathbb{R}^n$, $\|f(x) - \hat{f}(x)\| \le \varepsilon_{data}(K)\,\|x\|$, with the explicit bound $\varepsilon_{data}(K) := \bar{W}\ \|Z^\dagger\|\ \left\|\begin{bmatrix} I \\ K \end{bmatrix}\right\|$. In particular, once $\bar{W}$ is certified and Z is computed from data, the constant $\varepsilon_{data}(K)$ is computable.

Proof. From [eq:data-eq] and [eq:Theta-hat],
$$\hat{\Theta} = X^+ Z^\dagger = (\Theta Z + W)Z^\dagger = \Theta(ZZ^\dagger) + WZ^\dagger.$$
Since Z has full row rank, $ZZ^\dagger = I$ on $\mathbb{R}^{n+m}$, hence
$$\Theta - \hat{\Theta} = -WZ^\dagger.$$
Therefore, for any $x \in \mathbb{R}^n$,
$$f(x) - \hat{f}(x) = \left(\Theta - \hat{\Theta}\right)\begin{bmatrix} I \\ K \end{bmatrix} x = -WZ^\dagger \begin{bmatrix} I \\ K \end{bmatrix} x.$$
Taking norms and using submultiplicativity yields
$$\|f(x) - \hat{f}(x)\| \le \|W\|\ \|Z^\dagger\|\ \left\|\begin{bmatrix} I \\ K \end{bmatrix}\right\|\ \|x\|.$$
Now apply [eq:Wbound] to obtain [eq:lem:relerr]–[eq:lem:epsdata].

Remark 19 (Why this lemma is not a cosmetic rewrite). The conclusion [eq:lem:relerr] is the exact hypothesis needed in Theorem 11. The hypothesis [eq:Wbound] is a certified statement about the disturbance in the data equation. The lemma converts one into the other with an explicit constant [eq:lem:epsdata]. This step is absent in typical learning-control narratives, which often stop at a regression bound for $\widehat{\boldsymbol{\Theta}}$. The conversion also displays the conditioning factor $\| \boldsymbol{Z}^{\dagger} \|$. It follows that poor excitation makes $\boldsymbol{\varepsilon_{data}}(\boldsymbol{K})$ large even if $\bar{\boldsymbol{W}}$ is small. This is a quantitative obstruction, not a philosophical one.

Remark 20 (Relation to trajectory-based data-driven control). Trajectory-based methods use data matrices and rank conditions to replace model knowledge by data constraints. Lemma 18 is compatible with that viewpoint: it states that the quantity which matters for Theorem 11 is a certified operator-norm residual bound propagated through the pseudoinverse of the data matrix. When the disturbance model gives only probabilistic bounds, the present paper does not claim determinism. One must then state a confidence level explicitly, which is excluded by the present "no speculation" program.

A margin theorem combining the Lyapunov certificate with the data bound

We now combine Theorem 11 with Lemma 18 in the quadratic regime.

Proposition 21 (A computable allowable residual bound). Assume the quadratic certificate regime of Proposition 13. Let $\boldsymbol{\mathcal{X}} \subseteq \mathbb{B}(\boldsymbol{R})$ and let $\boldsymbol{V}(\boldsymbol{x}) = \boldsymbol{x}^{\top}\boldsymbol{P}\boldsymbol{x}$ with $\boldsymbol{P} \succ \boldsymbol{0}$. Assume that the proxy closed-loop map $\hat{\boldsymbol{f}}(\boldsymbol{x}) = (\widehat{\boldsymbol{A}} + \widehat{\boldsymbol{B}}\boldsymbol{K})\boldsymbol{x}$ satisfies $(\widehat{\boldsymbol{A}} + \widehat{\boldsymbol{B}}\boldsymbol{K})^{\top}\boldsymbol{P}(\widehat{\boldsymbol{A}} + \widehat{\boldsymbol{B}}\boldsymbol{K}) - \boldsymbol{P} \preccurlyeq -\boldsymbol{c}\boldsymbol{I}$ for some $\boldsymbol{c} > \boldsymbol{0}$. Assume Assumption 16–Definition 17 and define $\boldsymbol{\varepsilon_{data}}(\boldsymbol{K})$ by [eq:lem:epsdata]. If $\bar{\boldsymbol{W}} < \bar{\boldsymbol{W}}_{\star} := \frac{\boldsymbol{c}}{\boldsymbol{2\lambda_{max}(P)R}} \cdot \frac{\boldsymbol{1}}{\|\boldsymbol{Z}^{\dagger}\| \left\| \begin{bmatrix} \boldsymbol{I} \\ \boldsymbol{K} \end{bmatrix} \right\|}$, then the true closed loop $\boldsymbol{f}(\boldsymbol{x}) = (\boldsymbol{A} + \boldsymbol{B}\boldsymbol{K})\boldsymbol{x}$ is exponentially stable on $\boldsymbol{\mathcal{X}}$.

All quantities in [eq:allow:Wstar] are computable from $(\boldsymbol{P}, \boldsymbol{c}, \boldsymbol{R}, \boldsymbol{K})$ and the data matrix $\boldsymbol{Z}$, except $\bar{\boldsymbol{W}}$ which is an external certified noise bound.

Proof. By Lemma 18, [eq:lem:relerr] holds with $\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon_{data}}(\boldsymbol{K})$. By Proposition 13, the quadratic perturbation modulus on $\boldsymbol{\mathcal{X}}$ is $\boldsymbol{L_V^{(2)}} = \boldsymbol{2\lambda_{max}}(\boldsymbol{P})\boldsymbol{R}$. Thus Theorem 11 applies if $\boldsymbol{\varepsilon_{data}}(\boldsymbol{K}) < \boldsymbol{c}/\boldsymbol{L_V^{(2)}}$. This inequality is equivalent to [eq:allow:Wstar] by [eq:lem:epsdata]. □

Remark 22. Proposition 21 has the form robust control demands: a single explicit inequality which can be checked numerically. It differs from classical robust stability results because the uncertainty size is not postulated. It is inferred from a data residual bound, and the inference is explicit. This is the only step that makes the result "data-driven" in a mathematically relevant sense.

Computable robustness margins for switched/networked implementations

We now give an explicit dwell-time margin. The statement is quantitative. No "small enough" remains. The proof is a single inequality chase.

The starting point is classical: stability of switched systems under average dwell time. The present result differs in two respects. First, it keeps the constants explicit in a form suited to verification. Second, it allows each mode to be known only through a data-driven proxy, with an explicit per-mode error budget.

Definition 23 (Average dwell time). Let $\boldsymbol{\sigma}: \mathbb{N} \to \{\boldsymbol{1}, \dots, \boldsymbol{M}\}$ be a switching signal. Let $\boldsymbol{N_{\sigma}}(\boldsymbol{k_0}, \boldsymbol{k})$ denote the number of switches on $\{\boldsymbol{k_0}, \boldsymbol{k_0} + \boldsymbol{1}, \dots, \boldsymbol{k} - \boldsymbol{1}\}$. Given $\boldsymbol{N_0} \in \mathbb{N}$ and $\boldsymbol{N_a} > \boldsymbol{0}$, we say that $\boldsymbol{\sigma}$ has average dwell time $\boldsymbol{N_a}$ with chatter bound $\boldsymbol{N_0}$ if $\boldsymbol{N_{\sigma}}(\boldsymbol{k_0}, \boldsymbol{k}) \leq \boldsymbol{N_0} + \frac{\boldsymbol{k-k_0}}{\boldsymbol{N_a}}$ for all $\boldsymbol{0} \leq \boldsymbol{k_0} < \boldsymbol{k}$.

Assumption 24 (Mode-wise proxy certificates and cross-comparability). For each mode $\boldsymbol{i} \in \{\boldsymbol{1}, \dots, \boldsymbol{M}\}$ let $\hat{\boldsymbol{f}}_{\boldsymbol{i}}: \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{X}}$ be a proxy map and let $\boldsymbol{V_i}(\boldsymbol{x}) := \boldsymbol{x}^{\top}\boldsymbol{P_i}\boldsymbol{x}$ with $\boldsymbol{P_i} \succ \boldsymbol{0}$. Assume there exist numbers $\boldsymbol{\alpha_i} \in (\boldsymbol{0}, \boldsymbol{1})$ such that for all $\boldsymbol{x} \in \boldsymbol{\mathcal{X}}$, $\boldsymbol{V_i}\left(\hat{\boldsymbol{f}}_{\boldsymbol{i}}(\boldsymbol{x})\right) \leq (\boldsymbol{1} - \boldsymbol{\alpha_i})\,\boldsymbol{V_i}(\boldsymbol{x})$. Assume also that there exists $\boldsymbol{\mu} \geq \boldsymbol{1}$ such that for all $\boldsymbol{i}, \boldsymbol{j}$ and all $\boldsymbol{x} \in \boldsymbol{\mathcal{X}}$, $\boldsymbol{V_j}(\boldsymbol{x}) \leq \boldsymbol{\mu}\,\boldsymbol{V_i}(\boldsymbol{x})$.

Assumption 25 (Mode-wise relative model error). For each $\boldsymbol{i}$ the true map $\boldsymbol{f_i}: \boldsymbol{\mathcal{X}} \to \boldsymbol{\mathcal{X}}$ satisfies a certified relative error bound $\|\boldsymbol{f_i}(\boldsymbol{x}) - \hat{\boldsymbol{f}}_{\boldsymbol{i}}(\boldsymbol{x})\| \leq \boldsymbol{\varepsilon_i}\|\boldsymbol{x}\| \quad (\boldsymbol{x} \in \boldsymbol{\mathcal{X}})$.

Theorem 26 (Explicit average-dwell-time margin under certified proxy error). Consider the switched true system $\boldsymbol{x_{k+1}} = \boldsymbol{f_{\sigma(k)}}(\boldsymbol{x_k})$, and its proxy family $\{\hat{\boldsymbol{f}}_{\boldsymbol{i}}\}$. Assume Assumption 24 and Assumption 25. Let $\boldsymbol{R} > \boldsymbol{0}$ satisfy $\boldsymbol{\mathcal{X}} \subseteq \mathbb{B}(\boldsymbol{R})$. For each $\boldsymbol{i}$, define $\bar{\boldsymbol{\lambda}}_{\boldsymbol{i}} := \boldsymbol{\lambda_{max}}(\boldsymbol{P_i})$, $\quad \underline{\boldsymbol{\lambda}}_{\boldsymbol{i}} := \boldsymbol{\lambda_{min}}(\boldsymbol{P_i})$, $\quad \boldsymbol{L_i^{(2)}} := \boldsymbol{2}\bar{\boldsymbol{\lambda}}_{\boldsymbol{i}}\boldsymbol{R}$, so that $\boldsymbol{L_i^{(2)}}$ is the quadratic perturbation modulus from Proposition 13. Define the robust per-mode contraction factors $\beta_i := (1-\alpha_i) + \frac{L_i^{(2)}\varepsilon_i}{\underline{\lambda}_i}$. Assume $\beta_i < 1$ for all i. Set $\beta_{max} := \max_{1\le i\le M} \beta_i \in (0,1)$. If the switching signal σ has average dwell time N_a with chatter bound N_0 and $N_a > N_a^\star := \frac{\log\mu}{-\log\beta_{max}}$, then the origin of [eq:switched:true] is exponentially stable on $\mathcal{X}$. More precisely, for any trajectory remaining in $\mathcal{X}$, $\|x_k\| \le \sqrt{\frac{\max_i \bar{\lambda}_i}{\min_i \underline{\lambda}_i}}\, \mu^{N_0/2}\, \beta_{max}^{(k-k_0)/(2N_a)} \,\|x_{k_0}\| \qquad (k \ge k_0)$.

All constants in [eq:switched:beta]–[eq:switched:Na-star] are computable from (P_i, α_i), the radius R, and the certified error levels ε_i.

Proof. Fix i and $x \in \mathcal{X}$. By Proposition 13 with $V_i(x) = x^\top P_i x$,

$$\left|V_i(f_i(x)) - V_i\left(\hat{f}_i(x)\right)\right| \le L_i^{(2)} \|f_i(x) - \hat{f}_i(x)\|.$$

Using [eq:switched:relerr] and $\|x\|^2 \le V_i(x)/\underline{\lambda}_i$ yields

$$V_i(f_i(x)) \le V_i\left(\hat{f}_i(x)\right) + L_i^{(2)}\varepsilon_i \|x\|$$
$$\le V_i\left(\hat{f}_i(x)\right) + \frac{L_i^{(2)}\varepsilon_i}{\underline{\lambda}_i} V_i(x).$$

Combine [eq:switched:pert] with the proxy contraction [eq:switched:proxy-dec] to get

$$V_i(f_i(x)) \le \left((1-\alpha_i) + \frac{L_i^{(2)}\varepsilon_i}{\underline{\lambda}_i}\right) V_i(x) = \beta_i V_i(x).$$

Thus, along the true trajectory, at time k with mode $i = \sigma(k)$,

$$V_{\sigma(k)}(x_{k+1}) \le \beta_{\sigma(k)}\, V_{\sigma(k)}(x_k) \le \beta_{max} V_{\sigma(k)}(x_k).$$

At a switch from mode i to mode j, the Lyapunov index changes. The comparability [eq:switched:mu] gives

$$V_j(x) \le \mu V_i(x).$$

Let $S := N_\sigma(k_0, k)$ be the number of switches on $\{k_0, \dots, k-1\}$. Iterating [eq:switched:one-step] and inserting [eq:switched:switch-jump] at each switch yields

$$V_{\sigma(k)}(x_k) \le \mu^S\, \beta_{max}^{k-k_0}\, V_{\sigma(k_0)}(x_{k_0}).$$

Using Definition 23 we have $S \le N_0 + (k-k_0)/N_a$, hence

$$V_{\sigma(k)}(x_k) \le \mu^{N_0} \left(\mu^{1/N_a}\beta_{max}\right)^{k-k_0} V_{\sigma(k_0)}(x_{k_0}).$$

The condition [eq:switched:Na-star] is equivalent to $\mu^{1/N_a}\beta_{max} < 1$. Therefore $V_{\sigma(k)}(x_k)$ decays exponentially. Convert V_i to $\|x\|^2$ using $\underline{\lambda}_i\|x\|^2 \le V_i(x) \le \bar{\lambda}_i \|x\|^2$ to obtain [eq:switched:exp]. □

Verifiable hypotheses for the dwell-time margin

For Theorem 26 the checkable items are as follows.

1. Proxy contractions. For each mode i, compute $P_i \succ 0$ and α_i such that [eq:switched:proxy-dec] holds on $\mathcal{X}$. In linear modes $\hat{f}_i(x) = \hat{A}_i x$, this is the LMI/eigenvalue check
$$\hat{A}_i^\top P_i \hat{A}_i \preccurlyeq (1-\alpha_i) P_i.$$
- This is a standard synthesis/verification pattern.
2. Comparability constant. Compute any $\mu \ge 1$ satisfying [eq:switched:mu]. For quadratic $V_i(x) = x^\top P_i x$ it suffices to take
$$\mu \ge \max_{i,j} \frac{\lambda_{max}(P_j)}{\lambda_{min}(P_i)},$$
- which is explicit.
3. Certified mode-wise errors. For each i, provide a certified ε_i for [eq:switched:relerr]. A data route for LTI modes is given by Lemma 18. Trajectory-based routes provide the same object, but with different data matrices.
4. Compute β_i and the dwell-time threshold. Compute β_i from [eq:switched:beta] and verify [eq:switched:beta<1]. Then compute $N_a^\star$ from [eq:switched:Na-star].

Remark 27. Networked control with bounded delays and packet drops is often modeled as switching among a finite family of discrete-time maps on an augmented state. In that reduction, Theorem 26 gives an explicit margin in the form of an average dwell time constraint and certified per-mode proxy errors. This is not a survey claim. It is a template for a fully checkable corollary once the network mechanism is encoded as a finite mode set.

Remark 28. Average dwell time stability is classical. The new content here is the explicit appearance of $(\varepsilon_i, \|Z^\dagger\|)$-type terms inside β_i. It forces the correct conclusion: poor excitation or large certified residuals enlarge β_i and thus enlarge $N_a^\star$. This is a quantitative obstruction. It cannot be removed by rhetoric.

Secondary theorems and corollaries

We now connect the spine to event-triggered and networked implementations. The principle is the same.

One bounds the induced model mismatch in a relative form and checks an explicit margin.

Event-triggered implementation: no-Zeno and a computable decay margin

We pass to continuous time for this corollary. Zeno is a continuous-time phenomenon. Discrete-time "event-triggering" is vacuous because time is already quantized.

Assumption 29 (True and proxy continuous-time closed loops). Let $\mathcal{X} \subset \mathbb{R}^n$ be compact, $\mathbf{0} \in \mathcal{X}$. Let $g, \hat{g}: \mathcal{X} \to \mathbb{R}^n$ be locally Lipschitz vector fields with solutions remaining in $\mathcal{X}$ on their interval of existence. The true and proxy closed loops are $\dot{x} = g(x)$, $\dot{\hat{x}} = \hat{g}(\hat{x})$. Assume a certified relative vector-field error on $\mathcal{X}$: $\|g(x) - \hat{g}(x)\| \leq \varepsilon\|x\| \qquad (x \in \mathcal{X})$.

Assumption 30 (Quadratic proxy Lyapunov inequality). Let $V(x) = x^\top P x$ with $P \succ 0$. Assume there exists $c > 0$ such that for all $x \in \mathcal{X}$, $\nabla V(x) \cdot \hat{g}(x) \leq -c\|x\|^2$.

Remark 31. Quadratic V is used here for the same reason as in Section 3.3: it yields explicit moduli for ∇V. No qualitative smoothness is invoked.

We consider event-triggered control as follows. Let $u = \kappa(x)$ be the feedback law defining g. At event times $\{t_k\}$, the controller samples $x(t_k)$ and holds the input constant between events. Equivalently, the plant evolves under an input based on the last sample, generating a measurement error

$$e(t) := x(t_k) - x(t), \qquad t \in [t_k, t_{k+1}).$$

A standard trigger enforces a relative bound on $\|e(t)\|$ in terms of $\|x(t)\|$.

Assumption 32 (Trigger-compatible dissipation inequality). Assume there exist numbers $c > 0$ and $\gamma \geq 0$ such that, along the event-held closed-loop dynamics, $\dot{V}(x(t)) \leq -c\|x(t)\|^2 + \gamma\|e(t)\|^2 \qquad$ whenever $x(t) \in \mathcal{X}$.

Definition 33 (Relative event-trigger). Fix $\sigma \in (0,1)$. Define event times recursively by $t_0 = 0$ and $t_{k+1} := \inf\{t > t_k : \|e(t)\| \geq \sigma\|x(t)\|\}$.

The next theorem has two claims. First, it yields an explicit decay margin under a checkable inequality. Second, it yields a strictly positive inter-event lower bound, hence no Zeno.

Theorem 34 (Event-triggered exponential stability with explicit no-Zeno margin). Assume Assumption 29 and Assumption 32. Let $V(x) = x^\top P x$ with $P \succ 0$, and let $\underline{\lambda} = \lambda_{min}(P)$, $\bar{\lambda} = \lambda_{max}(P)$. Assume the trigger Definition 33 with some $\sigma \in (0,1)$. If $\gamma\sigma^2 < c$, then along every event-triggered trajectory remaining in $\mathcal{X}$, $\dot{V}(x(t)) \leq -(c - \gamma\sigma^2)\|x(t)\|^2$. Consequently, $x(t)$ converges exponentially to $\mathbf{0}$ on $\mathcal{X}$, with explicit rate $\|x(t)\| \leq \sqrt{\frac{\bar{\lambda}}{\underline{\lambda}}}\, exp \left(-\frac{c-\gamma\sigma^2}{2\bar{\lambda}}\, t\right) \|x(0)\|$.

Assume in addition that g is Lipschitz on $\mathcal{X}$ with modulus L_g and that the held-input closed-loop error satisfies $\frac{d}{dt}\|e(t)\| \leq L_g(\|x(t)\| + \|e(t)\|) \qquad$ for a.e. $t \in [t_k, t_{k+1})$. Then the inter-event times satisfy the explicit bound $t_{k+1} - t_k \geq \tau_\star := \frac{1}{L_g} log(1+\sigma)$, hence Zeno behavior is excluded.

Proof. On $[t_k, t_{k+1})$ the trigger [eq:trigger] enforces $\|e(t)\| \leq \sigma\|x(t)\|$. Insert this into [eq:ct:dissip] to obtain

$$\begin{aligned}\dot{V}(x(t)) &\leq -c\|x(t)\|^2 + \gamma\sigma^2\|x(t)\|^2 \\ &= -(c - \gamma\sigma^2)\|x(t)\|^2,\end{aligned}$$

which is [eq:event:Vdec]. Since $V(x) \leq \bar{\lambda}\|x\|^2$ and $\underline{\lambda}\|x\|^2 \leq V(x)$, we have $\|x\|^2 \geq V/\bar{\lambda}$. Thus

$$\dot{V} \leq -\frac{c - \gamma\sigma^2}{\bar{\lambda}} V.$$

Integrating yields $V(x(t)) \leq e^{-(c-\gamma\sigma^2)t/\bar{\lambda}} V(x(0))$, and [eq:event:rate] follows.

For the inter-event bound, define $\eta(t) := \|e(t)\|/\|x(t)\|$ for $t \in [t_k, t_{k+1})$ with $x(t) \neq \mathbf{0}$. At $t = t_k$ one has $e(t_k) = \mathbf{0}$, hence $\eta(t_k) = \mathbf{0}$. Under [eq:event:edot] and Lipschitzness of g on $\mathcal{X}$, one obtains the differential inequality

$$\dot{\eta}(t) \leq L_g(1 + \eta(t))$$

in the sense of upper Dini derivatives. This is the standard comparison step used in event-triggering. Solving yields $\eta(t) \leq e^{L_g(t-t_k)} - 1$. An event occurs only when $\eta(t) \geq \sigma$. Thus $\sigma \leq e^{L_g(t_{k+1}-t_k)} - 1$, which is equivalent to [eq:event:tau]. □

Verifiable hypotheses for Theorem 34

The checkable items are as follows.

1. Quadratic V and dissipation constants. Exhibit $P \succ 0$ and compute (c, γ) such that [eq:ct:dissip] holds on $\mathcal{X}$. In linear cases this is an LMI/eigenvalue check with explicit constants.
2. Trigger parameter. Choose $\sigma \in (0, 1)$ and verify the strict margin [eq:event:sigma]. Then the decay constant is $\tilde{c} := c - \gamma\sigma^2 > 0$, which is explicit.
3. No-Zeno modulus. Compute any Lipschitz modulus L_g of the closed-loop vector field on $\mathcal{X}$. Then $\tau_\star$ in [eq:event:tau] is explicit. If L_g cannot be certified, the program stops. A simulation-based estimate is not accepted.

Remark 35. The proof isolates the only delicate point: [eq:event:edot]. One must bound the growth of the measurement error between events. This is where many informal arguments break. If one writes $\dot{e} = -\dot{x}$ and then asserts $\|\dot{x}\| \leq L\|x\|$ without producing L, one has not proved no Zeno. The present theorem requires L_g explicitly.

A corollary linking event-triggering to the spine margin

We now state the promised link. The event-triggered implementation induces an additional modeling error because the input is held. If this induced error is bounded by a certified relative constant, then Theorem 11 applies on the sampled map. This is the same logic as before.

Corollary 36 (Spine margin under certified hold-induced mismatch). Let $\hat{f}$ be a proxy discrete-time closed-loop map admitting a quadratic certificate as in Section 3.3, with constants (P, c, R) and allowable error $\varepsilon_\star = \frac{c}{2\lambda_{max}(P)R}$. Assume the event-triggered implementation produces a true discrete-time return map f on event times such that $\|f(x) - \hat{f}(x)\| \leq \varepsilon_{hold}\|x\| \quad (x \in \mathcal{X})$, with a certified ε_{hold}. If $\varepsilon_{hold} < \varepsilon_\star$, then the event-time sampled dynamics is exponentially stable on $\mathcal{X}$.

Proof. This is Theorem 11 in the quadratic regime, with $\varepsilon = \varepsilon_{hold}$. □

Remark 37. Corollary 36 reduces event-triggered stability certification to one number: ε_{hold}. Bounding ε_{hold} is a modeling step. It depends on the plant and on the hold mechanism. The paper does not pretend this bound is automatic. It must be supplied as a certificate, or the conclusion must be weakened. This is the verifiability discipline.

We record that explicit robustness-margin formulations have recently been pursued in learning-based control from a different angle.

Extensions

Switched certificates from data.

Recent arXiv work proposes neural multiple Lyapunov function verification for switched systems. We only use it as a point of comparison for verifiable hypotheses.

Corollaries toward the two non-spine topics

The spine theorem Theorem 11 is a perturbation result. It is indifferent to the source of the perturbation. This permits a clean extension principle:

Reduce the implementation mechanism (sampling, delay, packet drops, triggering) to a certified relative mismatch bound $\|f - \hat{f}\| \leq \varepsilon\|x\|$ on a compact set $\mathcal{X}$. Then check the single inequality $\varepsilon < \varepsilon_\star$.

We state one explicit corollary for sampled-data implementations and one for bounded delays. Both are quantitative. Both expose what must be computed.

Sampling constraints as a certified model mismatch

Let $\dot{x} = g(x)$ be a continuous-time closed loop. Sampling with zero-order hold induces a discrete-time return map. A common informal argument asserts that the sampled map is "close" to the Euler step. Closeness is not a certificate. We now state a certificate with explicit constants.

Assumption 38 (Certified Lipschitz and linear-growth bounds on $\mathcal{X}$). Let $\mathcal{X} \subset \mathbb{R}^n$ be compact and forward invariant for $\dot{x} = g(x)$. Assume $g(0) = 0$. Assume there exist computable constants $L_g, G \geq 0$ such that for all $x, z \in \mathcal{X}$, $\|g(x) - g(z)\| \leq L_g\|x - z\|$, and for all $x \in \mathcal{X}$, $\|g(x)\| \leq G\|x\|$.

Let $\phi_h(x)$ denote the exact flow map at time $h > 0$, i.e. the solution at time h starting from x. Define the proxy one-step map

$$\hat{f}_h(x) := x + hg(x).$$

Lemma 39 (A certified relative bound for flow–Euler mismatch). Assume Assumption 38. For any $h > 0$

such that $\boldsymbol{\phi_t(x) \in \mathcal{X}}$ for $\boldsymbol{t \in [0, h]}$ and all $\boldsymbol{x \in \mathcal{X}}$, one has $\boldsymbol{\|\phi_h(x) - \hat{f}_h(x)\| \le \varepsilon_{samp}(h)\, \|x\|}$ $\quad(\boldsymbol{x \in \mathcal{X}})$, with the explicit function $\boldsymbol{\varepsilon_{samp}(h) := \frac{G}{L_g}\left(e^{L_g h} - 1 - L_g h\right)}$ $\quad\boldsymbol{(L_g > 0)}$, and in the case $\boldsymbol{L_g = 0}$ the bound becomes $\boldsymbol{\varepsilon_{samp}(h) = 0}$.

Proof. Fix $\boldsymbol{x \in \mathcal{X}}$ and write $\boldsymbol{x(t) := \phi_t(x)}$. Then

$$\boldsymbol{\phi_h(x) - \hat{f}_h(x) = \int_0^h \Big(g\big(x(t)\big) - g(x)\Big)\, dt.}$$

By [eq:sampling:Lg],

$$\boldsymbol{\|\phi_h(x) - \hat{f}_h(x)\| \le \int_0^h L_g\, \|x(t) - x\|\, dt.}$$

Also,

$$\boldsymbol{x(t) - x = \int_0^t g\,\big(x(s)\big)\, ds,}$$

hence using [eq:sampling:G] and the standard Grönwall estimate for Lipschitz ODEs on compact sets,

$$\boldsymbol{\|x(t) - x\| \le \int_0^t \|g\big(x(s)\big)\|\, ds \le \int_0^t G\, \|x(s)\|\, ds \le \frac{G}{L_g}\left(e^{L_g t} - 1\right)\|x\|.}$$

Insert this bound into the previous inequality to obtain

$$\boldsymbol{\|\phi_h(x) - \hat{f}_h(x)\| \le \int_0^h L_g \cdot \frac{G}{L_g}\left(e^{L_g t} - 1\right) dt\, \|x\| = \frac{G}{L_g}\left(e^{L_g h} - 1 - L_g h\right)\|x\|.}$$

This is [eq:sampling:mismatch]–[eq:sampling:epsh]. We now connect this mismatch to the spine margin.

Corollary 40 (A computable maximum sampling period). Assume the quadratic certification regime of Section 3.3 for a proxy discrete-time map $\boldsymbol{\hat{f}}$ with allowable relative error $\boldsymbol{\varepsilon_\star > 0}$. Assume Assumption 38 for a continuous-time closed loop $\boldsymbol{\dot{x} = g(x)}$ on $\boldsymbol{\mathcal{X}}$. Let $\boldsymbol{f_h := \phi_h}$ and let $\boldsymbol{\hat{f}_h}$ be defined by [eq:sampling:euler]. If the proxy certificate and discretization are chosen so that $\boldsymbol{\hat{f}_h = \hat{f}}$ on $\boldsymbol{\mathcal{X}}$, then the sampled true map $\boldsymbol{f_h}$ is exponentially stable on $\boldsymbol{\mathcal{X}}$ provided $\boldsymbol{\varepsilon_{samp}(h) < \varepsilon_\star}$, where $\boldsymbol{\varepsilon_{samp}(h)}$ is given by [eq:sampling:epsh]. In particular, any $\boldsymbol{h}$ satisfying [eq:sampling:hstar] is an admissible sampling period.

Proof. Lemma 39 provides the certified relative error bound $\boldsymbol{\|f_h(x) - \hat{f}_h(x)\| \le \varepsilon_{samp}(h)\|x\|}$ on $\boldsymbol{\mathcal{X}}$. Then Theorem 11 applies in the quadratic regime once [eq:sampling:hstar] holds. ☐

Remark 41. The corollary is not a numerical recipe. It is a logical reduction: sampling is acceptable if it induces a mismatch smaller than the allowable margin. All constants are explicit. Nothing is tuned by rhetoric.

Bounded delays as mode uncertainty with an explicit margin

We now treat bounded input/measurement delay. The standard route models delay as an augmented system or as a switching family. We state the reduction at the level needed for certification.

Consider a discrete-time family $\boldsymbol{x_{k+1} = f(x_k, u_k)}$ with a static feedback $\boldsymbol{u_k = \kappa(x_k)}$. With a delay of $\boldsymbol{d \in \{0, 1, \dots, D\}}$ steps, the implemented input is $\boldsymbol{u_k = \kappa(x_{k-d})}$. On an augmented state $\boldsymbol{\xi_k := (x_k, x_{k-1}, \dots, x_{k-D})}$ this becomes a delay-free map. The delay then appears as switching among finitely many maps indexed by $\boldsymbol{d}$.

Corollary 42 (Delay bound via the dwell-time margin). Assume the augmented-state reduction yields a finite family of true maps $\boldsymbol{\{f_d\}_{d=0}^{D}}$ on a compact set $\boldsymbol{\mathcal{X}_{aug}}$. Assume that for each delay mode $\boldsymbol{d}$ there exists a proxy map $\boldsymbol{\hat{f}_d}$ and a quadratic certificate satisfying Assumption 24 on $\boldsymbol{\mathcal{X}_{aug}}$ with explicit constants $\boldsymbol{(P_d, \alpha_d)}$, and certified mode-wise errors $\boldsymbol{\varepsilon_d}$ satisfying Assumption 25. If the resulting $\boldsymbol{\beta_d}$ in [eq:switched:beta] satisfy $\boldsymbol{\beta_d < 1}$ for all $\boldsymbol{d}$ and the switching induced by the network satisfies an average dwell time constraint $\boldsymbol{N_a > N_a^\star}$ as in Theorem 26, then the delayed closed loop is exponentially stable on $\boldsymbol{\mathcal{X}_{aug}}$.

In particular, if the network guarantee implies that switches among delays occur no faster than a computable $\boldsymbol{N_a(D)}$, then any $\boldsymbol{D}$ satisfying $\boldsymbol{N_a(D) > N_a^\star}$ is an admissible delay bound.

Proof. This is a direct application of Theorem 26 to the augmented switched model. ☐

Remark 43. The corollary is conditional. It is conditional on a finite-mode reduction and on a certified per-mode error budget. If a delay distribution is only described probabilistically, the paper does not claim determinism. One must then state a probability level explicitly, which is excluded by the present program.

Verifiable hypotheses for extensions

We list what must be checked for the extensions above.

1. Sampling extension. One must compute constants (L_g, G) satisfying [eq:sampling:Lg]–[eq:sampling:G] on the chosen compact set $\mathcal{X}$. Then $\varepsilon_{samp}(h)$ in [eq:sampling:epsh] is explicit. The admissible sampling periods are those satisfying [eq:sampling:hstar]. If L_g cannot be certified, the program stops.
2. Delay extension. One must supply a finite-mode reduction (augmented state) and then verify the inputs required by Theorem 26: mode-wise proxy contractions [eq:switched:proxy-dec], comparability [eq:switched:mu], and certified mode-wise relative errors [eq:switched:relerr].
3. No substitution by simulation. A simulated trajectory is not a bound. A fitted Lipschitz constant is not a bound. A confidence interval is not a bound unless the confidence level is stated and the theorem is probabilistic. The present paper is deterministic.

Excluded cases under the verifiability program

This subsection states, without mitigation, which commonly used assumptions are excluded and why. The exclusions are logical, not stylistic. Each excluded case fails to produce a checkable inequality of the form $\varepsilon < \varepsilon_\star$ required by the spine theorem.

Unknown or uncaptured Lipschitz constants

Many arguments assume the existence of a Lipschitz constant for a vector field, a closed-loop map, or a Lyapunov gradient, but do not compute it on a specified compact set. Such an assumption is not verifiable. Without a certified modulus, inequalities such as $\|g(x) - g(z)\| \le L\|x - z\|$ or $\left|V(f(x)) - V\left(\hat{f}(x)\right)\right| \le L_V\|f(x) - \hat{f}(x)\|$ have no numerical content. They cannot be used to derive $\varepsilon_\star$. Accordingly, any claim relying on an unspecified Lipschitz constant is excluded.

Additive error bounds presented as stability guarantees

Bounds of the form

$$\|f(x) - \hat{f}(x)\| \le \delta$$

are frequently used as surrogates for robustness. They are insufficient for exponential stability. Near the origin, an additive disturbance dominates any contraction and produces invariant neighborhoods. The correct conclusion is input-to-state stability with an ultimate bound, not asymptotic convergence. If ISS is claimed, it must be stated explicitly with its gain. The present paper excludes the substitution of additive error bounds for relative ones.

Simulation-based validation

Simulation is an exploratory tool. It is not a certificate. A finite set of trajectories cannot bound $sup_{x\in\mathcal{X}}\|f(x) - \hat{f}(x)\|$. Nor can it verify a strict Lyapunov decrease inequality on a continuum. Any argument whose final justification is "observed in simulation" is excluded.

Probabilistic guarantees without stated confidence

Learning-based control often provides bounds that hold with high probability. Such results are meaningful when the probability level is stated and propagated through the analysis. If the probability level is omitted, or if the deterministic conclusion is stated without qualification, the claim is incomplete. The present paper is deterministic. Probabilistic extensions are excluded unless the confidence parameter is explicit and retained throughout.

Neural-network certificates without global bounds

Neural networks are frequently advertised as Lyapunov functions or controllers. Unless one computes certified bounds on their gradients or Jacobians on a compact set, they do not yield explicit constants. Pointwise verification, sampling, or heuristic regularization does not produce a modulus. Accordingly, neural-network-based certificates are excluded unless accompanied by certified global bounds.

Delay and network models without finite reduction

Time-varying delays, packet drops, or scheduling effects are often described informally. If they cannot be reduced to a finite family of modes or to a certified dwell-time constraint, they cannot be handled by Theorem 26. Claims of "robustness to arbitrary delays" without a numerical bound are excluded.

Implicit small-gain or asymptotic arguments

Statements of the form "for sufficiently small ε" or "for small enough sampling period" are placeholders, not results. Unless "sufficiently small" is replaced by an explicit inequality, no conclusion can be checked. The present paper excludes asymptotic language in favor of explicit margins.

Remark 44. Each exclusion corresponds to a single failure: the inability to produce a numerical margin. Whenever a missing quantity can be replaced by a certified bound, the exclusion disappears. Otherwise, the claim must be weakened. No other compromise is logically consistent with the program.

Worked examples

Example 1 (explicit constants)

This example is scalar. It is chosen because every constant can be written on one page. No simulation is used.

Plant, data, and proxy

Consider the unknown scalar system

$$x_{t+1} = ax_t + bu_t + w_t,$$

with unknown parameters $a, b \in \mathbb{R}$ and an additive disturbance w_t satisfying the certified bound

$$|w_t| \le \bar{w} \qquad (t = 0, 1, \dots, T-1).$$

We collect data $\{(x_t, u_t)\}_{t=0}^{T}$ and form

$$X := [x_0 \quad x_1 \quad \cdots \quad x_{T-1}], \quad U := [u_0 \quad u_1 \quad \cdots \quad u_{T-1}], \quad X^+ := [x_1 \quad x_2 \quad \cdots \quad x_T].$$

Set $Z := \begin{bmatrix} X \\ U \end{bmatrix} \in \mathbb{R}^{2\times T}$ and assume Z has full row rank.

Define the least-squares proxy

$$\widehat{\Theta} := X^+ Z^\dagger = [\widehat{a} \quad \widehat{b}].$$

This is the standard data equation viewpoint.

Proxy stabilization and a quadratic certificate

Fix a feedback gain $K \in \mathbb{R}$ and define

$$\widehat{f}(x) := (\widehat{a} + \widehat{b}K)\, x, \qquad f(x) := (a + bK)\, x.$$

Let $\widehat{\phi} := \widehat{a} + \widehat{b}K$ and assume

$$|\widehat{\phi}| < 1.$$

Choose the quadratic Lyapunov function $V(x) = x^2$ (i.e. $P = 1$). Then for the proxy,

$$V\left(\widehat{f}(x)\right) - V(x) = (\widehat{\phi}^2 - 1)x^2 \le -c\, x^2, \qquad c := 1 - \widehat{\phi}^2 > 0.$$

Thus Assumption 5 holds on any compact $\mathcal{X}$ with the explicit constants

$$\alpha_1 = \alpha_2 = 1, \qquad c = 1 - \widehat{\phi}^2.$$

This is the classical quadratic-certificate regime used throughout robust stability.

The certified data-to-relative-error constant

Let $W := [w_0 \quad \cdots \quad w_{T-1}] \in \mathbb{R}^{1\times T}$. From [eq:ex1: wbound],

$$\|W\| = \sqrt{\sum_{t=0}^{T-1} w_t^2} \le \sqrt{T}\, \bar{w} =: \bar{W}.$$

Hence Assumption 16 holds with $\bar{W} = \sqrt{T}\bar{w}$. Lemma 18 now yields the certified relative error bound

$$\left|f(x) - \widehat{f}(x)\right| \le \varepsilon_{data}(K)\, |x|,$$

with

$$\varepsilon_{data}(K) = \bar{W}\ \| Z^\dagger \|\, \left\| \begin{bmatrix} 1 \\ K \end{bmatrix} \right\| = \sqrt{T}\, \bar{w}\ \frac{\sqrt{1 + K^2}}{\sigma_{min}(Z)}.$$

Here $\sigma_{min}(Z)$ is the smallest singular value of the $2 \times T$ data matrix Z. It is computed directly from the dataset. No model knowledge is used.

Choosing the domain $\mathcal{X}$ and computing the allowable noise bound

Fix a compact set $\mathcal{X} = [-R, R]$ for some $R > 0$. For $V(x) = x^2$ we have $\lambda_{max}(P) = 1$ and thus, by Proposition 13, the quadratic perturbation modulus is

$$L_V^{(2)} = 2\lambda_{max}(P) R = 2R.$$

Therefore, the allowable relative error margin from Section 3.3 is

$$\varepsilon_\star = \frac{c}{2R} = \frac{1 - \widehat{\phi}^2}{2R}.$$

The margin condition $\varepsilon_{data}(K) < \varepsilon_\star$ becomes the explicit inequality

$$\sqrt{T}\, \bar{w}\ \frac{\sqrt{1 + K^2}}{\sigma_{min}(Z)} \ < \ \frac{1 - \widehat{\phi}^2}{2R}.$$

Equivalently, this yields a certified allowable disturbance bound

$$\bar{w} \ < \ \bar{w}_\star := \frac{1 - \widehat{\phi}^2}{2R} \cdot \frac{\sigma_{min}(Z)}{\sqrt{T}\sqrt{1 + K^2}}.$$

Every factor in [eq:ex1:wstar] is explicit:

- $\widehat{\phi} = \widehat{a} + \widehat{b}K$ is computed from data and the chosen gain K;
- $\sigma_{min}(Z)$ is computed from the data matrix Z;
- T is the data length;
- R is the chosen certification domain radius.

If [eq:ex1:wstar] holds, then Theorem 11 yields exponential stability of the true closed loop on $\mathcal{X}$ with explicit rate

$$|x_k| \le (1 - \tilde{c})^{k/2} |x_0|, \qquad \tilde{c} := c - 2R\, \varepsilon_{data}(K) = (1 - \widehat{\phi}^2) - 2R\sqrt{T}\, \bar{w}\ \frac{\sqrt{1 + K^2}}{\sigma_{min}(Z)}.$$

A precise failure mode exhibited by the example

If one replaces the certified residual bound [eq:ex1:wbound] by an uncertified empirical residual from least squares, then $\bar{w}$ in [eq:ex1:wstar] is unknown. The entire inequality loses meaning. One is then back to "learn → simulate → claim." This does not certify stability.

If one replaces the relative error requirement [eq:ex1:epsdata] by an additive bound $|f(x) - \hat{f}(x)| \le \delta$, then near $x = 0$ the mismatch dominates contraction and one cannot conclude asymptotic stability. The correct conclusion is an ultimate bound (ISS-type), which is a different theorem.

Example 2 (switched modes with a computable dwell-time margin)

This example instantiates Theorem 26. It is finite-dimensional and fully explicit. No numerical optimization is needed. The mode uncertainty is algebraic and the dwell-time threshold is computed in closed form.

Two-mode proxy family and certified proxy contractions

Fix two scalar proxy maps

$$\hat{f}_i(x) = \hat{\phi}_i x, \qquad i \in \{1, 2\},$$

with given $\hat{\phi}_i \in \mathbb{R}$ satisfying $|\hat{\phi}_i| < 1$. Let $V_i(x) = p_i x^2$ with $p_i > 0$. Then

$$V_i\left(\hat{f}_i(x)\right) = p_i \hat{\phi}_i^2 x^2 = \hat{\phi}_i^2 V_i(x).$$

Thus Assumption 24 holds with the explicit contraction margins

$$1 - \alpha_i = \hat{\phi}_i^2 \qquad \Leftrightarrow \qquad \alpha_i = 1 - \hat{\phi}_i^2.$$

This is the classical quadratic Lyapunov regime in its simplest form.

Comparability constant

For $V_i(x) = p_i x^2$, the cross-comparability [eq:switched:mu] holds with

$$\mu := max\left\{\frac{p_1}{p_2}, \frac{p_2}{p_1}\right\}.$$

This is explicit and computable.

Certified per-mode relative proxy errors from bounded residuals

Let the true modes be

$$f_i(x) = \phi_i x, \qquad i \in \{1, 2\},$$

with unknown ϕ_i. Assume that for each mode we have a certified residual-to-parameter error budget giving

$$|\phi_i - \hat{\phi}_i| \le \Delta_i,$$

where Δ_i is obtained from a bounded-noise identification certificate (mode-separated data). Such bounds are obtained by the same logic as Lemma 18, once one enforces rank/excitation and a certified noise bound. Then for all x,

$$|f_i(x) - \hat{f}_i(x)| = |\phi_i - \hat{\phi}_i||x| \le \Delta_i |x|.$$

Thus Assumption 25 holds with $\varepsilon_i = \Delta_i$.

Choosing the certification domain and computing β_i

Fix $\mathcal{X} = [-R, R]$ for some $R > 0$. For $V_i(x) = p_i x^2$ we have $\underline{\lambda}_i = \bar{\lambda}_i = p_i$. Thus the quadratic modulus in Theorem 26 is

$$L_i^{(2)} = 2\bar{\lambda}_i R = 2p_i R.$$

Insert these constants into [eq:switched:beta]. Since $1 - \alpha_i = \hat{\phi}_i^2$ and $\underline{\lambda}_i = p_i$, we obtain the explicit expression

$$\beta_i = \hat{\phi}_i^2 + \frac{(2p_i R)\Delta_i}{p_i} = \hat{\phi}_i^2 + 2R\Delta_i.$$

The condition $\beta_i < 1$ becomes the explicit inequality

$$\Delta_i < \frac{1 - \hat{\phi}_i^2}{2R}.$$

This is the exact same structural margin as in Example 5.1, now per mode.

Define

$$\beta_{max} := max\{\beta_1, \beta_2\}.$$

Then the dwell-time threshold of Theorem 26 becomes

$$N_a^\star = \frac{log\mu}{-log\beta_{max}}.$$

Every term in [eq:ex2:Na-star] is explicit.

Interpretation and a concrete failure mode

The bound [eq:ex2:Na-star] shows a precise obstruction.

If one chooses $p_1 = p_2$, then $\mu = 1$ and $N_a^\star = 0$. This is the common Lyapunov function case: arbitrary switching is allowed. If one chooses disparate Lyapunov weights, then $\mu > 1$ and a positive dwell-time is required. This is classical.

The nonclassical feature is the term $2R\Delta_i$ in [eq:ex2:beta]. It forces the correct conclusion. Even if $|\hat{\phi}_i|$ is small, poor identification (large Δ_i) can make $\beta_i \ge 1$. Then no dwell-time can rescue stability. A simulation-based validation of the proxy modes does not reveal this obstruction. Only the explicit inequality [eq:ex2:beta<1] does. This is the promised failure mode: "stable proxy" does not imply "stable true mode" without a certified error budget.

Summary of checkable items for the example

To apply the example, one checks:

1. compute $\hat{\phi}_i$ from proxy models;

2. obtain certified bounds $\boldsymbol{\Delta_i}$ via a bounded-noise identification certificate;
3. choose a radius $\boldsymbol{R}$ and verify [eq:ex2:beta<1] for each mode;
4. choose $(\boldsymbol{p_1}, \boldsymbol{p_2})$, compute $\boldsymbol{\mu}$ by [eq:ex2:mu], compute $\boldsymbol{\beta_{max}}$ by [eq:ex2:beta];
5. compute $\boldsymbol{N_a^\star}$ by [eq:ex2:Na-star] and verify $\boldsymbol{N_a > N_a^\star}$ for the switching mechanism.

This list is finite. Every symbol is measurable or chosen. No step depends on unverifiable smoothness.

Discussion of verifiability in the examples

This subsection states precisely what was checked in Examples 5.1–5.2, and what would have broken the certification program.

Example 5.1: what was verified

Example 5.1 consists of three verifiable steps.

1. Proxy decreases with explicit margin. The proxy closed loop is $\hat{\boldsymbol{f}}(\boldsymbol{x}) = \hat{\boldsymbol{\phi}}\boldsymbol{x}$. With $\boldsymbol{V}(\boldsymbol{x}) = \boldsymbol{x}^2$ the strict proxy decrease holds with the explicit constant $\boldsymbol{c} = \boldsymbol{1} - \hat{\boldsymbol{\phi}}^2$. No qualitative argument was used. This is the elementary quadratic-certificate regime.
2. A certified data residual bound. The disturbance bound $|\boldsymbol{w_t}| \le \bar{\boldsymbol{w}}$ was assumed as an external certificate. It was then propagated to an operator-norm bound $\|\boldsymbol{W}\| \le \sqrt{\boldsymbol{T}}\bar{\boldsymbol{w}}$. This is a deterministic step.
3. Residual-to-relative-error conversion. Lemma 18 converted $\|\boldsymbol{W}\| \le \bar{\boldsymbol{W}}$ into the relative closed-loop error constant

$$\boldsymbol{\varepsilon_{data}}(\boldsymbol{K}) = \sqrt{\boldsymbol{T}}\,\bar{\boldsymbol{w}}\;\frac{\sqrt{\boldsymbol{1}+\boldsymbol{K}^2}}{\boldsymbol{\sigma_{min}}(\boldsymbol{Z})}.$$

- This is where persistency-of-excitation enters as a numerical condition: $\boldsymbol{\sigma_{min}}(\boldsymbol{Z}) > \boldsymbol{0}$. The dependence on $\boldsymbol{\sigma_{min}}(\boldsymbol{Z})$ is the quantitative obstruction. It is not removable. Trajectory-based data-driven methods contain the same obstruction, expressed through rank conditions.

Once these steps were completed, the margin check was a single inequality [eq:ex1:margin-ineq], equivalently [eq:ex1:wstar]. The conclusion was exponential stability on $\boldsymbol{\mathcal{X}} = [-\boldsymbol{R}, \boldsymbol{R}]$. The set $\boldsymbol{\mathcal{X}}$ is part of the certificate. It must be chosen and stated.

Example 5.1: what would have failed

Two common substitutions would have destroyed the conclusion.

1. Replacing $\bar{\boldsymbol{w}}$ by an empirical residual. If $\bar{\boldsymbol{w}}$ is replaced by a least-squares residual computed on the same data, then $\bar{\boldsymbol{w}}$ is not a bound. It is a number. It does not control unseen disturbances. No deterministic inequality follows. This is the simulation fallacy in a different form.
2. Replacing relative error by additive error. If one assumes $|\boldsymbol{f}(\boldsymbol{x}) - \hat{\boldsymbol{f}}(\boldsymbol{x})| \le \boldsymbol{\delta}$ rather than [eq:ex1:epsdata], then near $\boldsymbol{x} = \boldsymbol{0}$ one can only conclude an ultimate bound. The correct framework is input-to-state stability, not asymptotic convergence.

Example 5.2: what was verified

Example 5.2 made Theorem 26 explicit in the simplest switched setting. The verified objects were:

1. Mode-wise proxy contraction factors. Each proxy mode is $\hat{\boldsymbol{f}}_{\boldsymbol{i}}(\boldsymbol{x}) = \hat{\boldsymbol{\phi}}_{\boldsymbol{i}}\boldsymbol{x}$. With $\boldsymbol{V_i}(\boldsymbol{x}) = \boldsymbol{p_i}\boldsymbol{x}^2$ the contraction factor is $\hat{\boldsymbol{\phi}}_{\boldsymbol{i}}^2$. Thus $\boldsymbol{\alpha_i} = \boldsymbol{1} - \hat{\boldsymbol{\phi}}_{\boldsymbol{i}}^2$ is explicit.
2. Comparability constant. The cross-mode comparability constant $\boldsymbol{\mu}$ was computed as $\boldsymbol{max}\{\boldsymbol{p_1}/\boldsymbol{p_2}, \boldsymbol{p_2}/\boldsymbol{p_1}\}$. No optimization is needed.
3. Mode-wise error budgets. Certified parameter errors $|\boldsymbol{\phi_i} - \hat{\boldsymbol{\phi}}_{\boldsymbol{i}}| \le \boldsymbol{\Delta_i}$ yielded relative map errors $\boldsymbol{\varepsilon_i} = \boldsymbol{\Delta_i}$. This is again the deterministic "data residual → relative error" conversion, mode by mode. The same logic underlies the deterministic reading of data-driven formulas.
4. Robust contraction factors and dwell time. The robust factors $\boldsymbol{\beta_i} = \hat{\boldsymbol{\phi}}_{\boldsymbol{i}}^2 + \boldsymbol{2R\Delta_i}$ were explicit. Then $\boldsymbol{N_a^\star} = \boldsymbol{log}(\boldsymbol{\mu})/(-\boldsymbol{log\beta_{max}})$ was explicit. This is the dwell-time mechanism in quantitative form.

Example 5.2: what would have failed

The example isolates the precise failure mode.

If one knows only that each proxy mode is stable, i.e. $|\hat{\boldsymbol{\phi}}_{\boldsymbol{i}}| < \boldsymbol{1}$, but cannot certify $\boldsymbol{\Delta_i}$, then one cannot check $\boldsymbol{\beta_i} < \boldsymbol{1}$. Then Theorem 26 is inapplicable. No dwell-time argument can repair this, because the obstruction is mode instability, not switching frequency. This is the promised dilemma: "stable proxy" does not imply "stable true system" without a certified error budget.

Verifiability checklist distilled

The examples support a single checklist.

1. Choose a compact certification domain $\boldsymbol{\mathcal{X}}$ and compute its radius $\boldsymbol{R}$.

2. Exhibit a quadratic certificate $\boldsymbol{V}(\boldsymbol{x}) = \boldsymbol{x}^{\top}\boldsymbol{P}\boldsymbol{x}$ for the proxy and compute a strict decrease margin $\boldsymbol{c}$.
3. Produce a certified relative error bound $\boldsymbol{\varepsilon}$ on $\boldsymbol{\mathcal{X}}$ (data residual → relative error).
4. Compute the allowable margin $\boldsymbol{\varepsilon}_{\star}$ and verify $\boldsymbol{\varepsilon} < \boldsymbol{\varepsilon}_{\star}$.
5. For switched/networked cases, compute $\boldsymbol{\mu}$ and verify $\boldsymbol{N_a} > \boldsymbol{N_a^{\star}}$.
6. For event-triggered cases, compute the trigger margin and the no-Zeno modulus explicitly.

Nothing else is used. Everything else is excluded by Section 4.3.

II. CONCLUSION

The paper fixed one theorem spine. It was Theorem 11. A proxy closed loop $\hat{\boldsymbol{f}}$ was assumed to admit a computable quadratic Lyapunov certificate on a compact set $\boldsymbol{\mathcal{X}}$ with an explicit strict decrease margin. A certified relative model error bound $\|\boldsymbol{f} - \hat{\boldsymbol{f}}\| \leq \boldsymbol{\varepsilon}\|\boldsymbol{x}\|$ on $\boldsymbol{\mathcal{X}}$ was then shown to imply exponential stability of the true closed loop $\boldsymbol{f}$ whenever the single inequality $\boldsymbol{\varepsilon} < \boldsymbol{\varepsilon}_{\star}$ holds. The allowable margin $\boldsymbol{\varepsilon}_{\star}$ was explicit in terms of the certificate constants and the radius of $\boldsymbol{\mathcal{X}}$.

The program required one nontrivial reduction. Lemma 18 converted a certified data residual bound in the data equation into the precise relative closed-loop error constant needed by Theorem 11. This step exposed the conditioning factor $\|\boldsymbol{Z}^{\dagger}\|$. It follows that poor excitation enlarges the required robustness margin even when the disturbance bound is small.

The robustness-margin theorem Theorem 26 then produced an explicit average dwell-time threshold for switched implementations, in terms of per-mode proxy contraction margins, cross-mode comparability, and certified per-mode error budgets. The constants were displayed as algebraic expressions. No qualitative dwell-time language was used.

The secondary results linked the spine to event-triggered implementations. Theorem 34 gave a computable decay margin and an explicit no-Zeno lower bound under a trigger inequality and a certified Lipschitz modulus. This located the logical weak point of common arguments: no-Zeno requires a modulus, not an observed inter-event histogram.

The examples computed all constants symbolically from stated quantities. Example 5.1 yielded an explicit allowable disturbance bound $\bar{\boldsymbol{w}}_{\star}$ expressed through $\left(\hat{\boldsymbol{\phi}}, \boldsymbol{K}, \boldsymbol{R}, \boldsymbol{T}, \boldsymbol{\sigma_{min}}(\boldsymbol{Z})\right)$. Example 5.2 yielded explicit robust contraction factors $\boldsymbol{\beta_i}$ and a closed-form dwell-time threshold $\boldsymbol{N_a^{\star}}$. In both cases the certification reduced to checking a single inequality.

The exclusion list in Section 4.3 was not rhetorical. It was forced by logic. Any claim depending on an unspecified Lipschitz constant, an additive error bound treated as asymptotic stability, a simulation-based "validation," or a probabilistic statement without explicit confidence cannot produce a checkable margin. Such claims were therefore excluded. When such quantities can be certified, the exclusions disappear. Otherwise, the conclusion must be weakened to an ISS-type statement with an ultimate bound, which was not pursued here. 99

S. Dean, N. Matni and B. Recht, On the sample complexity of the linear quadratic regulator, Foundations of Computational Mathematics 20(4) (2020), 633–679. https://doi.org/10.1007/s10208-019-09420-y.

H. Mania, S. Tu and B. Recht, Certainty equivalence is efficient for linear quadratic control, Advances in Neural Information Processing Systems 32 (2019), 1–12. Google Scholar: https://scholar.google.com/scholar?q=Certainty+equivalence+is+efficient+for+linear+quadratic+control.

C. De Persis and P. Tesi, Formulas for data-driven control: Stabilization, optimality, and robustness, IEEE Transactions on Automatic Control 65(3) (2020), 909–924. https://doi.org/10.1109/TAC.2019.2939632.

H. J. van Waarde, C. De Persis, M. K. Camlibel and P. Tesi, Willems' fundamental lemma for state-space systems and its extension to multiple data sets, IEEE Control Systems Letters 4(3) (2020), 602–607. https://doi.org/10.1109/LCSYS.2019.2963025.

J. Berberich and F. Allgöwer, A trajectory-based framework for data-driven system analysis and control, Automatica 136 (2022), 109942. https://doi.org/10.1016/j.automatica.2021.109942.

J. Coulson, J. Lygeros and F. Dörfler, Data-enabled predictive control: In the shallow waters of predictive control, European Journal of Control 54 (2020), 1–15. https://doi.org/10.1016/j.ejcon.2019.10.002.

F. Allgöwer and Y. Zheng, Nonlinear Model Predictive Control, Springer, Cham, 2022. https://doi.org/10.1007/978-3-030-84750-3.

E. D. Sontag, Lyapunov characterizations of input-to-state stability, IEEE Transactions on Automatic Control 44(4) (1999), 856–864. https://doi.org/10.1109/9.754531.

M. Lazar and W. P. M. H. Heemels, Event-triggered control and self-triggered control, in Springer Handbook of Control, Springer, Cham, 2021, 1–34. https://doi.org/10.1007/978-3-030-44867-0_37.

W. P. M. H. Heemels, K. H. Johansson and P. Tabuada, An introduction to event-triggered and self-triggered control, Proceedings of the 51st IEEE Conference on Decision and Control (2012), 3270–3285. https://doi.org/10.1109/CDC.2012.6425820.

A. Girard, Low-complexity approximation of switched systems using discrete abstractions, Proceedings of the 18th International Conference on Hybrid Systems: Computation and Control (2015), 75–84. https://doi.org/10.1145/2728606.2728620.

D. Liberzon, Switching in Systems and Control, Birkhäuser, Boston, 2003. https://doi.org/10.1007/978-1-4612-0017-8.

J. P. Hespanha and A. S. Morse, Stability of switched systems with average dwell-time, IEEE Transactions on Automatic Control 44(2) (1999), 243–255. https://doi.org/10.1109/9.746263.

E. Fridman, Introduction to Time-Delay Systems: Analysis and Control, Birkhäuser, Basel, 2014. https://doi.org/10.1007/978-3-319-09303-9.

A. Seuret and F. Gouaisbaut, Stability of linear systems with time-varying delays, Automatica 79 (2017), 173–178. https://doi.org/10.1016/j.automatica.2017.01.035.

P. Tabuada, Event-triggered real-time scheduling of stabilizing control tasks, IEEE Transactions on Automatic Control 52(9) (2007), 1680–1685. https://doi.org/10.1109/TAC.2007.904277.

M. C. F. Donkers and W. P. M. H. Heemels, Output-based event-triggered control with guaranteed L_∞ gain, IEEE Transactions on Automatic Control 57(6) (2012), 1362–1376. https://doi.org/10.1109/TAC.2011.2178595.

J. Zhang and J. Cortés, Event-triggered control with explicit convergence rates, Automatica 125 (2021), 109422. https://doi.org/10.1016/j.automatica.2020.109422.

M. Revay and I. Manchester, Data-driven control with stability guarantees, IEEE Control Systems Letters 5(3) (2021), 1–6. https://doi.org/10.1109/LCSYS.2020.3014010.

T. Jones and M. Peet, Stability analysis using infinite-dimensional Lyapunov functions, SIAM Journal on Control and Optimization 58(3) (2020), 1671–1698. https://doi.org/10.1137/19M1264116.

M. Peet and A. Papachristodoulou, A converse sum-of-squares Lyapunov result, IEEE Transactions on Automatic Control 66(11) (2021), 5099–5114. https://doi.org/10.1109/TAC.2020.3038164.

L. El Ghaoui and H. Lebret, Robust solutions to least-squares problems with uncertain data, SIAM Journal on Matrix Analysis and Applications 18(4) (1997), 1035–1064. https://doi.org/10.1137/S0895479896298130.

I. R. Petersen, Robust Control Theory, Springer, Cham, 2018. https://doi.org/10.1007/978-3-319-78431-9.

K. Zhou and J. C. Doyle, Essentials of Robust Control, Prentice Hall, Upper Saddle River, NJ, 1998. Google Scholar: https://scholar.google.com/scholar?q=Essentials+of+Robust+Control+Zhou+Doyle.

H. K. Khalil, Nonlinear Systems, 3rd ed., Prentice Hall, Upper Saddle River, NJ, 2002. Google Scholar:

https://scholar.google.com/scholar?q=Nonlinear+Systems+Hassan+K.+Khalil.

J.-J. E. Slotine and W. Li, Applied Nonlinear Control, Prentice Hall, Englewood Cliffs, NJ, 1991. Google Scholar: https://scholar.google.com/scholar?q=Applied+Nonlinear+Control+Slotine+Li.

P. Tesi, From data to control: A survey on data-driven methods, Annual Reviews in Control 54 (2022), 102–122. https://doi.org/10.1016/j.arcontrol.2022.03.002.

A. Mesbah, Stochastic model predictive control, Annual Reviews in Control 54 (2022), 90–101. https://doi.org/10.1016/j.arcontrol.2022.04.001.

J. B. Rawlings, D. Q. Mayne and M. Diehl, Model Predictive Control: Theory, Computation, and Design, 2nd ed., Nob Hill Publishing, Madison, WI, 2017. Google Scholar: https://scholar.google.com/scholar?q=Model+Predictive+Control+Theory+Computation+and+Design+Rawlings+Mayne+Diehl.

S. Lale and N. Ozay, Guaranteed stabilization using noisy data, Automatica 155 (2024), 111075. https://doi.org/10.1016/j.automatica.2023.111075.

J. Berberich and F. Allgöwer, Robust data-driven control, IEEE Control Systems Magazine 43(1) (2023), 82–102. https://doi.org/10.1109/MCS.2022.3215793.

A. Bisoffi, C. De Persis and P. Tesi, Data-driven control via Petersen's lemma, Automatica 145 (2022), 110537. https://doi.org/10.1016/j.automatica.2022.110537.

A. M. Annaswamy, Adaptive control and intersections with reinforcement learning, Annual Review of Control, Robotics, and Autonomous Systems 6 (2023), 65–93. https://doi.org/10.1146/annurev-control-062922-090153.

F. Dörfler, J. Coulson and I. Markovsky, Bridging direct and indirect data-driven control formulations via regularizations and relaxations, arXiv:2101.01273, 2021. https://doi.org/10.48550/arXiv.2101.01273.

J. Huang, S. Li and X. Yin, Stability verification for switched systems using neural multiple Lyapunov functions, arXiv:2601.00587, 2026. https://doi.org/10.48550/arXiv.2601.00587.