Pade-Approximation Approach to 1-D Heat Conduction Moving Boundary Problem in a Slab

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Abstract—**Problems whose solutions satisfy certain conditions on the boundary are called moving boundary problems, phase change problems, or Stephan Problems. The characteristic feature of these problems is the existence of a continuously moving boundary/interface that moves at a constant rate. Tracking the position and velocity of the moving boundary is an important part of the solution. Moving boundary problems arising in phase-change processes, such as melting and solidification, are commonly modelled using Stefan-type formulations. Exact analytical solutions exist only for simplified cases and typically involve transcendental functions, such as the error function. In this study, a semi-analytical solution for a one-dimensional heat equation with a moving boundary is developed using Pade approximations. The temperature distribution is expressed in a closed rational form by approximating the error function using a Pade approximant. This approach converts the classical transcendental Stefan condition into an algebraic equation, thereby simplifying the evaluation of the moving interface. The proposed method satisfies all boundary and interface conditions and provides accurate temperature and interface predictions with reduced computational effort.**

Index Terms—**Stefan problem, moving boundary, heat equation, Pade approximations, phase change, melting.**

I. INTRODUCTION

Solidification/melting problems are referred to as Stefan problems because they were first encountered by Physician Joseph Stefan, who proposed a model for the polar ice melting problem. The dynamics of the freezing process is an important phenomenon that is widely encountered in nature and in many engineering systems. Heat transfer problems associated with melting and solidification have attracted considerable attention over the past decade because of their extensive occurrence in energy conservation units, industrial refrigeration, crystal growth, geophysical science, welding, and casting. Some of the specific applications include surface ice distribution in the Polar Regions, food processing technology, and cryosurgical procedures [1].

These Problems are highly nonlinear in nature, and obtaining an analytical solution to these problems is difficult. Therefore, there are many methods to reduce the complexity of the solution of these problems, such as introducing approximations for temperature, so that initial, boundary, and interface conditions are applied to obtain analytical, semi-analytical, or numerical solutions. Furthermore, in this study, the Pade rational approximations are applied to reduce the nonlinear equations to a simple algebraic equation. The Pade approximations are the "best" approximations of a function by a rational function of a given order [2]. Pade rational approximations, which represent solutions using ratios of polynomials, offer high accuracy with few degrees of freedom and are particularly effective for approximating functions that exhibit gradients or boundary layers. This study proposes a compact Pade method for Stefan problems that tracks temperature and interface motion with a small number of dynamic coefficients. The method efficiently captures the nonlinear conduction behavior and latent-heat effects.

II. LITERATURE REVIEW

Stephan problems are nonlinear; hence, analytical solutions have mathematical difficulties. Analytical solutions have been reported for one- and two-dimensional geometries, and it is comparatively difficult to solve these equations in higher dimensions.

In addition, the temperature in the medium is assumed to be uniform, with constant thermal properties [3]. Within these confines, a succession of models of increasingly complicated phase change processes are presented. A review of a long bibliography on moving and free boundary problems for phase-change materials (PCM) for the heat equation is discussed in [5]. Goodman [6], Reynolds and Dolton [7], and Gupta and Banik [8] investigated an approximate analytical method that yielded solutions of Stefan problems in simple closed forms. Soloman et al. [9] presented a similarity solution for a one-phase solidification problem. He considered a mushy region. The problem is solved for two moving boundaries, one in the solid phase and the other in the mushy region, along with the temperature. The classical Lame-Clapeyron [10] solution is a particular case of the solution given by Solomon [9]. Ice storage in rectangular units is widely used in air-conditioning cool storage systems. Fang, G.Y et al. [11] examined solidification properties in a rectangular encloser and obtained a closed form analytical solution for the one-dimensional solidification problem. He considered that the transfer of heat within the material was purely by conduction and neglected the stagnant liquid phase. This model uses first-order accuracy solutions to obtain second- and higher-order solutions. Linear and quadratic approximate temperature distributions were considered to obtain the analytical solutions for the temperature distribution, position of the interface, and interface velocity. The effect of the Stefan number on various process parameters was analyzed, and it was found that the Stefan number is directly proportional to the solidification thickness.

III. MATHEMATICAL FORMULATION

3.1 Governing Equation

Consider a semi-infinite material initially at the melting temperature T_m. At time $t > 0$ the boundary at $x = 0$ is raised to a temperature $T_0 > T_m$, where T_0 is initiating melting temperature.

The heat conduction equation in the liquid region is

$$\frac{\partial T}{\partial t} = \alpha \frac{\partial^2 T}{\partial x^2}, \; 0 < x < s(t), \; t > 0 \quad (1)$$

where α is the thermal diffusivity and $s(t)$ is the moving interface position.

3.2 Initial and Boundary Conditions

Initial condition:

$$T(x,0) = T_m \,, x > 0 \quad (2)$$

Boundary conditions:

$$T(0,t) = T_0 \,, t > 0 \quad (3)$$

$$T(s(t),t) = T_m \quad (4)$$

Stefan condition at the moving boundary:

$$\rho L \frac{ds}{dt} = k \frac{\partial T}{\partial x}\bigg|_{x=s(t)} \quad (5)$$

Where ρ is density, L is latent heat and k is thermal diffusivity.

3.3 Similarity Solution

Introducing the similarity variable

$$\eta = \frac{x}{2\sqrt{\alpha t}} \quad (6)$$

the temperature field can be written as

$$T(x,t) = T_m + (T_0 - T_m)\frac{f(\eta)}{f(\lambda)} \quad (7)$$

$\lambda = 2\alpha t s(t)$ is the dimensionless interface location.

For the classical Stefan problem,

$$f(\eta) = \mathrm{erf}(\eta) \quad (8)$$

and the interface position is

$$s(t) = 2\lambda\sqrt{\alpha t} \quad (9)$$

As the similarity solution depends on the error function, and error function is a transcendental equation not suitable for the coupling problems such as conduction-convection, convection-radiation, radiation-conduction etc. In addition, the computational cost increases for repeated simulations and the difficulty level increases for inverse problems as well. In this situation Pade approximations reduce the difficulty level of the solution and are most suitable for heat transfer problems.

IV. PADE APPROXIMATIONS FOR THE TEMPERATURE FIELD

4.1 Pade 2nd order Approximation

The error function is approximated using a Pade (2,2) approximant:

$$erf(\eta) = \frac{2}{\sqrt{\pi}} \frac{\eta + \frac{1}{3}\eta^3}{1 + \frac{1}{3}\eta^2} \quad (10)$$

This rational approximation preserves the odd symmetry of the error function and provides good accuracy over the range of interest.

substituting the Pade approximation into the similarity solution yields

$$T(x,t) = T_m + (T_0 - T_m)\frac{\dfrac{\eta + \frac{1}{3}\eta^3}{1 + \frac{1}{3}\eta^2}}{\dfrac{\lambda + \frac{1}{3}\lambda^3}{1 + \frac{1}{3}\lambda^2}} \qquad (11)$$

$$\mathrm{T(x,t) = T_m + (T_0 - T_m)}\frac{\left(\eta + \frac{1}{3}\eta^3\right)\left(1 + \frac{1}{3}\lambda^2\right)}{\left(\eta + \frac{1}{3}\eta^3\right)\left(1 + \frac{1}{3}\lambda^2\right)} \qquad (12)$$

Were

$$\eta = \frac{x}{2\sqrt{\alpha t}}$$

This expression satisfies the boundary conditions:

$$T(0,t) = T_0$$

$$T(s(t),t) = T_m$$

and provides a closed-form rational temperature field. The temperature gradient at the interface is

$$\left.\frac{\partial T}{\partial x}\right|_{x=s(t)} = \frac{(T_0 - T_m)}{2\sqrt{\alpha t}}\frac{d}{d\eta}\left.\frac{\dfrac{\eta + \frac{1}{3}\eta^3}{1 + \frac{1}{3}\eta^2}}{\dfrac{\lambda + \frac{1}{3}\lambda^3}{1 + \frac{1}{3}\lambda^2}}\right|_{\eta=\lambda} \qquad (13)$$

Substituting into the Stefan condition yields

$$\rho L\lambda = k\frac{(T_0 - T_m)}{\sqrt{\pi\alpha}}\;\frac{1 + \frac{1}{3}\lambda^2}{1 + \lambda^2} \qquad (14)$$

This equation is algebraic in λ, unlike the classical transcendental Stefan equation, and can be solved efficiently.

4.2 Pade 3rd order Approximation

We can choose Pade higher order approximations for higher accuracy.

The Pade (3,3) approximation

$$\mathrm{erf}(\eta) = \frac{2}{\sqrt{\pi}}\frac{\eta(1 + a_1\eta^2 + a_2\eta^4)}{1 + b_1\eta^2 + b_2\eta^4} \qquad (15)$$

The temperature field for the above approximation becomes

$$T(x,t) = T_m + (T_0 - T_m)\frac{\eta(1 + a_1\eta^2 + a_2\eta^4)(1 + b_1\lambda^2 + b_2\lambda^4)}{\lambda(1 + b_1\eta^2 + b_2\eta^4)(1 + a_1\lambda^2 + a_2\lambda^4)} \qquad (16)$$

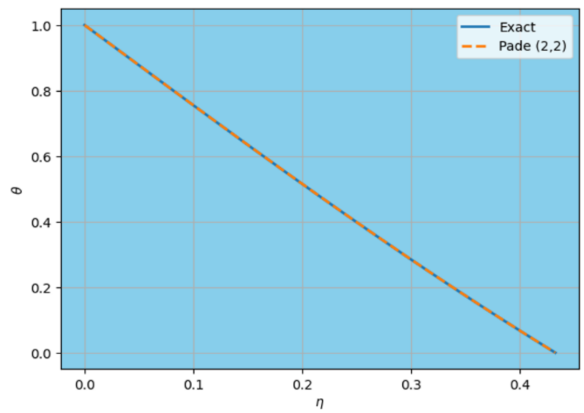

Figure 1: Exact Solution vs Pade(2,2)

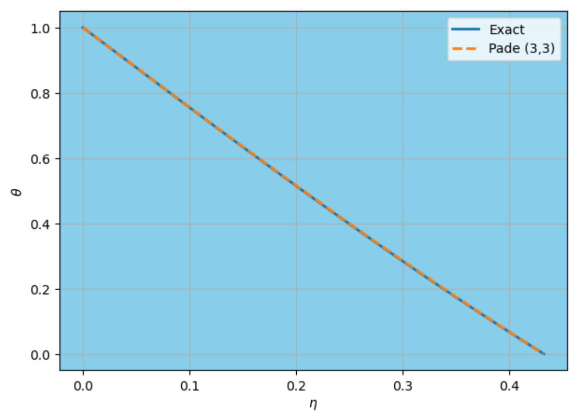

Figure2: Exact Solution vs Pade(3,3)

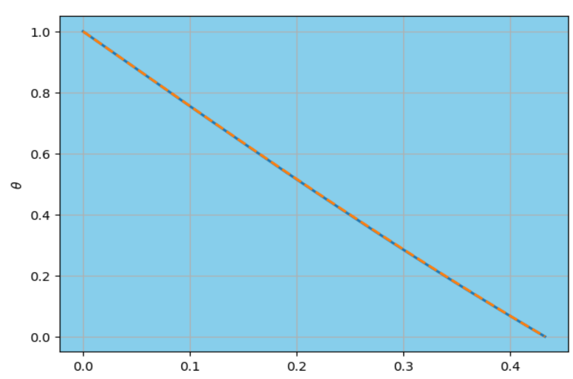

Figure3: Pade(2,2) vs Pade(3,3)

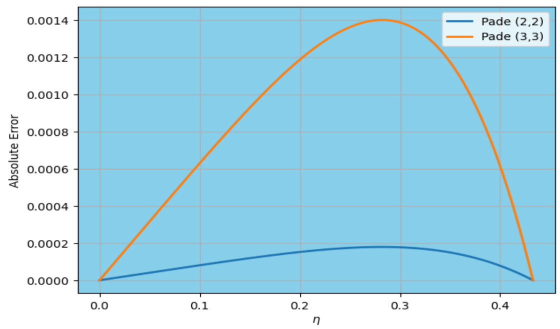

Figure4: Absolute Error Distribution

Figure 1 and 2 shows comparison between the exact solution and the Pade approximations for second and third order for the dimensionless temperature distribution. The two curves exhibit almost the same behaviour throughout the computational domain, demonstrating the remarkable accuracy of the Pade approximation. The approximation successfully satisfies the boundary conditions and accurately predicts the position of the moving interface, confirming its suitability for modelling one-dimensional Stefan problems. Figure 3 presents a comparison between the Pade (2,2) and Pade (3,3) approximations for the dimensionless temperature distribution θ as a function of the similarity variable η. The negligible difference between the second- and third-order approximations suggests that Pade (2,2) is sufficient to accurately describe the thermal field and moving interface position. Therefore, higher-order approximations provide little additional accuracy while increasing computational complexity.

Figure 4 depicts the absolute error distribution for the Pade (2,2) and Pade (3,3) approximations. The errors vanish at both the fixed boundary and the moving interface, indicating exact satisfaction of the boundary conditions. The maximum error occurs in the interior region near $\eta \approx 0.3$. The Pade (2,2) approximation exhibits a peak error of approximately 1.8×10−4, while the Pade (3,3) approximation reaches about 1.4×10−3. Although both approximations provide excellent accuracy, the Pade (2,2) approximation demonstrates superior performance and offers an optimal balance between computational efficiency and solution accuracy.

RMS Error Pade (2,2) = 0.00012483446005316298

RMS Error Pade (3,3) = 0.0009760734325966325

Maximum Error Pade (2,2) = 0.00017913650674228343

Maximum Error Pade (3,3) = 0.0014006504162553668

V. CONCLUSIONS

A semi-analytical Pade approximation method has been developed for solving a one-dimensional moving boundary heat conduction problem. The method accurately approximates the temperature distribution and simplifies the determination of the moving interface. Due to its simplicity, accuracy, and computational efficiency, the Pade approximation approach is well suited for thermal energy storage, melting and solidification problems, and engineering design applications. The comparative study of the exact solution, Pade (2,2), and Pade (3,3) approximations demonstrates that the Pade approximation technique is a reliable analytical tool for moving boundary heat conduction problems. Among the approximations considered, Pade (2,2) offers the best combination of simplicity, computational efficiency, and accuracy, making it particularly suitable for engineering and thermal energy storage applications involving phase-change processes.

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