

# Common Fixed-Point Theorems for Rational Contractions in Banach Spaces

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**Abstract**—This paper develops new common fixed-point theorems for rational contractions in Banach spaces. Rational contractions, defined via nonlinear rational inequalities, extend the classical Banach contraction principle and provide a richer framework for fixed point analysis. We establish the existence and uniqueness of common fixed points for pairs of commuting rational contractions, and further extend these results to measurable multifunction through the Kuratowski–Ryll–Nardzewski measurable selection theorem. Our approach unifies and generalizes earlier fixed-point results, offering a novel perspective on rational contraction mappings. Applications to iterative algorithms, equilibrium problems, control theory, and differential inclusions demonstrate the theoretical and practical relevance of the results.

**Index Terms**—Banach space, rational contraction, common fixed point, measurable multifunction, equilibrium problems.

## I. INTRODUCTION

Fixed point theory is one of the most powerful tools in nonlinear functional analysis. The classical Banach contraction principle ensures that every contraction mapping on a complete metric space has a unique fixed point. This principle has been extended in numerous directions, including non-expansive mappings, quasi-contractions, and multifunction's. However, rational contractions — mappings defined by rational inequalities — provide a richer framework. They generalize contractions by introducing nonlinear rational expressions in the contractive condition. Despite their potential, common fixed-point results for rational contractions remain largely unexplored.

The motivation for this paper is twofold:

1. To establish new common fixed-point theorems for rational contractions in Banach spaces.

2. To extend these results to measurable multifunctions, thereby connecting rational contractions with measurable selection theory.

Our results unify and extend earlier work on fixed point theory and open new avenues for applications in optimization, economics, and control theory.

## II. PRELIMINARIES

We recall some basic definitions and notations.

**Definition 2.1 (Banach Space):**

A Banach space is a complete normed linear space.

**Definition 2.2 (Rational Contraction):**

A mapping  $T: X \rightarrow X$  is called a rational contraction if there exist constants  $\alpha, \beta > 0$  such that

$$\|Tx - Ty\| \leq \frac{\alpha \|x - y\|}{1 + \beta \|x - y\|}, \forall x, y \in X.$$

**Definition 2.3 (Common Fixed Point):**

A point  $x_0 \in X$  is called a common fixed point of two mappings  $T, G: X \rightarrow X$  if

$$Tx_0 = Gx_0 = x_0.$$

**Lemma 2.4:** Every rational contraction is a strict contraction for sufficiently small distances, i.e.,

$$\|Tx - Ty\| < \|x - y\|, \text{ for } \|x - y\| \text{ small.}$$

## III. MAIN RESULTS

**Theorem 3.1 (Existence of Common Fixed Point)**

Let  $T, G: X \rightarrow X$  be two commuting rational contractions on a Banach space  $X$ . Then there exists a unique common fixed point  $x_0 \in X$ .

**Proof:**

- Construct iterative sequence:

$$d_{n+1} = \frac{1}{2}(Td_n + Gxd_n).$$

- Using rational contraction inequality, show that  $\|d_{n+1} - d_{n+2}\| \leq \frac{\alpha}{1 + \beta} \|d_n - d_{n+1}\|$ .
- Hence  $\{x_n\}$  is a Cauchy sequence.
- By completeness of Banach space,  $\{x_n\}$  converges to some  $x_0 \in X$ .
- Since  $T$  and  $G$  commute,  $Tx_0 = Gx_0 = x_0$ .
- Uniqueness follows from the strict contractive nature of rational contractions.

**Theorem 3.2 (Extension to Multifunctions)**

Let  $T, G: X \rightarrow 2^X$  be measurable multifunctions satisfying rational contraction conditions. Then there exists a measurable selector  $f$  such that  $f(x_0) = x_0$  is a common fixed point.

**IV. PROOF SKETCH:**

- 4.1. Apply Kuratowski–Ryll–Nardzewski measurable selection theorem to obtain measurable selectors.
- 4.2. Construct iterative sequence using rational contraction inequality.
- 4.3. Convergence yields measurable common fixed point.

**Applications**

- Optimization: Rational contractions model iterative algorithms with nonlinear convergence rates.
- Economics: Common fixed points represent equilibria in markets with rational preference structures.
- Control Theory: Stability analysis of systems with rational feedback mappings.
- Differential Inclusions: Existence of solutions via rational contraction fixed points.

4.1. Kuratowski–Ryll–Nardzewski measurable selection theorem to obtain measurable selectors: -

Proof: -

1. Let  $X$  be a Banach space with its Borel  $\sigma$ -algebra. Consider multifunctions  $T, G: X \rightarrow 2^X$ . Assume:
  - For each  $x \in X$ ,  $T(x)$  and  $G(x)$  are nonempty closed subsets of  $X$ .
  - Both  $T$  and  $G$  are measurable multifunctions (i.e., for every open set  $U \subseteq X$ ,  $\{x \in X: T(x) \cap U \neq \emptyset\}$  is measurable).

2. Rational Contraction Condition: For selectors  $t \in T(x), g \in G(y)$ , we assume  $\|t - g\| \leq \frac{\alpha \|x - y\|}{1 + \beta \|x - y\|}, \alpha, \beta > 0$ .

This ensures that the multifunctions are “contractive” in rational sense.

3. Apply Kuratowski–Ryll–Nardzewski: Since  $T$  and  $G$  are measurable multifunctions with closed nonempty values in a Polish space (Banach space is Polish if separable), the theorem guarantees existence of measurable selectors:

- There exists  $f_T: X \rightarrow X$  such that  $f_T(x) \in T(x)$ .
- There exists  $f_G: X \rightarrow X$  such that  $f_G(x) \in G(x)$ .

4. Construct Iterative Sequence: Define

$$x_{n+1} = \frac{1}{2}(f_T(x_n) + f_G(x_n)).$$

Since  $f_T, f_G$  are measurable, the sequence  $\{x_n\}$  is measurable.

5. Cauchy Property: Using rational contraction inequality,

$$\|x_{n+1} - x_{n+2}\| \leq \frac{\alpha}{1 + \beta} \|x_n - x_{n+1}\|.$$

Hence  $\{x_n\}$  is Cauchy.

6. Convergence:

By completeness of Banach space,  $\{x_n\}$  converges to some  $x_0 \in X$ .

7. Verification of Fixed Point:

Since  $f_T(x_0) \in T(x_0)$  and  $f_G(x_0) \in G(x_0)$ , and the rational contraction ensures uniqueness, we have

$$f_T(x_0) = f_G(x_0) = x_0.$$

Thus  $x_0$  is a common fixed point.

4.2. Construct iterative sequence using rational contraction inequality

1. Initial Point: Choose any  $x_0 \in X$ .
2. Define Iteration: For two commuting rational contractions  $T, G: X \rightarrow X$ , define the sequence:

$$x_{n+1} = \frac{1}{2}(Tx_n + Gx_n)$$

3. Estimate Difference: Consider

$$\|x_{n+1} - x_{n+2}\| = \left\| \frac{1}{2}(Tx_n + Gx_n) - \frac{1}{2}(Tx_{n+1} + Gx_{n+1}) \right\|.$$

Simplify:

$$\|x_{n+1} - x_{n+2}\| \leq \frac{1}{2} (\|Tx_n - Tx_{n+1}\| + \|Gx_n - Gx_{n+1}\|).$$

4. Apply Rational Contraction Inequality: For  $T$ :

$$\|Tx_n - Tx_{n+1}\| \leq \frac{\alpha \|x_n - x_{n+1}\|}{1 + \beta \|x_n - x_{n+1}\|}.$$

Similarly for  $G$ :

$$\|Gx_n - Gx_{n+1}\| \leq \frac{\alpha \|x_n - x_{n+1}\|}{1 + \beta \|x_n - x_{n+1}\|}.$$

5. Combine Estimates:

$$\|x_{n+1} - x_{n+2}\| \leq \frac{\alpha}{1 + \beta} \|x_n - x_{n+1}\|.$$

6. Contraction Factor: Since  $\frac{\alpha}{1 + \beta} < 1$ , the sequence differences shrink geometrically.

7. Cauchy Property: Therefore,  $\{x_n\}$  is a Cauchy sequence in Banach space  $X$ .

8. Convergence: By completeness of Banach space,  $\{x_n\}$  converges to some  $x_0 \in X$ .

9. Verification: Taking limit,

$$x_0 = \frac{1}{2}(Tx_0 + Gx_0).$$

Since  $T$  and  $G$  commute and are rational contractions, this implies

$$Tx_0 = Gx_0 = x_0.$$

10. Uniqueness: Suppose  $y_0$  is another common fixed point. Then

$$\|x_0 - y_0\| = \|Tx_0 - Ty_0\| \leq \frac{\alpha \|x_0 - y_0\|}{1 + \beta \|x_0 - y_0\|}.$$

This forces  $\|x_0 - y_0\| = 0$ . Hence  $x_0 = y_0$ .

4.3 Convergence yields measurable common fixed point

1. Define the Iterative Sequence

- Start with an arbitrary point  $x_0 \in X$ .

- Construct the sequence

$$x_{n+1} = \frac{1}{2}(f_T(x_n) + f_G(x_n)),$$

where  $f_T, f_G$  are measurable selectors of the multifunctions  $T, G$ .

2. Apply Rational Contraction Inequality

- For each mapping, the rational contraction condition gives:

$$\|f_T(x_n) - f_T(x_{n+1})\| \leq \frac{\alpha \|x_n - x_{n+1}\|}{1 + \beta \|x_n - x_{n+1}\|},$$

and similarly, for  $f_G$ .

- Combining these, we obtain:

$$\|x_{n+1} - x_{n+2}\| \leq \frac{\alpha}{1 + \beta} \|x_n - x_{n+1}\|.$$

3. Show the Sequence is Cauchy

- Since  $\frac{\alpha}{1 + \beta} < 1$ , the differences shrink geometrically.

- Therefore,  $\{x_n\}$  is a Cauchy sequence in the Banach space  $X$ .

4. Use Completeness of Banach Space

- Because  $X$  is complete, every Cauchy sequence converges.

- Thus, there exists a limit point  $x_0 \in X$  such that

$$\lim_{n \rightarrow \infty} x_n = x_0.$$

5. Measurability of the Limit

- Each step of the sequence is constructed using measurable selectors  $f_T, f_G$ .

- Hence, the sequence  $\{x_n\}$  is measurable.

- In a Polish space (Banach space with separability), the pointwise limit of measurable functions is measurable.

- Therefore, the limit  $x_0$  is measurable.

6. Verify Fixed Point Property

- Taking the limit in the iteration:

$$x_0 = \frac{1}{2}(f_T(x_0) + f_G(x_0)).$$

- This implies

$$f_T(x_0) = f_G(x_0) = x_0.$$

- Since  $f_T(x_0) \in T(x_0)$  and  $f_G(x_0) \in G(x_0)$ , we conclude

$$x_0 \in T(x_0) \text{ and } x_0 \in G(x_0).$$

7. Uniqueness of the Fixed Point

- Suppose  $y_0$  is another common fixed point. Then

$$\begin{aligned} \|x_0 - y_0\| &= \|f_T(x_0) - f_T(y_0)\| \\ &\leq \frac{\alpha \|x_0 - y_0\|}{1 + \beta \|x_0 - y_0\|}. \end{aligned}$$

- This inequality forces  $\|x_0 - y_0\| = 0$ .

- Hence,  $x_0 = y_0$ .

Applications

- Optimization: Rational contractions model iterative algorithms with nonlinear convergence rates.

- Economics: Common fixed points represent equilibria in markets with rational preference structures.
- Control Theory: Stability analysis of systems with rational feedback mappings.
- Differential Inclusions: Existence of solutions via rational contraction fixed points.

## V. CONCLUSION

In this work, we introduced and proved new common fixed-point theorems for rational contractions in Banach spaces, extending classical fixed-point theory and connecting it with measurable selection principles. By establishing existence and uniqueness results for both single-valued and multifunctional settings, we provided a unified framework that bridges pure and applied mathematics. The applications to optimization, economics, and control theory highlight the practical impact of rational contractions. Future research may explore stochastic rational contractions, fuzzy Banach spaces, and algorithmic implementations in machine learning and computational mathematics, thereby broadening the scope and utility of the theory.

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