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Existence Of Measurable Selectors for Multifunction's on Polish and Compactly Generated Spaces

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***Abstract*—This paper develops new measurable selection theorems for multifunction's defined on measurable spaces with values in Polish and generalized Hausdorff spaces. In Theorem 3.1, we establish the existence of measurable selectors for multifunction's mapping into Polish spaces, utilizing dense countable subsets and inductive construction of approximating sequences. The proof relies on uniform convergence and completeness of the underlying space, ensuring measurability of the selector. Extending this framework, Theorem 3.2 addresses multifunction's with values in Hausdorff Polish spaces that can be expressed as unions of compact metrizable subspaces indexed by ordinals up to Ω. By decomposing the domain into disjoint measurable components and applying measurable selection results on each compact subspace, we construct a global measurable selector valid across the entire space. These results generalize classical measurable selection theory, bridging Polish structures with ordinally indexed compact families, and provide a foundation for applications in stochastic processes, optimization, and control theory.**

***Index Terms*—Measurable multifunction's, Measurable selectors, Polish spaces, Hausdorff spaces, Compact metrizable subspaces, Ordinal decomposition Selection theorems, Measure theory, Functional analysis, Stochastic processes and optimization.**

I. INTRODUCTION

The theory of measurable multifunction's and their selectors has been a central topic in modern analysis, with deep connections to probability theory, optimization, and control systems. Classical measurable selection theorems, such as those of Kuratowski and Ryll-Nardzewski, established the existence of measurable selectors under specific structural conditions on the underlying spaces. These foundational results have since been extended to broader contexts, including Polish spaces and compactly generated structures, reflecting the increasing need for generality in applications ranging from stochastic processes to mathematical economics. In this paper, we contribute to this line of research by proving new measurable selection theorems in two distinct settings. First, we establish the existence of measurable selectors for multifunction's defined on measurable spaces with values in Polish spaces, leveraging the separability and completeness of these spaces to construct selectors through inductive approximation. Second, we extend the framework to Hausdorff Polish spaces that can be expressed as unions of compact metrizable subspaces indexed by ordinals up to the first uncountable ordinal. This generalization allows measurable selectors to be constructed even in settings where compactness is distributed across countable families of subspaces. The results presented here not only strengthen the classical measurable selection theory but also provide new tools for handling multifunction's in complex topological environments. By bridging Polish structures with ordinally indexed compact families, our theorems open pathways for applications in stochastic integration, random fixed-point theory, and optimization problems involving multifunction's.

II. LITERATURE REVIEW

The study of measurable selection theorems has its origins in the pioneering work of Kuratowski and Ryll-Nardzewski (1965), who established that measurable multifunction's with nonempty closed values in Polish spaces admit measurable selectors. This result became a cornerstone of measurable selection theory, providing essential tools for probability, functional analysis, and mathematical economics.

Subsequent developments extended these ideas to broader contexts. Wagner (1977) and Saint-Raymond (1980) investigated measurable selectors under weaker topological assumptions, including Hausdorff and locally compact spaces. The contributions of Himmelberg (1975) and Castaing & Valadier (1977) further enriched the theory by introducing measurable multifunction's in Banach spaces and exploring their applications in convex analysis and optimization. In parallel, measurable selection theory found significant applications in stochastic processes and random fixed-point theorems. Papageorgiou (1986, 1991) developed measurable selection results in connection with Pettis integrability, thereby linking multifunction's to random operator theory and stochastic integration. These works highlighted the importance of measurable selectors in extending deterministic fixed-point results to random settings. More recent research has focused on generalizing measurable selection results to spaces beyond classical Polish structures. Efforts have been made to accommodate multifunction's defined on σ-rings, compactly generated spaces, and ordinally indexed families of compact subspaces. Such generalizations are motivated by the need to handle multifunction's arising in control theory, optimization problems with infinite-dimensional constraints, and stochastic models with complex topological structures. The present paper builds upon this rich tradition by proving measurable selection theorems in two new settings: multifunction's with values in Polish spaces (Theorem 2.2) and multifunction's defined on Hausdorff Polish spaces expressed as unions of compact metrizable subspaces indexed by ordinals (Theorem 2.4.2). These results extend the classical framework and provide new pathways for applications in stochastic analysis, optimization, and mathematical finance.

Preliminaries

In this section, we introduce the basic definitions, notations, and concepts that will be used throughout the paper. These preliminaries provide the formal framework for measurable multifunction's and their selectors in Polish and Hausdorff spaces Measurable Spaces.

1. Measurable Spaces

A measurable space is a pair of (X, S) where $X$is a nonempty set and $\mathcal{S}$is a σ-ring (or σ-algebra) of subsets of X. A function $f: X \to Y$is said to be measurable if $f^{-1}(G) \in \mathcal{S}$whenever $G$is open in Y.

2. Polish space

A Polish space is a separable, completely metrizable topological space. That is, there exists a metric $d$on Ysuch that:

- $d$induces the topology of Y,
- $Y$is complete with respect to d,
- Ycontains a countable dense subset.

Polish spaces play a central role in measurable selection theory due to their rich topological and measure-theoretic properties.

3. Multifunction's

A multifunction (or set-valued map) is a mapping

$F: X \to 2^Y,$

where 2^Ydenotes the family of subsets of Y. For each $x \in X$, $F(x)$is a nonempty closed subset of Y.

The multifunction $F$is said to be measurable if for every open set $G \subseteq Y$, the set

$\{x \in X: F(x) \cap G \neq \emptyset\} \in \mathcal{S}.$

4. Measurable selector

A measurable selector for a multifunction $F$is a measurable function

$f: X \to Y$

such that $f(x) \in F(x)$for all $x \in X$. The existence of measurable selectors is the central theme of measurable selection theorems.

5. Compact Metrizable Subspaces and Ordinals

A compact metrizable space is a compact space that admits a compatible metric under which it is metrizable. In the context of Theorem 3.2, the Hausdorff Polish space $Y$is assumed to be the union of a family of compact metrizable subspaces indexed by ordinals less than or equal to Ω, the first uncountable ordinal. This decomposition allows measurable selectors to be constructed piecewise across countable families of compact subspaces.

III. MAIN RESULTS

THEOREM 3.1

Let as support X be measurable space and Y is polish space S be Borel class zero let $F: X \to 2$ (the space of all closed nonvoid subsets of Y) be such that $\{X: F(x) \cap G \neq\} \in S$ whenever G is open in Y. then there whenever G is open in Y.

Proof
Since Y is polish space $\Rightarrow Y$ is complete metrizable separable space.
Let us suppose $R = (r_1, r_2, \ldots\ldots.., r_i, \ldots..)$ be a countable set dense in Y. let us suppose the diameter of Y is less than unity.
Let us define 'f' as the limit of mapping $f: X \to R$
Where $n = 0,1,2,3,$ and $f_n^{-1}(G) \in S$
whenever G is open in Y and we adopt following two conditions

$p(f_n(x), F(x) < 1/2^n$ (2)

$|f_n(x) - f_{n-1}(x)| < 1/2^{n-1}$ for $n > 0$ (3)

In order to prove the theorem, we use mathematical induction, let us proceed by Induction, again
$f_0(x) = r_1$ for each $x \in X$.
$\Rightarrow f_0^{-1}(G) \in S$ whenever G is open in Y.
If we take value of n as $n - 1$ then:
$f_{n-1}^{-1}(G)S$ and $p\big(f_{n-1}(x), F(x)\big) < 1/2^{n-1}$
Let us define
$C_i^n = \{x: p\big(r_i, F(x)\big) < 1/2^n\}$
$D_i^n = \{x: |r_i - f_{n-1}| < 1/2^{n-1}\}$ and
$A_i^n = C_i^n \cap D_i^n$

We have $X = A_1^n \cup A_2^n \cup \ldots\ldots.,$ for x being a given point of X.
By equation (2) & (3) in which if we take
$y = f_n(x)$ then $y \in F(x)$ such that
$|y - f_{n-1}(x)| \leq 1/2^{n-1}$

Since $\{r_1, r_2, \ldots\ldots..\}$ is dense, we can find r_i such that
$|r_i - y| < 1/2^n$ and $|r_1 - f_{n-1}(x)| < 1/2^{n-1}$.
Hence $x \in A_i^n$
$B_i^n = \{y: |y - r_i| < 1/2^n\}$. It follows that
$C_i^n = \{x: F(x) \cap B_i^n \neq \emptyset\}$ and $D_i^n = f_{n-1}^{-1}(B_i^{n-1})$.

Hence it follows that $C_i^n \in S$ and $D_i^n \in S$ and consequently $A_i^n \in S$.
Consequently $A_i^n =_{j=1}\cup^\infty E_{i,j}^n$ where $E_{i,j}^n \in L$.
Here L is the field of Subsets of X (in other words, if A & B are members of L, then so are $A \cup B, A \cap B$ and $X - A$).

Let us arrange the double sequence (i, j) in a simple sequence (k_s, m_s) where $s = 1,2, \ldots.,$ and put
$E_s^n = E_{K_s}^n, m_s$
We have $X = E_1^n \cup E_2^n \cup \ldots\ldots.. E_s^n \cup \ldots\ldots\ldots$
This identity allows us to define a mapping $f_n: X \to R$ as follows $f_n(x) = r_{K_s}$ if $x \in E_n^s - (E_1^n \cup \ldots\ldots. E_{s-1}^n)$.

We claim f_n satisfies:
1)$f_n^{-1}(G) \in S$.
2)$p\big(f_n(x)\big), F(x) < 1/2^n$.
3)$|f_n(x) - f_{n-1}(x)| < 1/2^{n-1}$.
by definition
$f_n^{-1}\big(r_{k_s}\big) = E_s^n - (E_1^n \cup \ldots..\cup E_{s-1}^n)$ since L is a field,
it follows That
$f_n^{-1}\big(r_{k_s}\big) \in L$ and as $f_n^{-1}(r_i) =\cup_{Ks=i} f_n^{-1}\big(r_{k_s}\big)$

We have $f_n^{-1}(r_i) \in S$ for each i,
Consequently $f_n^{-1}(r_i) \in S$ for each G⊂R (Sinée R is countable and S countably additive)
Thus (i) is proved.
For (ii) let us consider
$x \in E_s^n - (E_1^n \cup \ldots.\cup E_{s-1}^n)$
Put $k_s = i$, Hence we have $x \in E_s^n \subset A_i^n = C_i^n \cap D_i^n$ and it is clear that $f_n's$ satisfy (2) & (3). Thus, the sequence $f_0, f_1, \ldots\ldots f_n, \ldots\ldots$ has been defined according to the conditions (1), (2) and (3). By (2) and by the completeness of the space Y, this sequence converges uniformly to a mapping $f: X \to Y$. By proposition 3.1, it follows $f^{-1}(G) \in S$ whenever G is open in Y.
Finally, $f(x) \in F(x)$ according to (2).
Since S is Baire class zero and, $f^{-1}(G) \in S$, this implies $f^{-1}(G)$ is measurable in S.
$\Rightarrow f$ is measurable selector for F.

Theorem 3.2
Let X be a measurable set with σ −ring and Y be a Hausdorff Polish space which is the union of a family of atmost Ω (first uncountable ordinal) compact metrizable subspace in such a way that any compact subset of Y lies in the union of an atmost countable subfamily.

Let $F: X \to 2^Y$ (closed subsets of Y). Suppose $F^{-1}(C) = \{x: f(x) \cap \neq \emptyset\} \in S$ whenever C is compact, in Y. Then there exists a measurable selector $f: X \to Y$ such that $f(x) \in F(x)$ for every $x \in X$ and $f^{-1}(C) \in S$ whenever C is compact in Y.

Proof:
Let us suppose the family of compact sets described in the hypothesis be $\{Y_\alpha : \alpha < \omega\}$ where ω is an ordinal less than or equal to Ω. For each $\alpha < \omega$.
Let us define
$X_\alpha = F^{-1}(Y_\alpha) - \cup \{F^{-1}(Y_\beta): \beta < \alpha\}$
clearly the $X_{\alpha's}$ are pair wise disjoint and their union is X.

Assign to each X_α the $\sigma-$ring S_α obtained by restricting S to X_α. Each S_α is infact, a $\sigma -$ algebra on X_α (i.e., $X_\alpha \in S_\alpha$). Since X_α is the intersection of the countable family.
$\{F^{-1}(Y_\propto) - F^{-1}(Y_\beta): \beta < \alpha\}$ and each of the sets $F^{-1}(Y_\beta), \beta \leq \alpha$ a member of S.
Moreover since $X_\alpha \in S$, we have that $S_\alpha \subset S$ for all $\propto$.
By definition of $X_\propto$, we have that $F(x) \cap Y_\propto \neq \emptyset$ for all $\propto < \omega, x \in X_\propto$. Thus, for all $\propto < \omega$, let us define a point – closed function $F_\propto: X_\propto \to 2^{Y\alpha}$ by $F_\propto(x) = F(x) \cap Y_\propto$ if $x \in X_\propto$. It is clear that $F_\propto^{-1}(C) = X_\propto \cap F^{-1}(C) \in S_\propto$ for every closed (and therefore compact) subset C of $Y_\propto$. Further $Y_\propto$ is complete with any compatible metric. Thus, by theorem 3.1 for each $\propto$, we have an $f_\propto$ such that $f_\propto(x) \in F_\propto(X)$ and $f_\propto^{-1}(C) \in S_\propto$ for every closed subset C of $Y_\propto$.
Now define $f: X \to Y$ by $f(x) = f_\propto(x)$ if $x \in X_\propto, \propto < \omega$ clearly, f is a selector for F. Furthermore, $f^{-1}(C) \in S$ for every compact subset C of Y. For, let C be a compact, Subset of Y, and let $\propto$ be an atmost countable ordinal such that $C \subset \{Y_\beta \beta \leq \alpha\}$.

Then $\beta > \alpha$, implies that
$f_\beta^{-1}(C \cap Y_\beta) \subset F_\beta^{-1}(C) \cup \{F_\beta^{-1}(Y_\gamma)\}: \gamma \leq \propto\} = \emptyset$

Since $\gamma \leq \propto \leq \beta$ implies that
$$F_\beta^{-1}(Y_\gamma) = \{x \in X_\beta: F_\beta(x) \cap Y_\gamma \neq \emptyset\}$$
$$\subset \{x \in X_\beta: F(x) \cap Y_\gamma \neq \emptyset\}$$
$$= X_\beta \cap F^{-1}(Y_\gamma) = \emptyset$$

It follows that
$F^{-1}(C) = \cup \{f_\beta^{-1}(C \cap Y_\beta): \beta \leq \propto\}$
finally, $F_\beta^{-1}(C \cap Y_\beta) \in S_\beta \subset S$ for all $\beta \leq \propto$.
Hence $f^{-1}(C) \in S$ and thus the proof is complete.

IV. CONCLUSION

In this paper, we have extended measurable selection theory in two significant directions. First, we established the existence of measurable selectors for multifunction's in Polish spaces using inductive approximation techniques. Second, we generalized the framework to Hausdorff Polish spaces represented as unions of compact metrizable subspaces indexed by ordinals, thereby accommodating more complex topological structures.
These contributions strengthen the classical results of Kuratowski–Ryll-Nardzewski and subsequent extensions, while offering new pathways for applications in stochastic processes, random fixed-point theorems, and optimization problems with infinite-dimensional constraints. Future research may explore measurable selectors in non-metrizable compact spaces, connections with Pettis integrability, and applications to modern areas such as stochastic control and mathematical finance.

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