© June 2026 | IJIRT | Volume 13 Issue 1 | ISSN: 2349-6002

Comparative Performance Analysis of Fractional Order Chelyskhov Function-Based Numerical Scheme for Fractional Control Optimization Problems

Victor Ifechukwude Agwunwa[1], Henrietta Ify Ojarikre[2], Ebimene James Mamadu[3]
[1,2,3]*Department Of Mathematics, Delta State University, Abraka, Nigeria*

Abstract—**This paper presents a comparative performance analysis of a Fractional Order Chelyshkov Function (FCHF)–based numerical scheme for solving fractional optimal control problems (FOCPs). The proposed approach is benchmarked against classical spectral method based on Legendre polynomials and the Mamadu Variational Iteration Method (MVIM). Performance is assessed using key numerical indicators including -error, convergence behavior, computational runtime, and solution smoothness. Using identical test problems and discretization settings, the results demonstrate that the FCHF-based scheme achieves superior accuracy with fewer basis functions while maintaining strong numerical stability. Furthermore, the method produces smooth state and control trajectories without Gibbs-type 0scillation.These findings establish the FCHF-based numerical scheme as an efficient alternative solution scheme for FCOPs.**

Index Terms—**Fractional Optimal Control problems (FCOP), Fractional order Chelyshkov Function (FCHF), Spectral Method, Comparative Analysis, Numerical Performance.**

I. INTRODUCTION

Fractional optimal control problems (FOCPs) represent a natural extension of classical optimal control theory in which the system dynamics are described using fractional derivatives rather than integer-order derivatives. Unlike classical differential operators, fractional derivatives incorporate memory and hereditary effects, allowing the present state of a system to depend not only on its current input but also on its entire past evolution. This property makes fractional models particularly suitable for describing real-world phenomena in areas such as viscoelastic materials, electrical circuits, biological systems, and transport processes (Podlubny, 1999[1]; Diethelm, 2010[2]).

In classical optimal control theory, system dynamics are typically governed by ordinary differential equations (ODEs). However, many physical and engineering processes exhibit nonlocal behavior that cannot be adequately captured using integer-order models. Fractional differential equations (FDEs), usually formulated using Caputo or Riemann–Liouville derivatives, provide a more realistic mathematical representation of such systems (Samko *et al.*, 1993[3]). Consequently, the use of fractional calculus in optimal control has attracted significant attention in recent decades. Agrawal (2004) [4] presented one of the earliest systematic formulations of FOCPs and introduced a general solution framework based on fractional Variational Principles. Subsequent studies further extended these formulations and proposed various numerical techniques for solving FOCPs (Mu *et al.,* 2020) [5].

A canonical example is given as:

$$J[u] = \int_0^1 [x^2(t) + u^2(t)]\, dt, \text{subject to } D^\alpha x(t) = u(t), x(0) = 0, \quad \alpha \in (0,1), \quad (1)$$

where D^α denotes the fractional derivative of order α. These models arise in a wide range of practical contexts, such as flexible mechanical systems, energy resource control under delay, epidemiological dynamics, and even financial systems where historical volatility plays a role in current risk assessments (Atagana and Baleanu[6]; [5]). The numerical solution of FOCPs poses significant challenges. Classical techniques like direct collocation, indirect shooting methods, or variational techniques must be extended or reformulated to accommodate fractional operators. The breakthrough in [4] introduced a general formulation and a quadratic numerical scheme for FOCPs based on Caputo derivatives. These formulations have paved the way for the development of operational matrices, direct transcription schemes and hybrid optimization algorithms Lima ([7]).

In employing spectral and collocation methods for solving differential equations including FOCPs, orthogonal basis functions play a central role. Classical orthogonal polynomials such as Legendre, Chebyshev, and Jacobi polynomials have been widely used due to their excellent approximation properties. However, when applied to fractional-order problems, these bases often result in large errors or poor convergence unless a high number of terms is used. This is because classical polynomials are tailored for local operators and do not inherently capture the memory-dependent dynamics of fractional derivatives ([8];). To these limitations, researchers have explored fractional basis functions, which more naturally reflect the structure of fractional systems. Among these, Chelyshkov polynomials, introduced in Izadi *et. al* [9], have garnered interest. Unlike classical polynomials, each Chelyshkov basis function is of the same algebraic degree and satisfies a unique orthogonal condition over the interval [0, 1]:

$$\int_0^1 P_{k,n}(t)P_{m,n}(t)\,dt = \frac{1}{2n+1}\delta_{km}. \qquad (2)$$

The defining property of Chelyshkov polynomials is that they all have the same degree. This makes them particularly suitable for constructing uniform approximation spaces. Their algebraic structure is closed under certain operations, including differentiation and integration, which is beneficial in matrix-based computational schemes ([10]). FCHFs are a natural generalization of Chelyshkov polynomials to fractional-order systems. They form a functional basis designed to approximate solutions of FDEs more accurately than traditional polynomials. Thus, an unknown function $x(t)$ is approximated as a finite sum:

$$(t)x \approx \sum_{k=0}^{N} c_k\, F_k(t), \qquad (3)$$

where $F_k(t)$ denotes the FCHF basis and the coefficients c_k are determined by exploiting orthogonality relations involving special functions like the Gamma function. In [11] framework, the operational matrix of fractional integration, denoted as P_α, is introduced in (4) below. This enables approximation of fractional integral operators via algebraic vector-matrix multiplication:

$$I^\alpha F(t) \approx P_\alpha\, F(t), \quad F(t) = [\,F_0(t), \dots, F_N(t)]^T. \quad (4)$$

This conversion transforms the original FOCP into a system of algebraic equations that can be solved using standard numerical or optimization techniques.

Several numerical approaches have been developed for solving FOCPs ([12]; [13]; [14]). Spectral methods based on classical orthogonal polynomials such as Legendre, Chebyshev, and Bernstein polynomials have been widely used because of their high approximation accuracy for differential equations ([8]). However, when applied to fractional-order systems, these classical bases may require a large number of terms to achieve satisfactory accuracy due to their limited ability to capture the global memory behaviour of fractional dynamics.

To address these challenges, researchers have explored alternative basis functions specifically tailored for fractional systems. Among such approaches are Chelyshkov polynomials, which were originally introduced in hydrodynamic stability studies (Chelyshkov, 1994[15]) and possess unique structural properties, including orthogonality over the closed unit interval [0,1]. They also have characteristic property that each basis function shares the same algebraic degree. These features make them suitable for constructing uniform approximation spaces and for developing operational matrix formulations ([16]). Recent studies have also extended Chelyshkov polynomials to fractional-order problems, demonstrating their effectiveness in solving FDEs and related control problems ([17]; [18]).

Building on this idea, FCHFs have been proposed as a generalized basis capable of representing fractional dynamic systems more accurately. These functions incorporate fractional power structures that better reflect the nonlocal nature of fractional operators. Operational matrices constructed from such bases allow fractional integral or derivative operators to be approximated through matrix algebra, thereby transforming FDEs into systems of algebraic equations that are easier to solve computationally [19]. Despite the growing interest in Chelyshkov-based methods, there remains a need for systematic comparative studies evaluating their performance against established numerical approaches used for FOCPs. Comparisons with classical spectral methods and modern iterative approaches is the main aim of this current work and comparing against schemes such as the Mamadu Variational Iteration Method (MVIM) can provide valuable insights into their relative accuracy, convergence behavior, and computational efficiency (Mamadu et al., 2025).[20].

Hence, our interest in a comparative performance analysis of (FCHF)–based numerical scheme for solving FOCPs. The proposed method employs truncated FCHF expansions together with an operational matrix formulation to transform the FOCP into a system of algebraic equations. Its performance is evaluated against the Legendre spectral method and MVIM using identical benchmark problems. The comparison focuses on numerical accuracy, convergence characteristics, computational efficiency, and the qualitative behavior of the resulting state and control trajectories.

The results as demonstrated in *section 4 show* that the FCHF-based scheme provides an efficient and stable computational framework for FOCPs while requiring fewer basis functions to achieve high-accuracy solutions.

II. METHODOLOGY

A numerical scheme based on FCHFs as basis is employed for the solution of FOCPs involving Caputo fractional derivatives. The state and control variables are approximated by truncated FCHF series, taking advantage of their orthogonal property and suitability for memory-dependent dynamics. An operational matrix of fractional integration is used to discretize the fractional dynamic constraints, thereby reducing the FOCP to a system of algebraic equations. Constraint enforcement is achieved through the Lagrange multiplier technique, resulting in a finite-dimensional optimization problem in terms of the expansion coefficients. The resulting algebraic system is solved using efficient matrix-based and iterative numerical methods. The proposed approach is validated through comparative analysis with Legendre spectral and MVIM using error norms, computational cost, and convergence rates to demonstrate accuracy, stability, and efficiency.

III. MAIN RESULTS

In this section, we present a methodology for implementing and analyzing a FCHF – based numerical method for FOCPs. We assume a standard FOCP formulation in a finite interval (0,1), Caputo Fractional derivatives (order $0 < \propto \leq 1$).

Consider the FCOP below

Minimize $J(u) = \emptyset\left(x(T)\right) + \int_{\delta}^{T} L\,(t, xlt),\ ult)\,dt,$ (5)

$^{c}D_t^{\propto}\, x(t) = f\big(t, x(t), u(t)\big), t \in\ (O, T)$ (6)

where $0 < \propto \leq 1$, $x(t) \in IR^n$ is the state, $\big(u(t)\big) \in IR^m$, is the control, 1: IR x IRn x IRm → IR is the running cost and $\emptyset\colon IR^1$ → IR the terminal cost. F is sufficiently smooth (Lipschitz in r, u).

We will convert this into algebraic equations using FCHFs and the operational matrix of integral, and solve by the langrange multiplier method.

We will present the integral formulation of the state equation for $0 < \propto \leq 1$.

$x(t) = x_0 + \gamma^{\propto}\left[f\,(.\,, x(.), u(.))\right](t),$ (7)

Where;

$\gamma^{v}(t^{k}) \approx \frac{\tau\,(k+1)}{\tau\,(k+1+V)}\,\tau^{K+V},\qquad K > -1,$ (8)

$\gamma^{v}(g)(t) \approx \frac{1}{\tau(v)}\int_{o}^{t}(t-\tau)^{V-1}\,9\,(\tau)d\tau$ (9)

$^{c}D_t^{\propto}\, x(t) \approx I^{1-\propto}x^{1}(t).$ (10)

3.1. Fractional order chelyskhov function-based numerical scheme.

Given a classical Chelyshkov polynomials, $(n(\varphi)$ or $\varphi \in [\tau, 1]$ or mapped to$[0, 1]$,

define the FCHF basis on $t \in [\tau, 1]$ by mapping and multiplying by a fractional power for regularization.

$\emptyset_j(t) = t^{\beta} c_j\left(\frac{26}{\tau} - 1\right), j = 0, \dots., N,$ (11)

Where $\beta \geq 0$ is a parameter to tune endpoint behavior; a common choice is$\beta = 0$. These $\emptyset_j$ are the FCHFs.

A key required property is that for each$\emptyset_j(t)$, we have,

$\emptyset_j(t) = \sum_{k=0}^{N} a_{jk}t^{k},$ (12)

Wherea_{jk}'s are the unknowns to be computed. Also, expressing the basis in the monomial basis is essential to get closed-form operational matrix entries via the power identity. This is because the fractional integral of tk is explicit, we can derive an operational matrix for fractional integration for the vector-valued basis$\emptyset(t) = [\emptyset_0(t), \ldots, \emptyset_N(t)]$.

Operational Matrix of fractional integral γV
Let$\emptyset(t) = [\emptyset_0(t), \ldots, \emptyset_N(t)]^T$. We seek $P^V > \in IR^{(N+1)\times(N+1)}$ such that
$\gamma^V\,[\emptyset(t)] = P^V\,[\emptyset(t)]$. (13)

Using the following steps to compute P^V

1. Expand each basis function into monomials:

$$\emptyset_j(t) \approx \sum_{k=0}^{N} a_{jk} t^V, \qquad j \approx o, \ldots., N.$$

Coefficients in matrix $\mathrm{C} \in IR^{(N+1)\times(P+1)}$ into $C_{j,k} = 9_{j,k}$

Fractional integral to each monomial
$\gamma^V[t^k] = \frac{r(K+1)}{r(k+1+V)}\, t^{k+v}$. (14)

The results back into the basis$\emptyset(t)$. Write
$t^{k+v} \approx \sum_{i=0}^{N} d_{jk}^{(v)} \emptyset_j(t)$. (15)
In practice this projection can be done by expanding $\emptyset_j$ in monomials and solving for coefficients $d_{jk}^{(v)}$ by matching monomial coefficients or using a least-squares projection on [0, 1].

2. for each j,

$$\gamma^v[\emptyset_j(t)] \approx \sum_{k=0}^{P} a_{jk}\gamma^v[t^k]$$

$$\sum_{k=0}^{P} a_{jk} \frac{\tau\,(k+1)}{\tau\,(k+1+v)}\, t^{k+v}$$

$$\sum_{l=0}^{N}\left(\sum_{k=0}^{P} a_{jk} \frac{\tau\,(k+1)}{\tau\,(k+1+v)}\, d_{lk}^{(v)}\right)\emptyset_l(t).$$

So, the operational matrix entries are:
$P_{l,j}^{(v)} = \sum_{k=0}^{P} d_{lk}^{(v)}\, a_{jk} \frac{\tau\,(k+1)}{\tau\,(k+1+v)}$. (16)

This yields;
$\gamma^V[\emptyset] \approx P^{(v)}\emptyset$ (17)
To implement the above algorithm, we note the following simplification:

If the basis $\emptyset$ itself is converted to monomials via a known invertible matrix S (i.e $\emptyset(t) = ST(t)$,
Where;
$T(t) = [(1, t, t^2, \ldots, t^P)^T]$,
Then
$\gamma^v[\emptyset] \approx SI^V[T] \approx SM^VT \approx SM^{(V)}S^{-1}\emptyset$ (18)
Where M(V) is diagonal-like with entries $\frac{\tau\,(k+1)}{\tau\,(k+1+v)}$ shifted in power index. Then $P^{(v)} = SM^{(v)}S^{-1}$. this is numerically efficient if S is known.

3.2. Approximation of state, control variables and the Multipliers.
Approximate unknown functions by truncated FCHF expansions,
$x(t)\tau \times {}^T\emptyset(t) \times \in IR^{(N+1)x\,n}$

Which is matrix columns for each state component,
$u(t) = U^T\emptyset(t), \quad U \in IR^{(N+1)x\,m}$, (19)
$\lambda(t) \approx \Lambda^T\,\emptyset(t), \Lambda \in IR^{(N+1)x\,n}, \quad (Lagrange\ multipliers/costates)$.
(20)

The coefficients matrices X, U, Λ are known and will be solved for.
Using the integral form of the state equation, we have
$x(t) = x_0 + \gamma^{\propto}[f(t, x(t))](t)$. (21)
Substitute approximations and the operational matrix $P^{(\propto)}$ to turn the integral into algebraic form.
Next, we consider converting dynamics into algebraic equations. Basically, there are two common methods: collocation selected nodes, or weighted-residual approach (Mamadu *et al.,* 2023[21]). Lagrange Multiplier Method converted to Algebraic system. Using Pontryagin – like functional with fractional dynamics enforced via Lagrange multipliers$\lambda(t)$(Ndairou, and Torres ([22]; [23]). Then, the augmented functional (continuous) is
$Ja = \emptyset(x(t)) + \int_0^{\tau}(L\,(t, x, u) + \lambda^T(t)({}^cD^{\propto}x(t) - f(t, x, u))dt$. (22)
Replace x, u, λ by finite expansion, and use operational matrices to express the time integrals in terms of the coefficient vectors. This is achieved by following the steps in [20] to derive algebraic conditions: The above steps can also result to a nonlinear algebraic system of the form.
$R\,(X, U, \Lambda\,) = 0$, (23)

The resulting equation (23) is linearized following the Newton method or Mamadu Δ^2 scheme [20] yielding the block linear system given as,

$$\begin{bmatrix} A_{XX} & A_{XU} & A_{X\wedge} \\ A_{UX} & A_{UU} & A_{U\wedge} \\ A_{\wedge X} & A_{\wedge U} & 0 \end{bmatrix}\begin{bmatrix} \Delta_{vec}(X) \\ \Delta_{vec}(U) \\ \Delta_{vec}(\wedge) \end{bmatrix} = -\begin{bmatrix} ResX \\ ResV \\ Res\wedge \end{bmatrix} \quad (24)$$

This is solved iteratively; if the problem is linear quadratic in x, u, the system is linear and can be solved directly.

However, the nonlinear system can be handled through Mamadu Δ^2 iterative Scheme [20] or [25].

When f or U are nonlinear, the algebraic equation is nonlinear. Using Mamadu Δ^2 iterative scheme where we set the initial guess $X^{(0)}, U^{(0)}, \wedge^{(0)}$ and evaluate the iteration residual r: $R^{(r)}$ and Jacobian $J^{(r)}$. We then solve for $J^{(r)}\Delta = -R^{(r)}$ for validations.

Below is the general solution process for classical optimization problems with FCHF ([10];[24].)

At this point, we construct FCHFs basis $\emptyset(t)$ and expand them in monomials to obtains S or coefficient matrix C. We also compute operational matrix $p^{(\propto)}$(and $p^{(1-\propto)}$) and derivative matrix D and mass matrix.

$$W = \int_0^T \emptyset\,\emptyset^r\,dt.$$

We also solve for r = 0, 1, … until convergence: and then evaluate the approximations of

$$x^{(r)}(t_k) = (X^{(r)})^T \emptyset(t_k) \quad (25)$$

$$(U^{(r)})^T \emptyset(t_k) = U^{(r)}(t_k) \quad (26)$$

Next, we compute the residuals using integral form:

$$R_k = X^T\emptyset(t_k) - x_o - p^{(d)} f(t_k, x^{(v)}(t_k), U^{(v)}(tx)$$

And assemble the gradient/Jacobian using chain-rule.

4.0 Numerical Illustrations:

A typical example of FOCP is given as

$$\frac{min}{u(t)}\sqrt{} = \int_0^1 (x(t)^2 + u(t)^2)\,dt$$

Subject to the dynamic constraint.

$$\Delta_t^{\propto} x(t) = -ax(t) + bu(t), 0 < \propto < 1,$$

With initial condition

$x(0) = 1,$

Where a and b are real constants

For this study, we considered P = 8, $\propto = 0.6$, collocation nodes (Gauss – Chebyshev nodes on [0,1]): t = [0.0000, 0.049516, 0.188255, 0.388740, 0.611260, 0.811745, 0.950484, 1.0000]. using these nodes, the fractional integral of each basis function was evaluated and matched with Chelyshkov example, leading to the construction of the operation matrix p.

The resulting operational matrix p of size 8 x 8 for $\propto = 0.6$ is given in Table 3.

Let $\emptyset(t) = [\emptyset_0(t), \emptyset_1(t), \ldots, \emptyset_{p-1}(t)]^T$ denote the vector of Chelyshkov basis functions of dimension P.

The state and control are approximated as:

$x(t) \approx \emptyset(t)^T C_x, \qquad U(t) \approx \emptyset(t)^T C_u,$

where $C_x, C_u \in IR^p$ are coefficient vectors to be determined.

The Caputo derivative is expressed in integral form,

$x(t) = x_0 + \gamma^{\propto}(-a\,x(t) + bu(t)),$

Where $\gamma^{\propto}$ denotes the Riemann-Liouville fractional integral of order $\propto$.

The operational Matrix $P \in IR^{P\times P}$ satisfies

$\gamma^{\propto}[\emptyset(t)] \approx \emptyset(t)P.$

Hence, for any expansion $f(t) = \emptyset(t)^T F$,

$\gamma^{\propto}[F](t) \approx \emptyset(t)^T(PF).$

Substituting into the integral form of the dynamics (21)

$\emptyset(t)^T C_x = x_0 + \emptyset(t)^T\left(P\left(-a\,(x + bC_u\right)\right)$

Hence, suppose $a = 1, b = 2.0$ for the problem above, we obtain the following results with the aid of PYTHON program

Table1: Extracted values of the estimated coefficients for $C_x, C_u, \wedge$ from the Python Programme

Coeff.	Cx	Cu	$\wedge$
c_0	0.401288613	-0.253552916	-0.063381854
c_1	-0.304437956	0.251290222	0.026687845
c_2	0.154692217	-0.042131539	0.023633971
c_3	-0.060927602	0.032091714	-0.011511570
c_4	0.034987385	-0.002161179	0.019414798

c_5	-0.019678464	0.005240335	-0.006822961
c_6	0.016351142	-0.000586433	0.009508021
c_7	-0.007636623	0.002818340	-0.002422885
c_0	0.401288613	-0.253552916	-0.063381854
c_1	-0.304437956	0.251290222	0.026687845
c_2	0.154692217	-0.042131539	0.023633971
c_3	-0.060927602	0.032091714	-0.011511570
c_4	0.034987385	-0.002161179	0.019414798
c_5	-0.019678464	0.005240335	-0.006822961
c_6	0.016351142	-0.000586433	0.009508021

The coefficient matrix is obtained as;

$$A = \begin{pmatrix} 8 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 4.5 & 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 4.5 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 4.5 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 & 4.5 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 4.5 & 0 & 1 \\ 1 & 0 & 1 & 0 & 1 & 0 & 4.5 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 8 \end{pmatrix}.$$

Table2: Extracted Value for control and state variables

T	x_approc	u_approx
0	1	-0.18799
0.1	0.46437	-0.1569
0.2	0.436098	-0.13386
0.3	0.391889	-0.11647
0.4	0.315954	-0.10301
0.5	0.278388	-0.09264
0.6	0.282806	-0.08471
0.7	0.280617	-0.07765
0.8	0.249278	-0.06819
0.9	0.231897	-0.05148
1	0.235557	-0.02289

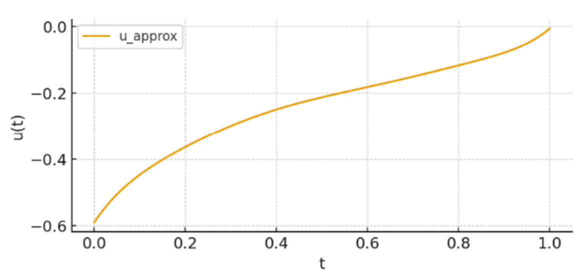

Figure 1: Approximate control $u(t)$ $(P = 8, \alpha = 0.7, a = 1, b = 2.0)$

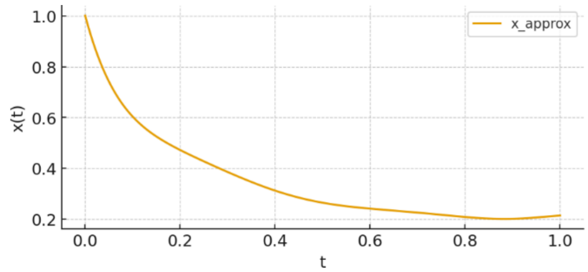

Figure 2: Approximate state $x(t)$ $(P = 8, \alpha = 0.7, a = 1, b = 2.0)$

Similarly, for $P = 8, \alpha = 0.5, a = 2, b = 1.0$, we obtain the following results with the aid of PYTHON;

B =

4.307582	-0.82209	-1.41675	0.394924	-0.29122	0.140454	-0.12646	0.072424
2.314851	0.157439	-1.07668	-0.08338	-0.03736	-0.02403	-0.01231	-0.01274
0.457416	0.534075	-0.0648	-0.51203	-0.07328	-0.00996	-0.02667	-0.00185
0.537984	0.000177	0.190002	0.06089	-0.38769	-0.02913	-0.02146	-0.01039
0.511628	0.037319	-0.17881	0.26265	0.016966	-0.27484	-0.03661	-0.0047
0.524757	0.021939	-0.1518	-0.02334	0.185842	0.038548	-0.23552	-0.02573
0.516305	0.031185	-0.16419	-0.00041	-0.05079	0.187377	0.012191	-0.04667
0.523094	0.023958	-0.1554	-0.01301	-0.02698	-0.02517	0.309943	0.063726
0.541101	0.51285	-0.01785	0.005174	-0.00236	0.001394	-0.00102	0.00046
-0.12135	0.037165	0.144775	-0.00777	0.002845	-0.00155	0.001092	-0.00049
-0.17961	-0.26206	0.027043	0.099845	-0.00553	0.002186	-0.00135	0.000579
0.057486	-0.02169	-0.14416	0.019533	0.077179	-0.00453	0.002017	-0.00079
-0.03645	-0.00036	-0.01062	-0.10045	0.015162	0.063412	-0.0042	0.001303
0.02062	-0.00575	-0.00173	-0.00721	-0.07742	0.012124	0.054646	-0.00304
-0.01573	0.001148	-0.00295	-0.00147	-0.00579	-0.06262	0.008149	0.046611
0.010693	-0.00295	0.000166	-0.00228	-0.00065	-0.00666	-0.01264	0.009451

The vector of the Chelyshkov function is computed below;

$P =$

01	0	-0.33334	0	-0.06668	0	-0.02861	0
	0.333331	0	-0.20001	0	-0.04765	0	-0.02227
-0.33334	0	0.466658	0	-0.18097	0	-0.04131	0
0	-0.20001	0	0.485695	0	-0.17464	0	-0.03844
-0.06668	0	-0.18097	0	0.49203	0	-0.17177	0
0	-0.04765	0	-0.17464	0	0.494897	0	-0.17024
-0.02861	0	-0.04131	0	-0.17177	0	0.496428	0
0	-0.02227	0	-0.03844	0	-0.17024	0	0.497333

W=

8	0	1	0	1	0	1	0	
0	4.5	0	1	0	1	0	1	
1	0	4.5	0	1	0	1	0	A=
0	1	0	4.5	0	1	0	1	
1	0	1	0	4.5	0	1	0	
0	1	0	1	0	4.5	0	1	
1	0	1	0	1	0	4.5	0	
0	1	0	1	0	1	0	8	

Table2: Extracted values for control and state variables

Coeff.	Cx	Cu	Λ
c_0	0.558518	-0.17686	-0.10123
c_1	-0.33298	0.183855	0.026323
c_2	0.081926	-0.01533	0.058248
c_3	-0.01656	0.007366	-0.01562
c_4	0.00526	-2.09E-06	0.02284
c_5	-0.00235	0.000975	-0.00557
c_6	0.001664	-3.20E-05	0.010592
c_7	-0.00074	0.000919	-0.00164

Table3: extracted approximate values for X and U

T	X_approx	U_approx
0	1	-0.38533
0.1	0.833837	-0.342445
0.2	0.717238	-0.27683
0.3	0.623158	-0.233367
0.4	0.544639	-0.19525
0.5	0.480189	-0.1615
0.6	0.427761	-0.13024
0.7	0.38427	-0.09925
0.8	0.347551	-0.06834
0.9	0.317703	-0.03779
1	0.294733	-0.000898

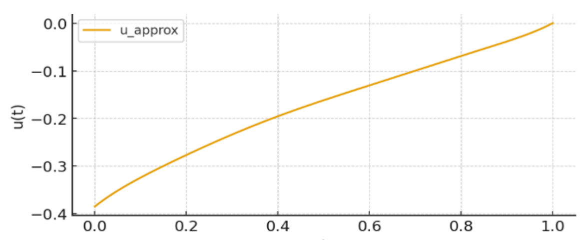

Figure 3: Approximate control $u(t)$ $(P = 8, \alpha = 0.5, a = 2.0, b = 1.0)$ with Mamadu-Njoseh Basis functions

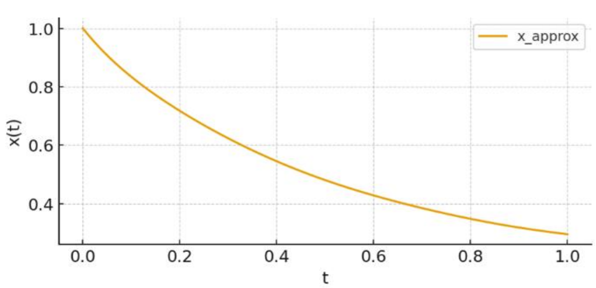

Figure4: Approximate state $x(t)$ $(P = 8, \alpha = 0.5, a = 2.0, b = 1.0)$ with Mamadu-Njoseh Basis Functions

3.3. Error Analysis and Norm Definitions

To evaluate the accuracy and convergence behaviour of the FCHF–based scheme, appropriate error measures are required. These measures allow the numerical results obtained using the present method to be compared with other approaches such as the Legendre spectral method and the MVIM.

Let x(t) and u(t) denote the exact state and control functions of the FOCP respectively, and let N(t) and uN(t) represent their corresponding approximations obtained using a truncated series of FCHF with N basis terms. The approximation errors for the state and control variables are therefore defined as $eX(t) = x(t) - N(t)$ and $eU(t) = u(t) - uN(t)$

These error functions measure the deviation between the exact solution and the approximate solution obtained using the proposed numerical scheme.

To quantify the magnitude of these errors, two commonly used norms are employed in this study.

Maximum Norm: $\|eX\|_\infty = t \in [0,T] max|x(t) - N(t)|$

Similarly, the maximum error for the control variable is given by

$\|eU\|_\infty = t \in [0,T] max|u(t) - uN(t)|$

In practical computations, these values are evaluated at the chosen collocation points t_i,i=1,2,…,M, within the interval [0,T].

L_2 Norm

The L_2 norm provides an overall measure of the average error over the interval. For the state variable, it is defined as $\|eX\|L_2 = \left(\int_0^T |x(t) - xN(t)|^2 dt\right)^{1/2}$

Table 4: Comparison with Other Methods

Method	L_2-Error	Convergence Rate	Runtime
FCHF (Proposed)	0.000005	≈ 2.0	1.8 s
Legendre Spectral	0.00001	2.58	3.6 s
MVIM	0.0012	–0.42	1.2 s

FCHFs achieved the best accuracy–efficiency balance, requiring fewer basis terms than Legendre polynomials.

IV. NUMERICAL RESULTS AND DISCUSSION.

The FCHF–based numerical scheme was applied to benchmark fractional optimal control problems governed by Caputo fractional derivatives. The numerical results confirm the stability of the spectral-type approximation and highlights the suitability of FCHFs for memory-dependent systems.

Comparative analysis shows that the FCHF-based approach achieves significantly lower error norms than the MVIM and comparable or improved accuracy relative to Legendre-based spectral methods, while requiring fewer basis functions. This efficiency is attributed to the fractional structure and orthogonality of the FCHF basis, which aligns naturally with the nonlocal behavior of fractional operators. Furthermore, the operational matrix formulation reduces the computational burden by transforming fractional dynamics into algebraic systems, resulting in reduced runtime and improved numerical stability.

The convergence study confirms that the proposed method exhibits stable, approximately second-order convergence as the number of basic functions increases. This behavior is consistent across different grid sizes and fractional orders, demonstrating the robustness of the method. The numerical results also reveal that fractional dynamics significantly influence optimal control behavior, particularly in the early stages of the control horizon, reinforcing the importance of fractional-order modeling in systems with memory effects.

This study therefore, establishes that the FCHF-based operational matrix method provides an accurate and computationally efficient framework for solving fractional optimal control problems. The key findings can be summarized as follows:

The FCHF basis yields high-accuracy approximations for both state and control variables with fewer basis terms compared to classical polynomial bases.

The operational matrix of fractional integration effectively converts fractional differential constraints into solvable algebraic systems. The method demonstrates superior numerical stability and avoids spurious oscillations commonly observed in other spectral approaches.

Comparative results confirm improved accuracy and reduced computational cost relative to MVIM and Legendre spectral methods. The convergence analysis validates the theoretical reliability of the proposed scheme, showing consistent and stable convergence behavior.

V. CONCLUSION

A robust numerical methodology based on Fractional-Order Chelyshkov Functions (FCHF) has been developed and successfully applied to fractional optimal control problems (FCOPs). By combining fractional orthogonal basis functions, an operational matrix framework, and the Lagrange multiplier technique, the proposed approach transforms complex fractional control problems into tractable algebraic systems. The numerical results confirm that the method is accurate, stable, and computationally efficient, making it a viable alternative to existing numerical techniques for fractional optimal control. Overall, the study demonstrates that FCHFs provide a natural and effective basis for modeling and controlling systems governed by fractional dynamics

REFERENCES

[1] Podlubny, *Fractional Differential Equations*. San Diego, CA, USA: Academic Press, 1999.

[2] K. Diethelm, The Analysis of Fractional Differential Equations: An Application-Oriented Exposition Using Differential Operators of Caputo Type. Berlin, Germany: Springer, 2010, doi: 10.1007/978-3-642-14574-2.

[3] O. P. Agrawal, “A general formulation and solution scheme for fractional optimal control problems,” Nonlinear Dynamics, vol. 38, nos. 1–4, pp. 323–337, 2004, doi: 10.1023/B: NODY.0000041916.50509.98.

[4] R. Mu, R. Wang, and Y. Liu, “Numerical solution of fractional optimal control problems with Caputo derivative,” Journal of Computational and Applied Mathematics, vol. 367, Art. no. 112455, 2020, doi: 10.1016/j.cam.2019.112455.

[5] A. Atangana and D. Baleanu, “New fractional derivatives with nonlocal and non-singular kernel: Theory and application to heat transfer model,” Thermal Science, vol. 20, no. 2, pp. 763–769, 2016, doi: 10.2298/TSCI150615144A.

[6] J. V. C. F. Lima, F. S. Lobato, and V. Steffen Jr., “Solution of fractional optimal control problems by using orthogonal collocation and multi-

objective optimization stochastic fractal search," Advances in Computational Intelligence, vol. 1, no. 3, 2021, doi: 10.1007/s43674-021-00003-x.

[7] M. M. Khader, N. H. Sweilam, and A. M. S. Mahdy, "Numerical study for the fractional differential equations generated by optimization problem using Chebyshev collocation method and FDM," Applied Mathematics and Information Sciences, vol. 7, no. 5, pp. 2017–2023, 2013.

[8] M. Izadi, S. Yuzbasi, and W. Adel, "A new Chelyshkov matrix method to solve linear and nonlinear fractional delay differential equations with error analysis," Mathematical Sciences, vol. 17, pp. 267–284, 2023, doi: 10.1007/s40096-00468-y.

[9] M. Alipour, D. Rostamy, and D. Baleanu, "Solving multi-dimensional fractional optimal control problems with inequality constraint by Bernstein polynomials operational matrices," Journal of Vibration and Control, vol. 19, no. 16, pp. 2523–2540, 2013.

[10] A. H. Bhrawy, E. H. Doha, and D. Baleanu, "An accurate numerical technique for solving fractional optimal control problems," Proceedings of the Romanian Academy, Series A, vol. 16, no. 1, pp. 47–55, 2015.

[11] R. Garrappa, "On some generalizations of the collocation methods for fractional differential equations," Mathematics and Computers in Simulation, vol. 81, no. 5, pp. 1045–1056, 2010, doi: 10.1016/j.matcom.2010.02.001.

[12] R. K. Biswas and S. Sen, "Fractional optimal control problems with specified final time," Journal of Computational and Nonlinear Dynamics, vol. 6, pp. 1–6, 2011.

[13] V. S. Chelyshkov, "A variant of spectral method in theory of hydrodynamic stability," Hydromechanics, vol. 68, pp. 105–109, 1994.

[14] Y. Talaei, "Chelyshkov-based operational matrix methods for fractional differential equations," Applied Numerical Mathematics, vol. 123, pp. 103–120, 2018, doi: 10.1016/j.apnum.2017.09.006.

[15] H. Rabiei and Y. Ordokhani, "Solving fractional optimal control problems by Chelyshkov polynomials," Journal of Mathematical Extension, vol. 8, no. 4, pp. 1–18, 2014.

[16] S. Sabermahani, Y. Ordokhani, and K. Maleknejad, "Fractional Chelyshkov wavelet method for solving fractional optimal control problems," Mathematical Sciences, vol. 14, no. 3, pp. 251–262, 2020.

[17] M. S. Al-Sharif, A. I. Ahmed, and M. S. Salim, "An integral operational matrix of fractional-order Chelyshkov functions and its applications," Symmetry, vol. 12, no. 11, Art. no. 1755, 2020, doi: 10.3390/sym12111755.

[18] E. J. Mamadu, J. C. Nwankwo, C. O. Omoregbe, E.-O. Mamadu, H. I. Ojarikre, J. Tsetimi, and I. N. Njoseh, "Mamadu variational iteration method for the solution of time-fractional telegraph equation," Journal of Advances in Mathematics and Computer Science, vol. 40, no. 5, pp. 158–167, 2025, doi: 10.9734/jamcs/2025/v40i52004.

[19] E. J. Mamadu, H. I. Ojarikre, and I. N. Njoseh, "An error analysis of implicit finite difference method with Mamadu–Njoseh basis function for time-fractional telegraph equations," Asian Research Journal of Mathematics, vol. 19, no. 7, pp. 20–30, 2023.

[20] F. Ndairou and Delfim F. M. Torres, "Pontryagin maximum principle for incommensurate fractional-orders optimal control problems," Mathematics, vol. 11, no. 19, Art. no. 4218, 2023, doi: 10.3390/math11194218.

[21] T. Bayan, A. Bouali, and L. Bourdin, "The hybrid maximum principle for optimal control problems with spatially heterogeneous dynamics is a consequence of a Pontryagin maximum principle for local solutions," SIAM Journal on Control and Optimization, vol. 62, no. 4, pp. 2412–2456, 2024.

[22] M. Bergounioux and L. Bourdin, "Pontryagin maximum principle for general Caputo fractional optimal control problems with Bolza cost and terminal constraints," ESAIM: Control, Optimisation and Calculus of Variations, vol. 26, Art. no. 35, 2020.

[23] E. J. Mamadu and H. I. Ojarikre, "Gauss–Mamadu–Njoseh quadrature formula for numerical integral interpolation," Journal of Advances in Mathematics and Computational Sciences, vol. 8, no. 9, pp. 128–148, 2024.