Fractional Powers of the Generalized Hankel–Clifford Transformation: Spectral Realization, Symbolic Calculus, and Pseudo-Differential Operators

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***Abstract*—We construct the fractional powers hᵩμ,ν of the generalized Hankel–Clifford transformation as a genuine one-parameter unitary group generated by an explicit self-adjoint Bessel-type operator Λμ,ν, whose spectrum is {2n : n ∈ ℕ₀} and whose eigenfunctions are the Laguerre functions x^((μ+ν)/2) e^(−x) Lⁿᵛ(2x). The Mehler kernel of the family is derived, by a self-contained argument using only the Laguerre generating function and Weber's integral, rather than postulated. Three structural consequences follow. First, a normal form theorem: the parameter μ is removable by an explicit unitary conjugation, so that the whole family is unitarily equivalent to the one-parameter family indexed by the order ν alone; in particular the four-parameter transformations of the Hankel–Clifford literature carry no more information than the two-parameter ones. Second, the associated Zemanian-type space is characterized intrinsically as the intersection of the domains of all powers of Λ, equivalently as the space of rapidly decreasing Laguerre coefficients; on this description the fractional transform acts diagonally by unimodular scalars, so its invariance properties become immediate, and the θ-dependence of the space disappears. Third, we develop a symbolic calculus: for symbols independent of the space variable we obtain an exact *-algebra isomorphism onto the multiplier algebra, together with composition, adjoints, parametrices, ellipticity, hypoellipticity and mapping properties on a Hankel–Clifford Sobolev scale, and we compare the symbol classes with the Hörmander classes. For general symbols p(x,y) we give a short structural proof of continuity on the test space and of L²-boundedness on a projective tensor class, together with kernel estimates and an integral representation. Explicit reductions to the fractional cosine, sine and Fourier transforms are given, together with a worked resolvent computation. All proofs are given in full. An appendix records the corrections that the present normalization makes to some formulas in circulation, and the high-precision numerical verification of every operational identity used.**

***Index Terms*—Bessel-type operator, ellipticity and parametrix, fractional powers, Hankel–Clifford transformation, Laguerre functions, pseudo-differential operators, symbolic calculus, Zemanian spaces.**

I. INTRODUCTION

A. Background

The Hankel–Clifford transformation, introduced in the distributional setting by Betancor [1] and Méndez Pérez–Socas Robayna [10], and extended by Malgonde–Bandewar [6] and Pansare–Waphare [8], is the integral transformation with kernel $J_\nu(2\sqrt{xy})$ on the half line, weighted by a power of the variable. Its fractionalization follows the pattern established by Namias [7] and Kerr [4,5] for the Hankel transformation, and was carried out for Hankel–Clifford transformations by Prasad–Kumar [11,12] and Prasad–Mahato [13]; Torre [17] surveys the several possible fractionalizations. Pseudo-differential operators built on Bessel-type transformations go back to Pathak–Pandey [9], and have been studied in the Hankel–Clifford setting in [8,11,18].

The literature just cited defines the fractional transformation by writing down a Mehler-type kernel and then verifying operational formulas by direct computation. The present paper takes the opposite route. We first identify the self-adjoint operator whose unitary group the fractional transformation is, and then read off the kernel, the group law, the invariance of the test spaces and the whole symbolic calculus from the spectral decomposition. This is what makes the phrase "fractional power" literally correct here: h^θ is $e^{-i\theta\Lambda}$ for an explicit $\Lambda = \Lambda^*$, and $h^\theta|_{\theta=\pi/2}$ is the Hankel–Clifford transformation itself, which is a self-adjoint involution; so h^θ is the functional-calculus power $\left(h^{\pi/2}\right)^{2\theta/\pi}$ of that involution.

B. Notation

Throughout $I = (0,\infty)$, $D = d/dx$, $\mathbb{N}_0 = \{0,1,2,\dots\}$. Two real parameters are fixed: the *order* $\nu > -1$ and the *weight* $\mu \in \mathbb{R}$. In the notation of [6,8] these are $\nu = a - b$ and $\mu = \alpha - \beta$, and we write the generalized Bessel–Clifford function as

$$\mathcal{C}_{\mu,\nu}(u) \;=\; u^{-\mu/2} J_\nu(2\sqrt{u}), \qquad u > 0,$$

so that $\mathcal{C}_{\nu,\nu}$ is the classical Bessel–Clifford function of [1,10]. We record the two operational formulas, valid for all $r \in \mathbb{N}_0$ and immediate from the series for J_ν [10, §3]:

$$D_u^r\, \mathcal{C}_{\mu,\nu}(u) = (-1)^r \mathcal{C}_{\mu+r,\nu+r}(u),$$

$$D_u^r\big[u^{\mu+r}\mathcal{C}_{\mu+r,\nu+r}(u)\big] = u^\mu \mathcal{C}_{\mu,\nu}(u).$$

The fractional parameter is θ. Integral formulas are stated for $0 < \theta < \pi$; the operator family itself is defined for all $\theta \in \mathbb{R}$ and is π-periodic (Theorem 5.2). Constants denoted C may change from line to line; their dependence is displayed by subscripts wherever it matters.

C. Summary of results and relation to earlier work

We state at the outset what is new and what is not, since the point is easy to mistake.

What is not new. The transformation $h_{\mu,\nu}^\theta$ of Definition 5.1 coincides, after the identification $\mu = \alpha - \beta$, $\nu = a - b$, with the fractional Hankel–Clifford transformation of Prasad–Kumar [11,12]; Theorem 2.2 below shows that *any* transformation of this shape depending on four parameters (α, β, a, b) depends on them only through the two differences $\alpha - \beta$ and $a - b$, and that the first of these is itself removable by a unitary conjugation. A four-parameter formulation therefore does not generalize the two-parameter one, and we make no such claim. Likewise the operational formulas of Proposition 7.3 are, in the present normalization, those of [12].

What is new.

1. The spectral realization: the explicit generator $\Lambda_{\mu,\nu}$, its self-adjointness with a complete domain analysis, its simple discrete spectrum $\{2n\}$ with Laguerre eigenfunctions, and the identity $h^\theta = e^{-i\theta\Lambda}$ (§4–§5).
2. A self-contained derivation of the Mehler kernel (Theorem 3.5), from which the group law and the sharp period π follow as theorems rather than assertions.
3. The normal form theorem (Theorem 2.2).
4. The intrinsic characterization of the Zemanian-type space, and the fact that it does not depend on θ (Theorem 6.4, Proposition 6.2); this replaces several pages of iterated differential manipulation by a one-line argument.
5. The symbolic calculus of §8: composition, adjoints, ellipticity, parametrices, hypoellipticity, isomorphism theorems on a Sobolev scale, comparison with Hörmander classes, and L^2-boundedness.
6. The reductions of §9, including the reduction to the fractional Fourier transform.
7. The corrections of Appendix A.

Remarks 8.13 and 8.14 state precisely which results we do *not* claim, namely a composition formula with asymptotic expansion for symbols depending on both variables, and a Fredholm theory; both require two-parameter (SG, or Shubin) symbol classes that we do not develop here.

II. THE HANKEL–CLIFFORD TRANSFORMATION AND A NORMAL FORM

Definition 2.1. For f measurable on I with $\int_0^\infty (1+x)^{-1/4}\,|f(x)|\,dx < \infty$ put

$$(T_\nu f)(y) = \int_0^\infty J_\nu\left(2\sqrt{xy}\right) f(x)\,dx, \qquad y > 0.$$

For $\mu \in \mathbb{R}$ put

$$\big(h_{\mu,\nu}\varphi\big)(y) = y^\mu \int_0^\infty \mathcal{C}_{\mu,\nu}\,(xy)\,\varphi(x)\,dx.$$

For $\mu = \nu$ the second definition is the first generalized Hankel–Clifford transformation of [6,8]. Let

$$\mathcal{H}_\mu := L^2(I, x^{-\mu}\,dx), \qquad M_\mu : \mathcal{H}_\mu \to L^2(I,\,dx), \quad \big(M_\mu f\big)(x) = x^{-\mu/2} f(x).$$

M_μ is unitary, since $\int \left|x^{-\mu/2} f\right|^2 dx = \int |f|^2 x^{-\mu} dx$.

Theorem 2.2 (Normal form). For all $\mu \in \mathbb{R}$, $\nu > -1$ and all φ for which either side is defined,

$$h_{\mu,\nu} \;=\; M_\mu^{-1}\, T_\nu\, M_\mu.$$

Consequently, once T_ν is known to be a unitary involution of $L^2(I, dx)$ (Corollary 3.6), $h_{\mu,\nu}$ is a unitary involution of $\mathcal{H}_\mu$, and every spectral, mapping or algebraic property of $h_{\mu,\nu}$ on $\mathcal{H}_\mu$ is the corresponding property of T_ν on $L^2(I,\,dx)$.

Proof. By (1),

$$y^\mu \mathcal{C}_{\mu,\nu}(xy) = y^\mu (xy)^{-\mu/2} J_\nu\big(2\sqrt{xy}\big) = y^{\mu/2}\, x^{-\mu/2} J_\nu\big(2\sqrt{xy}\big),$$

so that

$$\big(h_{\mu,\nu}\varphi\big)(y) = y^{\mu/2}\int_0^\infty J_\nu\left(2\sqrt{xy}\right) x^{-\mu/2}\varphi(x)\,dx = y^{\mu/2}\big(T_\nu\big[M_\mu\varphi\big]\big)(y) = \big(M_\mu^{-1} T_\nu M_\mu \varphi\big)(y).$$

The final assertion is the statement that conjugation by a unitary preserves all such properties. ▫

Remark 2.3. The *second* Hankel–Clifford transformation of [6,8,11], $\big(h_{\mu,\nu}^{(2)}\psi\big)(y) = \int_0^\infty x^\mu\, \mathcal{C}_{\mu,\nu}(xy)\psi(x)\,dx$, satisfies

$$h_{\mu,\nu}^{(2)} = N^{-1}\, h_{\mu,\nu}\, N, \qquad (N\psi)(x) = x^\mu \psi(x),$$

because $\int x^\mu \mathcal{C}_{\mu,\nu}(xy)\psi\,dx = y^{-\mu}\cdot y^\mu \int \mathcal{C}_{\mu,\nu}(xy)(x^\mu\psi)\,dx$. Hence $h^{(2)}$ requires no separate treatment: *every* statement proved below for $h_{\mu,\nu}$ transfers to $h_{\mu,\nu}^{(2)}$ verbatim by this conjugation, with $\mathcal{H}_\mu$ replaced by $L^2(I, x^\mu\,dx)$. This removes at a stroke the customary duplication of the entire theory for the "second variant".

Remark 2.4. Suppose a transformation is written with four parameters, its kernel built from $\mathcal{C}_{\alpha,\beta,a,b}(u) = u^{(-\alpha+\beta)/2} J_{a-b}(2\sqrt{u})$ as in [6,8]. Then $\mathcal{C}_{\alpha,\beta,a,b} = \mathcal{C}_{\mu,\nu}$ with $\mu = \alpha - \beta$, $\nu = a - b$, so the transformation depends on (α, β, a, b) only through these two differences; and by Theorem 2.2 the dependence on μ is a unitary conjugation. Under the substitution $\alpha = \frac{1}{4} + \frac{\mu}{2}$, $\beta = \frac{1}{4} - \frac{\mu}{2}$, $a = \frac{1}{4} + \frac{\nu}{2}$, $b = \frac{1}{4} - \frac{\nu}{2}$ one recovers exactly the family of [11]. No generalization is obtained by adding the two redundant parameters.

III. THE LAGUERRE SYSTEM AND THE FRACTIONAL KERNEL

Everything analytic in this paper rests on the computation carried out in this section. For $\nu > -1$ and $n \in \mathbb{N}_0$ set

$$\mathbf{e}_n(x) = \mathbf{e}_n^\nu(x) = x^{\nu/2} e^{-x} L_n^\nu(2x), \qquad x > 0,$$

where L_n^ν is the Laguerre polynomial of degree n and index ν, and put $\mathbf{e}_n^{\mu,\nu}(x) = x^{\mu/2}\mathbf{e}_n^\nu(x) = x^{(\mu+\nu)/2} e^{-x} L_n^\nu(2x)$.

A. Orthogonality and completeness

Lemma 3.1. $\{\mathbf{e}_n^\nu\}_{n\geq 0}$ is a complete orthogonal system in $L^2(I, dx)$, and

$$\| \mathbf{e}_n^\nu \|_{L^2(I,dx)}^2 = 2^{-\nu-1} \frac{\Gamma(n+\nu+1)}{n!}.$$

Consequently $\{\mathbf{e}_n^{\mu,\nu}\}$ is a complete orthogonal system in $\mathcal{H}_\mu$ with the same norms, and we write $\tilde{\mathbf{e}}_n^{\mu,\nu} = \mathbf{e}_n^{\mu,\nu} / \| \mathbf{e}_n^\nu \|$ for the normalized system.

Proof. Substituting $t = 2x$,

$$\int_0^\infty \mathbf{e}_n^\nu \, \mathbf{e}_m^\nu \, dx$$
$$= \int_0^\infty x^\nu \, e^{-2x} L_n^\nu(2x) L_m^\nu(2x) \, dx$$
$$= 2^{-\nu-1} \int_0^\infty t^\nu \, e^{-t} L_n^\nu(t) L_m^\nu(t) \, dt,$$

and the last integral equals $\delta_{nm}\Gamma(n+\nu+1)/n!$ by the orthogonality relation for Laguerre polynomials [16, Ch. V]. For completeness, note that $f \mapsto 2^{1/2} f(t/2)$ is a unitary map $L^2(I, dx) \to L^2(I, dt)$ carrying $\mathbf{e}_n^\nu$ to a constant multiple of the Laguerre function $t^{\nu/2} e^{-t/2} L_n^\nu(t)$; the latter system is complete in $L^2(I, dt)$ for $\nu > -1$ [16, Ch. V]. The statement for $\mathcal{H}_\mu$ follows since $M_\mu \mathbf{e}_n^{\mu,\nu} = \mathbf{e}_n^\nu$ and M_μ is unitary. ▫

B. Two classical identities

Lemma 3.2 (Generating function). For $|w| < 1$ and $x > 0$, with $\beta = \beta(w) := \frac{1+w}{1-w}$ (so $\Re\beta > 0$),

$$\sum_{n=0}^\infty \mathbf{e}_n^\nu(x) \, w^n = (1-w)^{-\nu-1} \, x^{\nu/2} \, e^{-\beta x},$$

the series converging absolutely and uniformly on $\{|w| \leq r\} \times [\varepsilon, R]$ for every $r < 1$ and $0 < \varepsilon < R < \infty$.

Proof. The classical generating function [16, Ch. V] is $\sum_{n\geq 0} L_n^\nu(u) w^n = (1-w)^{-\nu-1} \exp(-uw/(1-w))$ for $|w| < 1$. Multiplying by $x^{\nu/2} e^{-x}$ and putting $u = 2x$ gives

$$\sum_n \mathbf{e}_n^\nu(x) w^n = x^{\nu/2}(1-w)^{-\nu-1} \exp\left(-x - \frac{2xw}{1-w}\right)$$
$$= x^{\nu/2}(1-w)^{-\nu-1} \exp\left(-x \, \frac{1+w}{1-w}\right),$$

since $1 + \frac{2w}{1-w} = \frac{1+w}{1-w}$. That $\Re\beta > 0$ for $|w| < 1$ follows from $\Re \frac{1+w}{1-w} = \frac{1-|w|^2}{|1-w|^2} > 0$. The convergence statement is the standard one for the Laguerre generating series. ▫

Lemma 3.3 (Weber's integral). Let $\nu > -1$, $y > 0$ and $\rho \in \mathbb{C}$ with $\Re\rho > 0$. Then

$$\int_0^\infty x^{\nu/2} \, e^{-\rho x} J_\nu(2\sqrt{xy}) \, dx = \rho^{-\nu-1} \, y^{\nu/2} \, e^{-y/\rho},$$

the integral converging absolutely.

Proof. Weber's classical formula [19, Ch. XIII] states $\int_0^\infty e^{-pt^2} J_\nu(at) t^{\nu+1} \, dt = \frac{a^\nu}{(2p)^{\nu+1}} e^{-a^2/(4p)}$ for $\Re p > 0$, $a > 0$, $\nu > -1$. Substituting $x = t^2$, $dx = 2t \, dt$, with $a = 2\sqrt{y}$ and $p = \rho$,

$$\int_0^\infty x^{\nu/2} \, e^{-\rho x} J_\nu(2\sqrt{xy}) \, dx$$
$$= 2 \int_0^\infty t^{\nu+1} \, e^{-\rho t^2} J_\nu(2\sqrt{y} \, t) \, dt$$
$$= 2 \, \frac{(2\sqrt{y})^\nu}{(2\rho)^{\nu+1}} e^{-4y/(4\rho)}$$
$$= \rho^{-\nu-1} y^{\nu/2} e^{-y/\rho}.$$

Absolute convergence near 0 uses $|J_\nu(2\sqrt{xy})| \leq C(xy)^{\nu/2}$, giving an integrand $O(x^\nu)$ with $\nu > -1$; near ∞ it uses $|J_\nu(u)| \leq Cu^{-1/2}$ and $\Re\rho > 0$. ▫

C. The fractional kernel acts diagonally

Definition 3.4. For $0 < \theta < \pi$ set $\gamma_{\nu,\theta} = e^{i(\nu+1)(\theta-\pi/2)}$ and

$$K_\nu^\theta(x, y) = \gamma_{\nu,\theta} \csc\theta \; e^{i(x+y)\cot\theta} J_\nu(2\sqrt{xy} \csc\theta),$$

and let $\mathcal{T}_\nu^\theta$ be the integral operator with this kernel, defined on the linear span $\mathcal{E}_\nu$ of $\{\mathbf{e}_n^\nu\}$ (on which the integral converges absolutely, by the estimates in Lemma 3.3).

Theorem 3.5 (Diagonalization). Let $\nu > -1$ and $0 < \theta < \pi$. Then for every $n \in \mathbb{N}_0$,

$$\mathcal{T}_\nu^\theta\, \mathbf{e}_n^\nu = e^{-2in\theta}\, \mathbf{e}_n^\nu.$$

Proof. Fix $y > 0$. Write $c = \cot\theta$, $\lambda = \csc^2\theta$ and, for $|w| < 1$,

$$F(w) := \int_0^\infty K_\nu^\theta\,(x,y) \left(\sum_{n\geq 0} \mathbf{e}_n^\nu\,(x) w^n \right) dx.$$

Step 1: interchange of sum and integral for small $|w|$. For $\nu > -1$ every binomial coefficient $\binom{n+\nu}{n-i}$ is positive, so from $L_n^\nu(u) = \sum_{i=0}^n \binom{n+\nu}{n-i} \frac{(-u)^i}{i!}$ we get $|L_n^\nu(u)| \leq L_n^\nu(-u)$ for $u \geq 0$; hence, by the generating function evaluated at the positive argument $-2x$,

$$\sum_{n\geq 0} |\,L_n^\nu(2x)|\,|w|^n \leq \sum_{n\geq 0} L_n^\nu\,(-2x)|w|^n = (1-|w|)^{-\nu-1} \exp\left(\frac{2x|w|}{1-|w|}\right).$$

Multiplying by $x^{\nu/2}e^{-x}$ gives

$$\sum_{n\geq 0} |\,\mathbf{e}_n^\nu(x)|\,|w|^n \leq (1-|w|)^{-\nu-1}\, x^{\nu/2}\, e^{-\kappa x}, \qquad \kappa := 1 - \frac{2|w|}{1-|w|},$$

and $\kappa > 0$ as soon as $|w| < 1/3$. Since $\left|K_\nu^\theta(x,y)\right| \leq C_{\theta,y}\min\{x^{\nu/2}, x^{-1/4}\}$ by the bounds on J_ν used in Lemma 3.3, the function $(x,n) \mapsto K_\nu^\theta(x,y)\mathbf{e}_n^\nu(x)w^n$ is absolutely summable-integrable for $|w| < 1/3$, and Fubini–Tonelli justifies the interchange.

Step 2: evaluation. By Lemma 3.2 and Step 1, for $|w| < 1/3$,

$$F(w) = \gamma_{\nu,\theta}\csc\theta\, e^{icy}\,(1-w)^{-\nu-1} \int_0^\infty x^{\nu/2}\, e^{-(\beta-ic)x} J_\nu\!\left(2\sqrt{x\cdot\lambda y}\right) dx,$$

where we used $\sqrt{xy}\csc\theta = \sqrt{x\cdot\lambda y}$. Put $\rho := \beta - ic$; then $\Re\rho = \Re\beta > 0$, so Lemma 3.3 gives

$$F(w) = \gamma_{\nu,\theta}\,\csc\theta\, e^{icy}\,(1-w)^{-\nu-1}\rho^{-\nu-1}(\lambda y)^{\nu/2} e^{-\lambda y/\rho}.$$

Step 3: simplification of ρ. Using $1 - ic = \frac{\sin\theta - i\cos\theta}{\sin\theta} = -ie^{i\theta}\csc\theta$ and $1 + ic = ie^{-i\theta}\csc\theta$,

$$\begin{aligned}(1-w)\rho &= (1+w) - ic(1-w) = (1-ic) + w(1+ic) \\ &= i\csc\theta\left(we^{-i\theta} - e^{i\theta}\right) \\ &= -i\csc\theta\, e^{i\theta}\left(1 - we^{-2i\theta}\right).\end{aligned}$$

Write $w' := we^{-2i\theta}$ (so $|w'| = |w|$) and $\beta' := \frac{1+w'}{1-w'}$. Then

$$\begin{aligned}(1-w)\rho &= -i\csc\theta\, e^{i\theta}(1-w') \\ &= e^{i(\theta-\pi/2)}\csc\theta\,(1-w'),\end{aligned}$$

since $-ie^{i\theta} = e^{i(\theta-\pi/2)}$.

Step 4: the exponent. We claim $\lambda/\rho - ic = \beta'$. Indeed, from (10), $\rho = \frac{i\csc\theta(we^{-i\theta} - e^{i\theta})}{1-w}$, so

$$\frac{\lambda}{\rho} - ic = \frac{\csc^2\theta\,(1-w)}{i\csc\theta\,(we^{-i\theta} - e^{i\theta})} - ic = \frac{i\csc\theta\,(1-w)}{e^{i\theta} - we^{-i\theta}} - ic = i\,\frac{\csc\theta(1-w) - \cot\theta\left(e^{i\theta} - we^{-i\theta}\right)}{e^{i\theta} - we^{-i\theta}}.$$

Now $\csc\theta - \cot\theta\, e^{i\theta} = \frac{1-\cos^2\theta}{\sin\theta} - i\cos\theta = \sin\theta - i\cos\theta = -ie^{i\theta}$, and $-\csc\theta + \cot\theta\, e^{-i\theta} = -\sin\theta - i\cos\theta = -ie^{-i\theta}$, so the numerator equals $-i\left(e^{i\theta} + we^{-i\theta}\right)$ and

$$\frac{\lambda}{\rho} - ic = \frac{e^{i\theta} + we^{-i\theta}}{e^{i\theta} - we^{-i\theta}} = \frac{1 + we^{-2i\theta}}{1 - we^{-2i\theta}} = \beta'.$$

Hence $e^{icy}e^{-\lambda y/\rho} = e^{-(\lambda/\rho - ic)y} = e^{-\beta' y}$.

Step 5: the prefactor. By (10),

$$(1-w)^{-\nu-1}\rho^{-\nu-1} = [(1-w)\rho]^{-\nu-1} = e^{-i(\nu+1)(\theta-\pi/2)}(\csc\theta)^{-\nu-1}(1-w')^{-\nu-1} = \overline{\gamma_{\nu,\theta}}\,(\csc\theta)^{-\nu-1}(1-w')^{-\nu-1},$$

where the branch is the principal one, legitimate because $(1-w)\rho$ stays in the right half plane rotated by $e^{i(\theta-\pi/2)}$ and $1 - w'$ lies in the right half plane. Since $(\lambda)^{\nu/2} = (\csc\theta)^\nu$, the prefactor in (9) equals

$$\gamma_{\nu,\theta}\,\csc\theta \cdot \overline{\gamma_{\nu,\theta}}\,(\csc\theta)^{-\nu-1} \cdot (\csc\theta)^\nu = 1,$$

because $\left|\gamma_{\nu,\theta}\right| = 1$. Combining Steps 4 and 5 with (9),

$$F(w) = (1-w')^{-\nu-1} y^{\nu/2} e^{-\beta' y} = \sum_{n\geq 0} \mathbf{e}_n^\nu\,(y)\,(w')^n = \sum_{n\geq 0} e^{-2in\theta}\,\mathbf{e}_n^\nu(y)\, w^n,$$

the middle equality being Lemma 3.2 applied at w'.

Step 6: conclusion. By Step 1, $F(w) = \sum_n \left(\mathcal{T}_\nu^\theta \mathbf{e}_n^\nu\right)(y) w^n$ for $|w| < 1/3$. Two power series in w agreeing on a disc have equal coefficients, which is (8). ▫

Corollary 3.6. $T_\nu \mathbf{e}_n^\nu = (-1)^n \mathbf{e}_n^\nu$ for all n, and T_ν extends from $\mathcal{E}_\nu$ to a unitary self-adjoint involution of $L^2(I, dx)$. Consequently $h_{\mu,\nu}$ is a unitary self-adjoint involution of $\mathcal{H}_\mu$, with $h_{\mu,\nu}\mathbf{e}_n^{\mu,\nu} = (-1)^n \mathbf{e}_n^{\mu,\nu}$.

Proof. At $\theta = \pi/2$ one has $\gamma_{\nu,\pi/2} = 1$, $\csc\theta = 1$, $\cot\theta = 0$, so $K_\nu^{\pi/2}(x,y) = J_\nu\left(2\sqrt{xy}\right)$ and $\mathcal{T}_\nu^{\pi/2} = T_\nu$; and $e^{-2in\pi/2} = (-1)^n$. Thus T_ν maps the complete orthogonal system $\{\mathbf{e}_n^\nu\}$ (Lemma 3.1) to itself with unimodular real eigenvalues ± 1; the linear extension is therefore isometric on $\mathcal{E}_\nu$, extends to a unitary operator, and satisfies $T_\nu^2 = \mathrm{Id}$ and $T_\nu^* = T_\nu^{-1} = T_\nu$. The statement for $h_{\mu,\nu}$ is Theorem 2.2. ▫

Corollary 3.7. For each $0 < \theta < \pi$, $\mathcal{T}_\nu^\theta$ extends from $\mathcal{E}_\nu$ to a unitary operator on $L^2(I, dx)$, and $\mathcal{T}_\nu^{\theta_1}\mathcal{T}_\nu^{\theta_2} = \mathcal{T}_\nu^{\theta_1+\theta_2}$ whenever all three angles lie in $(0, \pi)$.

Proof. By Theorem 3.5, $\mathcal{T}_\nu^\theta$ acts on the complete orthonormal system $\{\tilde{\mathbf{e}}_n^\nu\}$ by multiplication by the unimodular scalars $e^{-2in\theta}$; hence it is isometric on $\mathcal{E}_\nu$, and its continuous extension is unitary with adjoint acting by $e^{+2in\theta}$. Composition multiplies the eigenvalues, and $e^{-2in\theta_1}e^{-2in\theta_2} = e^{-2in(\theta_1+\theta_2)}$; two bounded operators agreeing on a complete orthonormal system coincide. ▫

IV. THE GENERATOR

Proposition 4.1 (The generator). Define, on $C_c^\infty(I)$,

$$\Lambda_{\mu,\nu} = -xD^2 + (\mu - 1)D + x + \frac{\nu^2 - \mu^2}{4x} - (\nu + 1).$$

Then

$$\Lambda_{\mu,\nu} = M_\mu^{-1}\Lambda_{0,\nu}M_\mu, \qquad \Lambda_{0,\nu} = -D(xD) + x + \frac{\nu^2}{4x} - (\nu + 1),$$

and

$$\Lambda_{\mu,\nu}\, \mathbf{e}_n^{\mu,\nu} = 2n\, \mathbf{e}_n^{\mu,\nu}, \qquad n \in \mathbb{N}_0.$$

Proof. Step 1: the conjugation (12). Let $f = x^{\mu/2}g$, i.e. $g = M_\mu f$. Then $g' = x^{-\mu/2}\left(f' - \frac{\mu}{2x}f\right)$, hence $xg' = x^{-\mu/2}\left(xf' - \frac{\mu}{2}f\right)$ and

$$(xg')' = x^{-\mu/2}[(xf' - \mu/2\, f)' - \frac{\mu}{2x}(xf' - \mu/2\, f)] = x^{-\mu/2}\left[xf'' + (1-\mu)f' + \frac{\mu^2}{4x}f\right].$$

Therefore

$$M_\mu^{-1}\Lambda_{0,\nu}M_\mu f = x^{\mu/2}\{ -x^{-\mu/2}[xf'' + (1-\mu)f' + \mu^2/4x\, f] +\big(x + \nu^2/4x - (\nu+1)\big)x^{-\mu/2}f\},$$

which is (11). It therefore suffices to prove $\Lambda_{0,\nu}\mathbf{e}_n^\nu = 2n\mathbf{e}_n^\nu$.

Step 2: Laguerre's equation in the variable x. Laguerre's differential equation is $u\ddot{L} + (\nu + 1 - u)\dot{L} + nL = 0$ for $L = L_n^\nu(u)$. Put $u = 2x$ and $L(x) := L_n^\nu(2x)$; then $\dot{L} = 1/2\, L'$, $\ddot{L} = 1/4\, L''$, so

$$2x \cdot 1/4\, L'' + (\nu + 1 - 2x)\, 1/2\, L' + nL = 0, \qquad \text{i.e.} \qquad xL'' + (\nu + 1 - 2x)L' + 2nL = 0.$$

Equivalently $BL = 2nL$, where $B := -xD^2 - (\nu + 1)D + 2xD$.

Step 3: the ground-state conjugation. Let $W(x) = x^{\nu/2}e^{-x}$, so $\mathbf{e}_n^\nu = WL$. We compute WBW^{-1}. Write $w := W'/W = \frac{\nu}{2x} - 1$, so $w' = -\frac{\nu}{2x^2}$; note W^{-1} acts on f by $g = W^{-1}f$, and

$$g' = W^{-1}(f' - wf), \qquad g'' = W^{-1}(f'' - 2wf' + (w^2 - w')f).$$

Hence

$$W\,B\,W^{-1}f = -x(f'' - 2wf' + (w^2 - w')f) - (\nu + 1)(f' - wf) + 2x(f' - wf).$$

The coefficient of f' is $2xw - (\nu + 1) + 2x = (\nu - 2x) - (\nu + 1) + 2x = -1$. The coefficient of f is

$$-x(w^2 - w') + (\nu + 1)w - 2xw = -x((\nu/2x - 1)^2 + \nu/2x^2) +(\nu + 1)(\nu/2x - 1) -(\nu - 2x).$$

Expanding, $-x\left(\frac{\nu^2}{4x^2} - \frac{\nu}{x} + 1 + \frac{\nu}{2x^2}\right) = -\frac{\nu^2}{4x} + \nu - x - \frac{\nu}{2x}$, and $(\nu + 1)\left(\frac{\nu}{2x} - 1\right) - \nu + 2x = \frac{\nu(\nu+1)}{2x} - (\nu + 1) - \nu + 2x$. Summing,

$$(-\frac{\nu^2}{4x} - \frac{\nu}{2x} + \frac{\nu^2 + \nu}{2x}) + (\nu - \nu - 1 - \nu) + (-x + 2x) = \frac{\nu^2}{4x} - (\nu + 1) + x.$$

Therefore $WBW^{-1} = -xD^2 - D + x + \frac{\nu^2}{4x} - (\nu + 1) = \Lambda_{0,\nu}$, and by Step 2

$$\Lambda_{0,\nu}\mathbf{e}_n^\nu = WBW^{-1}(WL) = W(BL) = 2n\,WL = 2n\,\mathbf{e}_n^\nu.$$

▫

Lemma 4.2 (Reduced form of the generator). Let $G = G_{\mu,\nu}$ denote multiplication by $x^{-(\mu+\nu)/2}$. Then

$$G\,\Lambda_{\mu,\nu}\,G^{-1} = -xD^2 - (\nu + 1)D + x - (\nu + 1).$$

In particular the reduced operator has polynomial coefficients: the singular term $\frac{\nu^2-\mu^2}{4x}$ of (11) cancels identically against the first-order term produced by the conjugation.

Proof. By (12), $G\Lambda_{\mu,\nu}G^{-1} = x^{-\nu/2}\Lambda_{0,\nu}x^{\nu/2}$, since the two factors $x^{\mp\mu/2}$ cancel. By Step 3 of the proof of Proposition 4.1, $\Lambda_{0,\nu} = WBW^{-1}$ with $W = x^{\nu/2}e^{-x}$ and $B = -xD^2 - (\nu + 1)D + 2xD$; hence $x^{-\nu/2}\Lambda_{0,\nu}x^{\nu/2} = e^{-x}Be^x$. Since $e^{-x}De^x = D + 1$,

$$e^{-x}Be^x = -x(D + 1)^2 - (\nu + 1)(D + 1) + 2x(D + 1) = -xD^2 - (\nu + 1)D + x - (\nu + 1),$$

the terms $-2xD$ and $+2xD$ cancelling and $-x + 2x = x$. ▫

Lemma 4.3 (Ground-state representation). For all $f \in C_c^\infty(I)$,

$$\langle \Lambda_{0,\nu} f, f\rangle_{L^2(I,dx)} = \int_0^\infty x\ \mathbf{e}_0^\nu(x)^2 \,|\left(\frac{f}{\mathbf{e}_0^\nu}\right)'(x)|^2\,dx \geq 0.$$

In particular $\Lambda_{0,\nu}$ is a nonnegative symmetric operator, and so is $\Lambda_{\mu,\nu}$ in $\mathcal{H}_\mu$.

Proof. Write $\Lambda_{0,\nu} = -D(xD) + V$ with $V(x) = x + \frac{\nu^2}{4x} - (\nu+1)$, and put $e := \mathbf{e}_0^\nu = x^{\nu/2}e^{-x} > 0$, which satisfies $\Lambda_{0,\nu}e = 0$ by Proposition 4.1, i.e. $(xe')' = Ve$. Symmetry is clear: for $f, g \in C_c^\infty(I)$,

$$\langle \Lambda_{0,\nu} f, g\rangle - \langle f, \Lambda_{0,\nu} g\rangle = [-x(f'\bar{g} - f\bar{g}')]_0^\infty = 0.$$

For (14), integrate by parts once: $\langle \Lambda_{0,\nu} f, f\rangle = \int (x|f'|^2 + V|f|^2)dx$. Substitute $f = eu$ with $u = f/e \in C_c^\infty(I)$. Since e is real,

$$x|f'|^2 = x(e'^2|u|^2 + ee'(|u|^2)' + e^2|u'|^2),$$

and integrating the middle term by parts,

$$\int xee'(|u|^2)'\,dx = -\int (xee')'|u|^2\,dx = -\int (xe')'e|u|^2\,dx - \int xe'^2|u|^2\,dx = -\int Ve^2|u|^2\,dx - \int xe'^2|u|^2\,dx,$$

all boundary terms vanishing since u has compact support in I. Hence $\int x|f'|^2 = \int xe^2|u'|^2 - \int Ve^2|u|^2$, and adding $\int V|f|^2 = \int Ve^2|u|^2$ gives (14). The statement for $\Lambda_{\mu,\nu}$ follows by unitary conjugation, (12). ▫

Theorem 4.4 (Spectral theorem for the generator). Let $\nu > -1$, $\mu \in \mathbb{R}$, and regard $\Lambda_{\mu,\nu}$ as an operator in $\mathcal{H}_\mu$ with initial domain $C_c^\infty(I)$.

1. Under the unitary VM_μ, where $(Vf)(t) = \sqrt{2t}f(t^2)$ is unitary from $L^2(I,dx)$ onto $L^2(I,dt)$,
$$VM_\mu\,\Lambda_{\mu,\nu}\left(VM_\mu\right)^{-1} = \frac{1}{4}[-\frac{d^2}{dt^2} + 4t^2 + \frac{\nu^2 - 1/4}{t^2}] - (\nu+1),$$
a shifted and rescaled radial harmonic oscillator.
2. $\Lambda_{\mu,\nu}$ is symmetric and nonnegative. It is essentially self-adjoint on $C_c^\infty(I)$ if and only if $\nu \geq 1$; for $-1 < \nu < 1$ its deficiency indices are (1,1) and we take throughout its Friedrichs extension. In either case the resulting self-adjoint operator is again denoted $\Lambda_{\mu,\nu}$.
3. $\{\tilde{\mathbf{e}}_n^{\mu,\nu}\}_{n\geq 0}$ is an orthonormal basis of $\mathcal{H}_\mu$ consisting of eigenvectors of $\Lambda_{\mu,\nu}$, $\mathrm{spec}(\Lambda_{\mu,\nu}) = \{2n : n \in \mathbb{N}_0\}$, every eigenvalue is simple, and $\Lambda_{\mu,\nu}$ has compact resolvent.

Proof. (a). V is unitary because $\int_0^\infty |f(x)|^2\,dx = \int_0^\infty |f(t^2)|^2\,2t\,dt = \int_0^\infty |(Vf)(t)|^2\,dt$. By (12) it suffices to conjugate $\Lambda_{0,\nu}$. Let $F(t) = f(t^2)$ and $g = Vf = \sqrt{2t}F$. From $x = t^2$ we get $\frac{d}{dx} = \frac{1}{2t}\frac{d}{dt}$, hence $xf' = \frac{t}{2}\dot{F}$ and

$$(xf')'|_{x=t^2} = \frac{1}{2t}\frac{d}{dt}\left(\frac{t}{2}\dot{F}\right) = \frac{\dot{F}}{4t} + \frac{\ddot{F}}{4},$$

so that $\left(\Lambda_{0,\nu}f\right)(t^2) = -1/4\left(\ddot{F} + \dot{F}/t\right) + \left(t^2 + \nu^2/4t^2 - (\nu+1)\right)F$. Now $F = (2t)^{-1/2}g$, and with $c = 2^{-1/2}$,

$$\dot{F} = c\left(t^{-1/2}\dot{g} - 1/2\,t^{-3/2}g\right), \qquad \ddot{F} = c(t^{-1/2}\ddot{g} - t^{-3/2}\dot{g} + 3/4\,t^{-5/2}g),$$

whence $\ddot{F} + \dot{F}/t = c\left(t^{-1/2}\ddot{g} + 1/4\,t^{-5/2}g\right)$. Multiplying $\left(\Lambda_{0,\nu}f\right)(t^2)$ by $\sqrt{2t}$ and using $\sqrt{2t}\,ct^{-1/2} = 1$ gives

$$\left(V\Lambda_{0,\nu}f\right)(t) = -1/4\,\ddot{g} - 1/16\,t^{-2}g + \left(t^2 + \frac{\nu^2}{4t^2} - (\nu+1)\right)g,$$

which is (15), since $1/4\,(\nu^2 - 1/4)t^{-2} = \nu^2/4\,t^{-2} - 1/16\,t^{-2}$ and $1/4 \cdot 4t^2 = t^2$.

(b). Symmetry and nonnegativity are Lemma 4.3. For the self-adjointness dichotomy we use (a). The Schrödinger operator $-\frac{d^2}{dt^2} + q(t)$ with $q(t) = 4t^2 + \frac{\nu^2 - 1/4}{t^2}$ is in the limit point case at $t = \infty$ because $q(t) \to +\infty$ [14, §X.1, Thm. X.8]; at $t = 0$ the classical criterion for the potential κt^{-2} states limit point exactly when $\kappa \geq 3/4$ [14, §X.1, Example 4], i.e. $\nu^2 - 1/4 \geq 3/4$, i.e. $\nu \geq 1$ (recall $\nu > -1$). In the limit circle case at one endpoint and limit point at the other, the deficiency indices are (1,1). Since the operator is nonnegative, the Friedrichs extension exists and is the unique nonnegative self-adjoint extension whose form domain is the closure of $C_c^\infty(I)$ in the form norm.

(c). Each $\mathbf{e}_n^{\mu,\nu}$ lies in $\mathcal{H}_\mu$ (Lemma 3.1) and satisfies $\Lambda_{\mu,\nu}\mathbf{e}_n^{\mu,\nu} = 2n\mathbf{e}_n^{\mu,\nu}$ pointwise (Proposition 4.1); moreover the quadratic form (14) is finite on it and it is approximated in form norm by C_c^∞ truncations (its ground-state quotient $\mathbf{e}_n^\nu/\mathbf{e}_0^\nu = L_n^\nu(2x)$ is a polynomial, and the weight $x\mathbf{e}_0^2$ is integrable against polynomial derivatives), so $\mathbf{e}_n^{\mu,\nu}$ belongs to the form domain and hence to the domain of the Friedrichs extension. Thus $\{\tilde{\mathbf{e}}_n^{\mu,\nu}\}$ is an orthonormal basis of eigenvectors by Lemma 3.1. Given a complete orthonormal system of eigenvectors, the spectrum is the closure of the set of eigenvalues, i.e. $\{2n\}$; and if $\Lambda\psi = 2n\psi$ then expanding ψ in the basis forces $\psi \in \mathbb{C}\tilde{\mathbf{e}}_n^{\mu,\nu}$, so eigenvalues are simple. Finally $(\Lambda+1)^{-1}$ is diagonal with eigenvalues $(2n+1)^{-1} \to 0$, hence compact. ▫

Remark. The heat semigroup $e^{-s\Lambda_{\mu,\nu}}$, $s > 0$, is trace class with $\mathrm{tr}e^{-s\Lambda} = \sum_{n\geq 0} e^{-2ns} = (1 - e^{-2s})^{-1}$.

V. FRACTIONAL POWERS

Definition 5.1. For $\theta \in \mathbb{R}$ define, by the functional calculus of Theorem 4.4,

$$h_{\mu,\nu}^{\theta} := e^{-i\theta\Lambda_{\mu,\nu}}, \qquad \text{equivalently} \qquad h_{\mu,\nu}^{\theta}\tilde{\mathbf{e}}_n^{\mu,\nu} = e^{-2in\theta}\,\tilde{\mathbf{e}}_n^{\mu,\nu}.$$

We write $h_\nu^\theta := h_{0,\nu}^\theta$.

Theorem 5.2 (Group law, normalization, periodicity). For all $\theta, \theta_1, \theta_2 \in \mathbb{R}$:

1. $h_{\mu,\nu}^{\theta}$ is unitary on $\mathcal{H}_\mu$ and $h_{\mu,\nu}^{\theta_1}h_{\mu,\nu}^{\theta_2} = h_{\mu,\nu}^{\theta_1+\theta_2}$;
2. $h_{\mu,\nu}^{0} = \mathrm{Id}$ and $h_{\mu,\nu}^{\pi/2} = h_{\mu,\nu}$;
3. $\theta \mapsto h_{\mu,\nu}^{\theta}$ has *sharp* period π; in particular $\left(h^\theta\right)^{-1} = h^{-\theta} = h^{\pi-\theta}$ and h^θ is also 2π-periodic;
4. $\theta \mapsto h_{\mu,\nu}^{\theta}$ is strongly continuous, and $\Lambda_{\mu,\nu} = i\frac{d}{d\theta}h_{\mu,\nu}^{\theta}|_{\theta=0}$ on $\mathrm{Dom}(\Lambda_{\mu,\nu})$;
5. for $0<\theta<\pi$, $h_{\mu,\nu}^{\theta} = M_\mu^{-1}\mathcal{T}_\nu^\theta M_\mu$, i.e. h^θ is the integral operator of Theorem 5.3 below.

Proof. Let $\Lambda = \Lambda_{\mu,\nu}$ and P_n the orthogonal projection onto $\mathbb{C}\tilde{\mathbf{e}}_n^{\mu,\nu}$; by Theorem 4.4(c), $\Lambda = \sum_n 2nP_n$ and $e^{-i\theta\Lambda} = \sum_n e^{-2in\theta}P_n$, the series converging strongly.

(a) The multipliers $e^{-2in\theta}$ are unimodular, so $\sum_n e^{-2in\theta}P_n$ is unitary with adjoint $\sum_n e^{2in\theta}P_n$; and $e^{-2in\theta_1}e^{-2in\theta_2} = e^{-2in(\theta_1+\theta_2)}$ termwise.

(b) At $\theta = 0$ all multipliers equal 1; at $\theta = \pi/2$ they equal $e^{-in\pi} = (-1)^n$, which by Corollary 3.6 are the eigenvalues of $h_{\mu,\nu}$ on the same basis. Two bounded operators agreeing on an orthonormal basis coincide.

(c) $e^{-2in(\theta+\pi)} = e^{-2in\theta}e^{-2in\pi} = e^{-2in\theta}$, so π is a period. If τ is a period then in particular $e^{-2i\tau} = 1$ (the multiplier at $n=1$), forcing $\tau \in \pi\mathbb{Z}$. Hence π is sharp. The identities for the inverse follow from (a) and $h^\pi = \mathrm{Id}$.

(d) For $\psi \in \mathcal{H}_\mu$, $\| h^\theta\psi - \psi \|^2 = \sum_n \left|e^{-2in\theta} - 1\right|^2 |c_n(\psi)|^2 \to 0$ as $\theta \to 0$ by dominated convergence. The generator statement is Stone's theorem, or directly: $\| \theta^{-1}\left(h^\theta\psi - \psi\right) + i\Lambda\psi \|^2 = \sum_n |\,\theta^{-1}\left(e^{-2in\theta}-1\right) + 2in|^2|c_n(\psi)|^2 \to 0$ for $\psi \in \mathrm{Dom}(\Lambda)$, again by dominated convergence, using $\left|\theta^{-1}\left(e^{-i\lambda\theta}-1\right) + i\lambda\right| \le 2|\lambda|$.

(e) By Corollary 3.7 the operator $M_\mu^{-1}\mathcal{T}_\nu^\theta M_\mu$ is unitary on $\mathcal{H}_\mu$ and, by Theorem 3.5, acts on $\tilde{\mathbf{e}}_n^{\mu,\nu}$ by $e^{-2in\theta}$, the same multipliers as $h_{\mu,\nu}^{\theta}$. ▫

Theorem 5.3 (Mehler kernel). Let $0<\theta<\pi$ and $\gamma_{\nu,\theta} = e^{i(\nu+1)(\theta-\pi/2)}$. Then for φ in the span of $\{\mathbf{e}_n^{\mu,\nu}\}$, and by continuity for all $\varphi \in \mathcal{H}_\mu$ in the mean,

$$\left(h_{\mu,\nu}^{\theta}\varphi\right)(y) = \int_0^\infty K_{\mu,\nu}^{\theta}(x,y)\varphi(x)\,dx, \qquad K_{\mu,\nu}^{\theta}(x,y) = \gamma_{\nu,\theta}\,(\csc\theta)^{\mu+1}\,y^\mu\,\mathcal{C}_{\mu,\nu}(xy\csc^2\theta)\,e^{i(x+y)\cot\theta},$$

and $K_{\mu,\nu}^{\theta}(x,y) = (y/x)^{\mu/2}K_\nu^\theta(x,y)$ with K_ν^θ as in (7). For $\theta \in \pi\mathbb{Z}$, $h^\theta = \mathrm{Id}$.

Proof. By Theorem 5.2(e), $h_{\mu,\nu}^{\theta} = M_\mu^{-1}\mathcal{T}_\nu^\theta M_\mu$, whose kernel is $y^{\mu/2}K_\nu^\theta(x,y)x^{-\mu/2} = (y/x)^{\mu/2}K_\nu^\theta(x,y)$. It remains to rewrite this in the form (17). From (1),

$$\mathcal{C}_{\mu,\nu}(xy\csc^2\theta) = (xy)^{-\mu/2}(\csc\theta)^{-\mu}J_\nu\left(2\sqrt{xy}\csc\theta\right),$$

so $J_\nu\left(2\sqrt{xy}\csc\theta\right) = (xy)^{\mu/2}(\csc\theta)^\mu\mathcal{C}_{\mu,\nu}(xy\csc^2\theta)$, and therefore

$$(y/x)^{\mu/2}K_\nu^\theta(x,y) = \gamma_{\nu,\theta}\csc\theta\,e^{i(x+y)\cot\theta} \times (y/x)^{\mu/2}(xy)^{\mu/2}(\csc\theta)^\mu \times \mathcal{C}_{\mu,\nu}(xy\csc^2\theta),$$

and $(y/x)^{\mu/2}(xy)^{\mu/2} = y^\mu$. The last assertion is Theorem 5.2(b),(c). ▫

Corollary 5.4 (Chirp factorization). Let $0<\theta<\pi$ and $\chi_\theta(x) = e^{ix\cot\theta}$. Then

$$\left(h_\nu^\theta\varphi\right)(y) = \gamma_{\nu,\theta}\csc\theta\;\chi_\theta(y)\;(T_\nu[\chi_\theta\varphi])(y\csc^2\theta).$$

Proof. In (7) write $e^{i(x+y)\cot\theta} = \chi_\theta(y)\chi_\theta(x)$ and $2\sqrt{xy}\csc\theta = 2\sqrt{x\cdot(y\csc^2\theta)}$; then

$$\left(h_\nu^\theta\varphi\right)(y) = \gamma_{\nu,\theta}\csc\theta\,\chi_\theta(y) \times \int_0^\infty J_\nu\left(2\sqrt{x\cdot y\csc^2\theta}\right)\chi_\theta(x)\varphi(x)\,dx,$$

which is the right-hand side of (18). ▫

VI. THE TEST SPACE, ITS INTRINSIC DESCRIPTION, AND DUALITY

Definition 6.1. $\mathcal{S}_{\mu,\nu}(I)$ is the space of $\varphi \in C^\infty(I)$ such that

$$\sigma_{q,k}(\varphi) := \sup_{x\in I}\left|x^qD^k\left(x^{-\frac{\mu+\nu}{2}}\varphi(x)\right)\right| < \infty$$

for all $q,k \in \mathbb{N}_0$, with the topology generated by $\{\sigma_{q,k}\}$.

This is the Zemanian-type space of [21,11], with the chirp factor suppressed. The suppression is legitimate:

Proposition 6.2 (The space does not depend on θ). For $c \in \mathbb{R}$ let $\sigma_{q,k}^c(\varphi) = \sup_x\left|x^qD^k\left(x^{-(\mu+\nu)/2}e^{-icx}\varphi(x)\right)\right|$. Then $\{\sigma_{q,k}^c\}$ and $\{\sigma_{q,k}\}$ are finite on the same functions and generate the same topology. In particular the spaces $H_{1,\alpha,\beta,a,b}^{\theta}(I)$ of the literature are, for every θ, one and the same space $\mathcal{S}_{\mu,\nu}(I)$.

Proof. Put $g = x^{-(\mu+\nu)/2}\varphi$. By Leibniz, $D^k(e^{-icx}g) = \sum_{m=0}^{k}\binom{k}{m}(-ic)^{k-m}e^{-icx}D^m g$, so

$$\sigma_{q,k}^c(\varphi) \le \sum_{m=0}^{k}\binom{k}{m}|c|^{k-m}\sigma_{q,m}(\varphi).$$

Applying the same inequality with c replaced by $-c$ and φ by $e^{-icx}\varphi$ gives the reverse estimate. Hence each system of seminorms is dominated by the other, so they define the same space with the same topology. ▫

Lemma 6.3 (Polynomial growth of the basis seminorms). For all $q, k \in \mathbb{N}_0$ there exist $C_{q,k} > 0$ and $A_{q,k} \in \mathbb{N}$ with

$$\sigma_{q,k}(\tilde{\mathbf{e}}_n^{\mu,\nu}) \le C_{q,k}\,(1+n)^{A_{q,k}} \qquad (n \in \mathbb{N}_0).$$

Proof. We use only two classical facts about Laguerre polynomials [16, Ch. V, Ch. VII]: the bound

$$e^{-t/2}|L_m^\lambda(t)| \le \frac{\Gamma(m+\lambda+1)}{m!\,\Gamma(\lambda+1)} \le C_\lambda(1+m)^\lambda \qquad (t \ge 0,\ \lambda \ge 0),$$

the three-term recurrence

$$t\,L_m^\lambda(t) = (2m+\lambda+1)L_m^\lambda(t) - (m+1)L_{m+1}^\lambda(t) - (m+\lambda)L_{m-1}^\lambda(t),$$

and the derivative and index-shift relations

$$\frac{d}{dt}L_m^\lambda(t) = -L_{m-1}^{\lambda+1}(t), \qquad L_m^\lambda = L_m^{\lambda+1} - L_{m-1}^{\lambda+1}.$$

Step 1: reduction to $\lambda \ge 0$. By the second relation in (22), any L_m^λ with $-1 < \lambda < 0$ is a difference of two Laguerre polynomials of index $\lambda + 1 \ge 0$; so it suffices to prove the required bounds for $\lambda \ge 0$ and to allow the final exponent $A_{q,k}$ to absorb a finite number of such reductions.

Step 2: the weighted sup norm. We claim that for $\lambda \ge 0$ and $q \in \mathbb{N}_0$,

$$\sup_{t>0} t^q e^{-t/2}|L_m^\lambda(t)| \le C_{q,\lambda}(1+m)^{q+\lambda}.$$

Indeed, iterating (21) q times expresses $t^q L_m^\lambda$ as a linear combination $\sum_{|j|\le q} a_j(m)L_{m+j}^\lambda$ with $|a_j(m)| \le C_{q,\lambda}(1+m)^q$; applying (20) to each term gives (23).

Step 3: derivatives. By definition $x^{-(\mu+\nu)/2}\mathbf{e}_n^{\mu,\nu}(x) = e^{-x}L_n^\nu(2x)$. Using the first relation in (22) and the chain rule, $D[L_n^\nu(2x)] = -2L_{n-1}^{\nu+1}(2x)$; hence by Leibniz

$$D^k[e^{-x}L_n^\nu(2x)] = \sum_{j=0}^{k}\binom{k}{j}(-1)^{k-j}(-2)^j\,e^{-x}L_{n-j}^{\nu+j}(2x).$$

Step 4: conclusion. Therefore, substituting $t = 2x$ and applying (23) with $\lambda = \nu + j$,

$$\sup_{x>0} x^q|D^k[e^{-x}L_n^\nu(2x)]| \le \sum_{j=0}^{k}\binom{k}{j}2^j\,2^{-q}\sup_{t>0} t^q e^{-t/2}|L_{n-j}^{\nu+j}(t)| \le C_{q,k,\nu}(1+n)^{q+\nu+k}.$$

Finally $\|\mathbf{e}_n^\nu\|^{-1} = O(n^{-\nu/2})$ by Lemma 3.1 and Stirling, so $\sigma_{q,k}(\tilde{\mathbf{e}}_n^{\mu,\nu}) \le C(1+n)^{q+k+\nu/2}$, which is the assertion. ▫

Theorem 6.4 (Intrinsic characterization). Let $c_n(\varphi) = \langle\varphi, \tilde{\mathbf{e}}_n^{\mu,\nu}\rangle_{\mathcal{H}_\mu}$ and let

$$s := \{(c_n) \subset \mathbb{C} : \|(c_n)\|_N := \sup_n (1+n)^N|c_n| < \infty\ \ \forall N \in \mathbb{N}_0\}.$$

Then

$$\mathcal{S}_{\mu,\nu}(I) = \bigcap_{k\ge0}\mathrm{Dom}(\Lambda_{\mu,\nu}^k) = \{\varphi \in \mathcal{H}_\mu : (c_n(\varphi)) \in s\},$$

with equivalent topologies (the last carrying $\{\|\cdot\|_N\}$). Consequently $\mathcal{S}_{\mu,\nu}(I)$ is a nuclear Fréchet space, finite linear combinations of the $\tilde{\mathbf{e}}_n^{\mu,\nu}$ are dense in it, and its dual is

$$\mathcal{S}_{\mu,\nu}'(I) = \left\{\sum_n d_n\,\tilde{\mathbf{e}}_n^{\mu,\nu} : |d_n| \le C(1+n)^N \text{ for some } C, N\right\},$$

the series converging in $\mathcal{S}'$.

Proof. Step 1: $\Lambda_{\mu,\nu}$ maps $\mathcal{S}_{\mu,\nu}$ continuously into itself. Write $g = x^{-(\mu+\nu)/2}\varphi$, so that $\sigma_{q,k}(\varphi) = \sup_x|x^q D^k g|$. By Lemma 4.2,

$$x^{-\frac{\mu+\nu}{2}}\Lambda_{\mu,\nu}\varphi = -xD^2g - (\nu+1)Dg + xg - (\nu+1)g,$$

an expression with *polynomial* coefficients: no negative power of x occurs. Applying D^k and Leibniz, and using $D^k(xD^bg) = xD^{k+b}g + kD^{k+b-1}g$, we obtain

$$\sigma_{q,k}(\Lambda_{\mu,\nu}\varphi) \le C_{k,\nu}(\sigma_{q+1,k+2}(\varphi) + \sigma_{q,k+1}(\varphi) + \sigma_{q+1,k}(\varphi) + \sigma_{q,k}(\varphi)) < \infty.$$

Hence $\Lambda_{\mu,\nu}\varphi \in \mathcal{S}_{\mu,\nu}$ and the map is continuous.

Step 2: $\mathcal{S}_{\mu,\nu} \subset \mathcal{H}_\mu$ and rapid decay of coefficients. If $\varphi \in \mathcal{S}_{\mu,\nu}$ then $|\varphi(x)| \le \sigma_{q,0}(\varphi)x^{(\mu+\nu)/2-q}$ for every q, so φ decays faster than any power at infinity, while near 0, $|\varphi(x)| \le \sigma_{0,0}(\varphi)x^{(\mu+\nu)/2}$; since $\nu > -1$, $\int_0^1 x^{-\mu}|\varphi|^2dx \le C\int_0^1 x^\nu\,dx < \infty$. Hence $\varphi \in \mathcal{H}_\mu$, and by Step 1 also $\Lambda^k\varphi \in \mathcal{H}_\mu$ for every k, i.e. $\mathcal{S}_{\mu,\nu} \subseteq \bigcap_k \mathrm{Dom}(\Lambda^k)$. By symmetry of Λ,

$$(2n)^k c_n(\varphi) = \langle\varphi, \Lambda^k\tilde{\mathbf{e}}_n^{\mu,\nu}\rangle = \langle\Lambda^k\varphi, \tilde{\mathbf{e}}_n^{\mu,\nu}\rangle, \quad \text{so} \quad (2n)^k|c_n(\varphi)| \le \|\Lambda^k\varphi\|_{\mathcal{H}_\mu}.$$

Thus $\left(c_n(\varphi)\right) \in s$, with each $\|\cdot\|_N$ dominated by a continuous seminorm of $\mathcal{S}_{\mu,\nu}$ (using Step 1 and $\| \psi \|_{\mathcal{H}_\mu} \leq C\, \sigma_{2,0}(\psi) + C\sigma_{0,0}(\psi)$, which follows from the decay estimates just given). The same computation shows $\bigcap_k \mathrm{Dom}(\Lambda^k) \subseteq \{\varphi: (c_n) \in s\}$.

Step 3: the reverse inclusion. Let $(c_n) \in s$ and put $\varphi_N = \sum_{n \leq N} c_n\, \tilde{\mathbf{e}}_n^{\mu,\nu}$. By Lemma 6.3, for each (q,k),

$$\sum_n |c_n|\ \sigma_{q,k}\left(\tilde{\mathbf{e}}_n^{\mu,\nu}\right) \leq C_{q,k} \sum_n |c_n|\,(1+n)^{A_{q,k}} \leq C_{q,k}\ \|\,(c_n)\,\|_{A_{q,k}+2} \sum_n (1+n)^{-2} < \infty.$$

Hence (φ_N) is Cauchy in every seminorm $\sigma_{q,k}$; since $\mathcal{S}_{\mu,\nu}$ is complete (a routine verification: a σ-Cauchy sequence converges locally uniformly together with all derivatives, and the limit satisfies the bounds), $\varphi := \lim \varphi_N$ exists in $\mathcal{S}_{\mu,\nu}$, and it has coefficients c_n because convergence in $\mathcal{S}_{\mu,\nu}$ implies convergence in $\mathcal{H}_\mu$. This proves both reverse inclusions and, together with the estimate just displayed, continuity of the inverse map $s \to \mathcal{S}_{\mu,\nu}$. Density of the finite combinations is contained in the argument.

Step 4: consequences. The coefficient map is thus a topological isomorphism $\mathcal{S}_{\mu,\nu} \cong s$. The space s is a nuclear Fréchet space whose dual is the space of sequences of polynomial growth, with the duality $\langle (d_n), (c_n) \rangle = \sum d_n c_n$; transporting this isomorphism gives the stated description of $\mathcal{S}'_{\mu,\nu}$. (This is the exact analogue, for the present basis, of the Hermite description of $\mathcal{S}(\mathbb{R})$ in [15].) ▫

Corollary 6.5 (Invariance). For every $\theta \in \mathbb{R}$, $h^{\theta}_{\mu,\nu}$ is a topological isomorphism of $\mathcal{S}_{\mu,\nu}(I)$ onto itself with inverse $h^{-\theta}_{\mu,\nu}$, and extends by transposition to a topological isomorphism of $\mathcal{S}'_{\mu,\nu}(I)$. The same holds for $F(\Lambda_{\mu,\nu})$ whenever F and $1/F$ are of polynomial growth on $\{2n\}$.

Proof. By Theorem 6.4 it suffices to check the statement on the sequence side. There h^θ acts by $c_n \mapsto e^{-2in\theta} c_n$, which satisfies $\| \left(e^{-2in\theta} c_n\right) \|_N = \| (c_n) \|_N$ for every N: it is an isometry for every defining seminorm of s, with inverse $c_n \mapsto e^{2in\theta} c_n$. For $F(\Lambda)$ the action is $c_n \mapsto F(2n) c_n$, and $\| (F(2n)c_n) \|_N \leq C \,\| (c_n) \|_{N+d}$ if $|F(2n)| \leq C(1+n)^d$; similarly for the inverse. The dual statements follow by transposition, which is continuous because the transpose of a topological isomorphism is one. ▫

Remark. Corollary 6.5 replaces the several pages of iterated differential manipulation customarily used for such invariance statements. The reason it is short is structural: on the Laguerre side the fractional transformation is *diagonal with unimodular entries*, so it cannot change any seminorm defined by coefficient decay.

VII. THE BESSEL–CLIFFORD OPERATOR AND A SOBOLEV SCALE

Definition 7.1. For $\mu \in \mathbb{R}$, $\nu > -1$ and $0 < \theta < \pi$ set, with $\chi_\theta(x) = e^{ix\cot\theta}$,

$$\Delta_{\mu,\nu} := xD^2 + (1-\mu)D - \frac{\nu^2 - \mu^2}{4x},$$

$$\Delta_{\mu,\nu,\theta} := \chi_\theta\, \Delta_{\mu,\nu}\, \chi_\theta^{-1}, \qquad \Delta^*_{\mu,\nu,\theta} := \overline{\chi_\theta}\, \Delta_{\mu,\nu}\, \overline{\chi_\theta}^{-1}.$$

Lemma 7.2 (Explicit form). With $c = \cot\theta$,

$$\Delta_{\mu,\nu,\theta} = xD^2 + [(1-\mu) - 2icx]D - [(1-\mu)ic + c^2 x + \frac{\nu^2 - \mu^2}{4x}],$$

and $\Delta^*_{\mu,\nu,\theta}$ is obtained by replacing c by $-c$ (equivalently, by conjugating all coefficients). Moreover $\Delta_{\mu,\nu} = M_\mu^{-1}\left[D(xD) - \frac{\nu^2}{4x}\right] M_\mu$ and $\Delta_{\mu,\nu,\pi/2} = \Delta_{\mu,\nu}$. Finally, with G the multiplication by $x^{-(\mu+\nu)/2}$ of Lemma 4.2,

$$G\,\Delta_{\mu,\nu}\,G^{-1} = xD^2 + (\nu+1)D, \qquad G\,\Delta^*_{\mu,\nu,\theta}\,G^{-1} = xD^2 + [(\nu+1) + 2icx]D + [ic(\nu+1) - c^2 x],$$

so the reduced operators again have polynomial coefficients and no singular term; consequently $\Delta_{\mu,\nu}$ and $\Delta^*_{\mu,\nu,\theta}$ map $\mathcal{S}_{\mu,\nu}(I)$ continuously into itself, by the estimate of Step 1 in the proof of Theorem 6.4.

Proof. Since $\chi_\theta D \chi_\theta^{-1} = D - ic$, we get

$$\chi_\theta \Delta_{\mu,\nu} \chi_\theta^{-1} = x(D-ic)^2 + (1-\mu)(D-ic) - \frac{\nu^2-\mu^2}{4x},$$

and $x(D-ic)^2 = xD^2 - 2icxD - c^2 x$, which gives (26). The conjugation identity for $\Delta_{\mu,\nu}$ is the computation of Step 1 in the proof of Proposition 4.1 with the sign of the second-order part reversed: conjugating $D(xD)$ by $x^{\mu/2}$ produces $xD^2 + (1-\mu)D + \frac{\mu^2}{4x}$, and subtracting $\frac{\nu^2}{4x}$ gives (24). For the first identity in (27), the same computation with μ replaced by $-\nu$ gives $x^{-\nu/2} D(xD) x^{\nu/2} = xD^2 + (1+\nu)D + \frac{\nu^2}{4x}$, whose singular term cancels against $-\frac{\nu^2}{4x}$; the second identity follows by replacing D with $D + ic$, as above. Finally $\cot(\pi/2) = 0$. ▫

Proposition 7.3 (Intertwining and operational formulas). Let $\varphi \in \mathcal{S}_{\mu,\nu}(I)$, $0 < \theta < \pi$ and $r \in \mathbb{N}_0$. Then

$$\begin{aligned} h_{\mu,\nu}\big(\Delta_{\mu,\nu}\varphi\big)(y) &= -y\,\big(h_{\mu,\nu}\varphi\big)(y),\\ h^{\theta}_{\mu,\nu}\big(\Delta^*_{\mu,\nu,\theta}\varphi\big)(y) &= -(y\csc^2\theta)\big(h^{\theta}_{\mu,\nu}\varphi\big)(y),\\ h^{\theta}_{\mu,\nu}\big((\Delta^*_{\mu,\nu,\theta})^r\varphi\big)(y) &= (-y\csc^2\theta)^r\big(h^{\theta}_{\mu,\nu}\varphi\big)(y). \end{aligned}$$

Proof. By Theorem 2.2 and Lemma 7.2 we may take $\mu = 0$; the general case follows by conjugating with M_μ.

Step 1: the kernel is an eigenfunction. Let $F(x) = J_\nu\big(2\sqrt{xy}\big)$ and put $u = 2\sqrt{xy}$, so $u^2 = 4xy$ and $\frac{\partial u}{\partial x} = \sqrt{y/x} = \frac{2y}{u}$. Then

$$xF'(x) = x\sqrt{y/x}\;F_u = \sqrt{xy}\,F_u = u/2\,F_u,$$

and

$$\begin{aligned} (xF')' &= \frac{2y}{u}\frac{d}{du}\Big(\frac{u}{2}F_u\Big)\\ &= \frac{2y}{u}\Big(\frac{1}{2}F_u + \frac{u}{2}F_{uu}\Big)\\ &= \frac{y}{u^2}(uF_u + u^2F_{uu}). \end{aligned}$$

Bessel's equation $u^2F_{uu} + uF_u + (u^2-\nu^2)F = 0$ now gives

$$\begin{aligned} (xF')' &= \frac{y}{u^2}(\nu^2-u^2)F = \frac{\nu^2 y}{4xy}F - yF\\ &= \frac{\nu^2}{4x}F - yF, \end{aligned}$$

that is, $\Delta_{0,\nu}F = D(xF') - \frac{\nu^2}{4x}F = -yF$.

Step 2: (28). $\Delta_{0,\nu} = D(xD) - \frac{\nu^2}{4x}$ is formally self-adjoint on $L^2(I, dx)$. For $\varphi \in \mathcal{S}_{0,\nu}$ two integrations by parts give

$$\begin{aligned} &\big(T_\nu\Delta_{0,\nu}\varphi\big)(y)\\ &= \int_0^\infty F\,(x)\big(\Delta_{0,\nu}\varphi\big)(x)\,dx\\ &= \int_0^\infty \big(\Delta_{0,\nu}F\big)\,(x)\varphi(x)\,dx\\ &= -y(T_\nu\varphi)(y), \end{aligned}$$

the boundary terms $[x(F\varphi' - F'\varphi)]_0^\infty$ vanishing because φ and all its derivatives decay faster than any power at ∞ and $\varphi(x) = O\big(x^{\nu/2}\big)$, $F(x) = O\big(x^{\nu/2}\big)$ as $x \to 0^+$ with $\nu > -1$.

Step 3: (29). By Corollary 5.4, with $\lambda = \csc^2\theta$,

$$\big(h^{\theta}_{\nu}\psi\big)(y) = \gamma_{\nu,\theta}\csc\theta\,\chi_\theta(y)(T_\nu[\chi_\theta\psi])(\lambda y).$$

Since $|\chi_\theta| = 1$ we have $\overline{\chi_\theta} = \chi_\theta^{-1}$, so (25) reads $\Delta^*_{0,\nu,\theta} = \chi_\theta^{-1}\,\Delta_{0,\nu}\,\chi_\theta$, that is,

$$\chi_\theta\,\Delta^*_{0,\nu,\theta}\varphi = \Delta_{0,\nu}(\chi_\theta\varphi).$$

Applying Step 2 to $\chi_\theta\varphi$,

$$\begin{aligned} &\big(T_\nu\big[\chi_\theta\Delta^*_{0,\nu,\theta}\varphi\big]\big)(w)\\ &= \big(T_\nu\Delta_{0,\nu}[\chi_\theta\varphi]\big)(w)\\ &= -w(T_\nu[\chi_\theta\varphi])(w). \end{aligned}$$

Evaluating at $w = \lambda y$ and multiplying by $\gamma_{\nu,\theta}\csc\theta\,\chi_\theta(y)$ yields

$$\begin{aligned} \big(h^{\theta}_{\nu}\Delta^*_{0,\nu,\theta}\varphi\big)(y) &= -\lambda y\,\big(h^{\theta}_{\nu}\varphi\big)(y)\\ &= -(y\csc^2\theta)\big(h^{\theta}_{\nu}\varphi\big)(y). \end{aligned}$$

Step 4: (30). By (27), $\Delta^*_{\mu,\nu,\theta}$ maps $\mathcal{S}_{\mu,\nu}$ into itself, so (29) may be iterated r times. ▫

Definition 7.4. For $s \in \mathbb{R}$ and $0 < \theta < \pi$ put

$$\mathcal{H}^s_\theta := \{f \in \mathcal{S}'_{\mu,\nu}(I):\ (1 + y\csc^2\theta)^{s/2}h^{\theta}_{\mu,\nu}f \in \mathcal{H}_\mu\},\qquad \| f \|_{s,\theta} = \| (1 + y\csc^2\theta)^{s/2}h^{\theta}_{\mu,\nu}f \|_{\mathcal{H}_\mu}.$$

Proposition 7.5. $\mathcal{H}^s_\theta$ is a Hilbert space; $\mathcal{H}^0_\theta = \mathcal{H}_\mu$; $\mathcal{H}^s_\theta \hookrightarrow \mathcal{H}^t_\theta$ continuously for $s \ge t$; $\mathcal{S}_{\mu,\nu}(I) \subseteq \bigcap_s\mathcal{H}^s_\theta$; and

$$\mathcal{H}^s_\theta = \mathrm{Dom}\Big(\big(\mathrm{Id} - \Delta^*_{\mu,\nu,\theta}\big)^{s/2}\Big) = \chi_\theta^{-1}\cdot\mathcal{H}^s_{\pi/2}$$

with equivalence of norms, where $\mathcal{H}^s_{\pi/2} = \mathrm{Dom}\Big(\big(\mathrm{Id} - \Delta_{\mu,\nu}\big)^{s/2}\Big)$. Under the unitary VM_μ of Theorem 4.4(a), the scale $\mathcal{H}^s_{\pi/2}$ corresponds to the domain scale of order $2s$ of the Bessel operator $-\frac{d^2}{dt^2} + \frac{\nu^2 - 1/4}{t^2}$.

Proof. Since $h^{\theta}_{\mu,\nu}$ is unitary on $\mathcal{H}_\mu$ and multiplication by $(1 + y\csc^2\theta)^{s/2}$ is a positive self-adjoint (unbounded for $s > 0$) operator, $\|\cdot\|_{s,\theta}$ is a Hilbert norm and $\mathcal{H}^s_\theta$ is complete, being the image under the unitary $\big(h^\theta\big)^{-1}$ of a weighted L^2 space. The monotonicity of the scale is $(1 + y\csc^2\theta)^{t/2} \le (1 + y\csc^2\theta)^{s/2}$ for $t \le s$. That $\mathcal{S} \subseteq \bigcap_s\mathcal{H}^s_\theta$ follows from Corollary 6.5 and the fact that $y^N\psi \in \mathcal{H}_\mu$ for all N when $\psi \in \mathcal{S}$.

By (29), h^θ conjugates $\Delta^*_{\mu,\nu,\theta}$ into multiplication by $-y\csc^2\theta$; hence it conjugates $(\mathrm{Id} - \Delta^*_\theta)^{s/2}$ into multiplication by $(1 + y\csc^2\theta)^{s/2}$, which is the first equality.

For the second, apply Corollary 5.4: $h^\theta = \gamma\csc\theta\,M_{\chi_\theta}\mathcal{D}_\lambda T_\nu M_{\chi_\theta}$ where $\mathcal{D}_\lambda g(y) = g(\lambda y)$ and $\lambda = \csc^2\theta$ (this is (18) rewritten). Since $|\chi_\theta| = 1$, M_{χ_θ} is unitary on $\mathcal{H}_\mu$, and since $\lambda > 0$ is fixed, $(1 + \lambda y)^{s/2}$ and $(1 + y)^{s/2}$ are comparable after the dilation. Hence $f \in \mathcal{H}^s_\theta$ iff $\chi_\theta f \in \mathcal{H}^s_{\pi/2}$, with comparable norms.

The last statement follows by the change of variables $x = t^2$ used in Theorem 4.4(a): that computation applied to $\Delta_{0,\nu} = D(xD) - \frac{\nu^2}{4x}$ gives $V\Delta_{0,\nu}V^{-1} = \frac{1}{4}\Big(\frac{d^2}{dt^2} - \frac{\nu^2 - 1/4}{t^2}\Big)$, and the covariable y becomes $\frac{1}{4}$ times the square of the Hankel covariable, so weights $(1 + y)^{s/2}$ become weights of order $2s$ in the latter. ▫

Remark 7.6 (Comparison with Hörmander **classes).** The factor 2 just obtained is the exchange rate between the two calculi. Under $x = t^2$ the Hankel–Clifford covariable y becomes (a multiple of) the square τ^2 of the Hankel covariable, so a symbol with $\left|D_y^s p(y)\right| \le C_s(1+y)^{l-s}$ corresponds to $\tilde{p}(\tau) = p(\tau^2/4)$ with $|D_\tau^s \tilde{p}(\tau)| \le C(1+\tau)^{2l-s}$, i.e. to a symbol of Hörmander class $S_{1,0}^{2l}$ in the covariable [3, Ch. XVIII]. The classes below are therefore the Bessel-adapted radial version of $S_{1,0}^{2l}$; they are not exotic classes in the sense of [3], and no $S_{\rho,\delta}^m$ phenomena with $\rho < 1$ occur. What they do *not* inherit is control of the behaviour in the space variable at infinity; that is why the weight $(1+x)^{-q}$ appears in Definition 8.1.

VIII. PSEUDO-DIFFERENTIAL OPERATORS

Throughout, $0<\theta<\pi$, $\mu \in \mathbb{R}$ and $\nu > -1$ are fixed. We write $h^\theta = h_{\mu,\nu}^\theta$, $\mathcal{S} = \mathcal{S}_{\mu,\nu}(I)$, $K^\theta = K_{\mu,\nu}^\theta$ and $\hat{\varphi}^{\,\theta} = h^\theta \varphi$.

A. Symbol classes and the operator

Definition 8.1. Let $l \in \mathbb{R}$, $q \in \mathbb{N}_0$.

1. $\mathcal{M}^l$ is the set of $p \in C^\infty([0,\infty))$ with $|D^s p(y)| \le C_s(p)(1+y)^{l-s}$ for all $s \in \mathbb{N}_0$ (*multiplier symbols*); compare the classes of [20,9]. Set $\mathcal{M}^\bullet = \bigcup_l \mathcal{M}^l$ and $\mathcal{M}^{-\infty} = \bigcap_l \mathcal{M}^l$.
2. S_q^l is the set of $p \in C^\infty(I \times [0,\infty))$ with
$$(1+x)^q |D_x^r D_y^s p(x,y)| \le C_{r,s}(p)\,(1+y)^{l-s}, \qquad r,s \in \mathbb{N}_0.$$
Write $S^l = S_0^l$.
3. $\mathcal{O}$ is the set of $a \in C^\infty(I)$ with $|D^r a(x)| \le C_r(a)(1+x)^{-r}$ for all $r \in \mathbb{N}_0$; set $\pi_N(a) = \max_{r\le N} C_r(a)$. Then $\mathcal{A}^l$ is the set of p admitting $p(x,y) = \sum_{j\ge0} a_j(x) b_j(y)$ with $b_j \in \mathcal{M}^l$, $a_j \in \mathcal{O}$, and $\sum_j \pi_N(a_j)\rho_N(b_j) < \infty$ for every N, where $\rho_N(b) = \max_{s\le N} C_s(b)$ (*tensor class*). The condition $a_j \in \mathcal{O}$ is the SG-type requirement that each x-factor behave like a symbol of order 0 in the space variable; it is what makes Theorem 8.9 available on the whole Sobolev scale.

Thus $\mathcal{M}^l \subset \mathcal{A}^l \subset S^l$, and symbols $p(x,y) = \sum_{i=0}^N a_i(x)(-y\csc^2\theta)^i$ lie in $\mathcal{A}^N$ whenever $a_i \in \mathcal{O}$.

Definition 8.2. For $p \in S_q^l$ and $\varphi \in \mathcal{S}$,
$$\left(h_p^\theta \varphi\right)(x) := \left[(h^\theta)^{-1}\left(p(x,\cdot)\,\hat{\varphi}^{\,\theta}\right)\right](x),$$
the inverse transform being applied in the second variable and the result evaluated on the diagonal. To write this as an integral, let
$$\widetilde{K}^\theta(x,y) := (xy)^{\mu/2} K_\nu^\theta(x,y) = x^\mu K_{\mu,\nu}^\theta(x,y)$$
be the kernel of $h_{\mu,\nu}^\theta$ *with respect to the measure* $x^{-\mu}\,dx$ *of* $\mathcal{H}_\mu$; it is symmetric, $\widetilde{K}^\theta(x,y) = \widetilde{K}^\theta(y,x)$, so that $\left(h^\theta\right)^{-1}$ has kernel $\overline{\widetilde{K}^\theta(x,y)}$ with respect to $y^{-\mu}\,dy$. Hence
$$\left(h_p^\theta \varphi\right)(x) = \int_0^\infty \overline{\widetilde{K}^\theta(x,y)}\; p(x,y)\,\hat{\varphi}^{\,\theta}(y)\,y^{-\mu}\,dy.$$

Proposition 8.3. If $p(x,y) = \sum_{i=0}^N a_i(x)(-y\csc^2\theta)^i$ then
$$h_p^\theta = \sum_{i=0}^N a_i(x)(\Delta_{\mu,\nu,\theta}^*)^i,$$
a linear differential operator of order $2N$ with the coefficients (26).

Proof. By (30), $(-y\csc^2\theta)^i \hat{\varphi}^{\,\theta} = h^\theta[(\Delta_\theta^*)^i \varphi]$, so
$$(h^\theta)^{-1}\left(p(x,\cdot)\hat{\varphi}^{\,\theta}\right)(z) = \sum_i a_i(x)(\Delta_\theta^*)^i \varphi(z);$$
setting $z=x$ gives the claim. ▫

B. Continuity on the test space

Lemma 8.4. Let $b \in \mathcal{M}^l$. Then multiplication by b maps $\mathcal{S}$ continuously into itself; more precisely, with $l_+ = \max(l,0)$ and $d = \lceil l_+ \rceil$,
$$\sigma_{q,k}(b\psi) \le 2^k \rho_k(b)\, C_l \sum_{m=0}^k \sum_{i=0}^d \sigma_{q+i,m}(\psi) \qquad (q,k \in \mathbb{N}_0).$$

Proof. Put $g = y^{-(\mu+\nu)/2}\psi$, so $y^{-(\mu+\nu)/2}(b\psi) = bg$. By Leibniz, $D^k(bg) = \sum_{m=0}^k \binom{k}{m}(D^{k-m}b)(D^m g)$, and $|D^{k-m}b(y)| \le \rho_k(b)(1+y)^{l-(k-m)} \le \rho_k(b)(1+y)^{l_+}$. Hence
$$y^q|D^k(bg)| \le \rho_k(b)\sum_{m=0}^k \binom{k}{m} y^q(1+y)^{l_+}|D^m g|.$$
Since $(1+y)^{l_+} \le (1+y)^d = \sum_{i=0}^d \binom{d}{i} y^i$, we get $y^q(1+y)^{l_+} \le C_l \sum_{i=0}^d y^{q+i}$, and taking suprema gives the stated inequality with $\sum_m \binom{k}{m} \le 2^k$. For $l<0$ one may take $d=0$. ▫

Theorem 8.5 (Continuity on $\mathcal{S}$). Let $p \in S_q^l$ with $q \ge 0$. Then h_p^θ maps $\mathcal{S}$ continuously into itself, and extends by transposition to a continuous linear map of $\mathcal{S}'$ into itself.

Proof. By Theorem 2.2 we may take $\mu = 0$ (conjugating by M_μ transforms symbol classes into themselves). Fix $\varphi \in \mathcal{S}$ and put $\hat{\varphi} = h^\theta\varphi$, which lies in $\mathcal{S}$ by Corollary 6.5. Define
$$\Theta: I \to \mathcal{S}, \qquad \Theta(x) := (h^\theta)^{-1}[p(x,\cdot)\,\hat{\varphi}\,].$$

Step 1: $\Theta(x) \in \mathcal{S}$ with uniform bounds. For each fixed x, (31) says $p(x,\cdot) \in \mathcal{M}^l$ with $\rho_N(p(x,\cdot)) \le (1+x)^{-q}\max_{s\le N} C_{0,s}(p)$. By Lemma 8.4, $p(x,\cdot)\hat{\varphi} \in \mathcal{S}$ with $\sigma_{m,k}(p(x,\cdot)\hat{\varphi}) \le (1+x)^{-q}\, C_{m,k}\, \Sigma_{m,k}(\hat{\varphi})$, where $\Sigma_{m,k}$ is a finite sum of seminorms $\sigma_{m',k'}$ with $m' \le m+d$, $k' \le k$. Applying Corollary 6.5 (continuity of $\left(h^\theta\right)^{-1}$ on $\mathcal{S}$) and again Corollary 6.5 to bound $\Sigma_{m,k}(\hat{\varphi})$ by a continuous seminorm of φ, we obtain a continuous seminorm $\Sigma'_{m,k}$ on $\mathcal{S}$ with
$$\sigma_{m,k}(\Theta(x)) \le C_{m,k}\,(1+x)^{-q}\,\Sigma'_{m,k}(\varphi) \qquad \text{for all } x > 0.$$

Step 2: Θ is smooth as an $\mathcal{S}$-valued map. We claim that for each r, Θ is r times differentiable with

$\Theta^{(r)}(x) = (h^\theta)^{-1}[D_x^r p(x,\cdot)\hat{\varphi}]$. It suffices to prove this for $r = 1$ and iterate, since $D_x p \in S_q^l$ as well. For $h \neq 0$,

$$\frac{p(x+h,y)-p(x,y)}{h} - D_x p(x,y) = \frac{1}{h}\int_0^h [D_x p(x+\tau,y) - D_x p(x,y)]\, d\tau = \frac{1}{h}\int_0^h \int_0^\tau D_x^2\, p(x+\varsigma,y)\, d\varsigma\, d\tau.$$

Differentiating s times in y under the integral signs and using (31) for $D_x^2 D_y^s p$ gives, for every $x > 0$ and every h with $|h| \le x/2$ (so that $x+\varsigma \ge x/2$ throughout, whence $(1+x+\varsigma)^{-q} \le (1+x/2)^{-q} \le 2^q(1+x)^{-q}$),

$$|D_y^s(\frac{p(x+h,\cdot)-p(x,\cdot)}{h} - D_x p(x,\cdot))(y)| \le \frac{|h|}{2}\, 2^q\, C_{2,s}(p)\, (1+x)^{-q}(1+y)^{l-s},$$

so the difference quotient converges to $D_x p(x,\cdot)$ in the topology of $\mathcal{M}^l$ (i.e. in every ρ_N). By Lemma 8.4 and continuity of $\left(h^\theta\right)^{-1}$ on $\mathcal{S}$, the corresponding difference quotients of Θ converge in $\mathcal{S}$. The same estimates as in Step 1 yield

$$\sigma_{m,k}\left(\Theta^{(r)}(x)\right) \le C_{m,k,r}\,(1+x)^{-q}\,\Sigma'_{m,k,r}(\varphi) \qquad \text{for all } x>0,\ r\in\mathbb{N}_0.$$

Step 3: restriction to the diagonal. By Definition 8.2, $\left(h_p^\theta\varphi\right)(x) = \Theta(x)(x)$. Put $\Psi(x,z) := z^{-\nu/2}\,\Theta(x)(z)$, so that $x^{-\nu/2}\left(h_p^\theta\varphi\right)(x) = \Psi(x,x)$. By Step 2, Ψ is smooth in (x,z) with $\partial_x^r\Psi(x,z) = z^{-\nu/2}\Theta^{(r)}(x)(z)$, and by definition of $\sigma_{m,k}$,

$$\sup_{z>0}|z^m\,\partial_x^r\,\partial_z^k\Psi(x,z)| = \sigma_{m,k}\left(\Theta^{(r)}(x)\right).$$

By the Leibniz rule for the diagonal restriction of a smooth function of two variables,

$$D_x^k[\Psi(x,x)] = \sum_{r+m=k}\binom{k}{r}(\partial_x^r\,\partial_z^m\Psi)(x,x).$$

Hence, using (37) at $z = x$ and then (36),

$$\sigma_{m_0,k}\left(h_p^\theta\varphi\right) = \sup_{x>0}|x^{m_0}D_x^k\Psi(x,x)| \le \sum_{r+m=k}\binom{k}{r}\sup_{x>0}\sigma_{m_0,m}\left(\Theta^{(r)}(x)\right) \le C_{m_0,k}\,\Sigma''_{m_0,k}(\varphi) < \infty,$$

with Σ'' a continuous seminorm on $\mathcal{S}$. Thus $h_p^\theta\varphi \in \mathcal{S}$ and $h_p^\theta\colon \mathcal{S}\to\mathcal{S}$ is continuous. The extension to $\mathcal{S}'$ is by transposition, which is continuous for the strong topologies because $\mathcal{S}$ is a Fréchet–Montel space (being isomorphic to s, Theorem 6.4). ▫

Remark. The proof occupies half a page and uses no iterated differential identities, because the two genuinely nontrivial facts were isolated beforehand: invariance of $\mathcal{S}$ under h^θ (Corollary 6.5, itself a consequence of diagonality on the Laguerre side) and stability of $\mathcal{S}$ under multiplication by symbols (Lemma 8.4).

C. Kernel and integral representation

Theorem 8.7 (Integral representation and kernel bound). Let $p \in S_q^l$ with $l < -1/2$. Then for $\varphi \in \mathcal{S}$,

$$\left(h_p^\theta\varphi\right)(x) = \int_0^\infty \mathcal{K}_p\,(x,z)\,\varphi(z)\,z^{-\mu}\,dz,$$

where

$$\mathcal{K}_p(x,z) = 2\,(xz)^{\mu/2}\,e^{i(z-x)\cot\theta} \times \int_0^\infty J_\nu\left(2w\sqrt{x}\right)J_\nu\left(2w\sqrt{z}\right) \times p(x,w^2\sin^2\theta)\,w\,dw,$$

the integral converging absolutely, and

$$|\mathcal{K}_p(x,z)| \le C\,(1+x)^{-q}(xz)^{\mu/2}[(xz)^{-1/4} + \min\{1,(xz)^{\nu/2}\}],$$

with C depending only on ν, l, θ and finitely many $C_{0,s}(p)$.

Proof. By (34), $\hat{\varphi}^{\,\theta}(y) = \int_0^\infty \tilde{K}^\theta\,(z,y)\varphi(z)z^{-\mu}\,dz$ and Fubini (justified below),

$$\left(h_p^\theta\varphi\right)(x) = \int_0^\infty \mathcal{K}_p\,(x,z)\varphi(z)z^{-\mu}\,dz, \qquad \mathcal{K}_p(x,z) = \int_0^\infty \overline{\tilde{K}^\theta(x,y)}\;p(x,y)\,\tilde{K}^\theta(z,y)\,y^{-\mu}\,dy.$$

By (33), $\overline{\tilde{K}^\theta(x,y)}\tilde{K}^\theta(z,y)y^{-\mu} = (xy)^{\mu/2}(zy)^{\mu/2}y^{-\mu}\overline{K_\nu^\theta(x,y)}K_\nu^\theta(z,y) = (xz)^{\mu/2}\overline{K_\nu^\theta(x,y)}K_\nu^\theta(z,y)$, the powers of y cancelling exactly. By (7), $\left|\gamma_{\nu,\theta}\right| = 1$ and the y-dependent chirps cancel, so

$$\overline{K_\nu^\theta(x,y)}K_\nu^\theta(z,y) = \csc^2\theta\,e^{i(z-x)\cot\theta}J_\nu\left(2\sqrt{xy}\csc\theta\right)J_\nu\left(2\sqrt{zy}\csc\theta\right).$$

Substituting $y = w^2\sin^2\theta$, so that $\sqrt{y}\csc\theta = w$ and $dy = 2w\sin^2\theta\,dw$, the factor $\csc^2\theta\cdot 2\sin^2\theta = 2$ remains and (38) follows.

For the estimate, use the elementary bounds $|J_\nu(u)| \le C_\nu u^\nu$ for $0 < u \le 1$ and $|J_\nu(u)| \le C_\nu u^{-1/2}$ for $u \ge 1$ [19, Ch. VII], and $|p(x,w^2\sin^2\theta)| \le C_{0,0}(p)(1+x)^{-q}(1+w^2\sin^2\theta)^l \le C(1+x)^{-q}(1+w)^{2l}$. Splitting the w-integral at $w_0 = \max\{1/\left(2\sqrt{x}\right), 1/\left(2\sqrt{z}\right), 1\}$: on (w_0,∞),

$$\int_{w_0}^\infty |J_\nu\left(2w\sqrt{x}\right)J_\nu\left(2w\sqrt{z}\right)|(1+w)^{2l}w\,dw \le C(xz)^{-1/4}\int_1^\infty (1+w)^{2l}\,dw < \infty$$

because $2l < -1$; on $(0, w_0)$ the integrand is bounded by $C\min\{1, (w^2\sqrt{xz})^{\nu}\}\, w$, whose integral is bounded by $C\min\{1, (xz)^{\nu/2}\}$ after the substitution $w \mapsto w\sqrt[4]{xz}$ when $\nu \geq 0$, and directly when $-1 < \nu < 0$. Adding the two contributions and restoring the factor $(xz)^{\mu/2}$ gives (39). Absolute convergence of the double integral, hence the use of Fubini, follows from the same bounds together with the rapid decay of φ. ▫

Remark. Sharper off-diagonal decay — of the form $|\mathcal{K}_p(x,z)| \leq C_N|\sqrt{x} - \sqrt{z}|^{-N}$, the natural distance being measured in the variable $t = \sqrt{x}$ of Theorem 4.4(a) — can be extracted from (38) by inserting the asymptotic expansion of J_ν and integrating by parts in w, each step gaining a factor $(\sqrt{x} - \sqrt{z})^{-1}$ and costing one w-derivative on the symbol. We do not carry this out, since no result below uses it, and the uniform treatment of the region $w \lesssim \max\{x^{-1/2}, z^{-1/2}\}$ requires more care than the gain justifies here.

L^2 D. boundedness

Theorem 8.9. Let $p = \sum_j a_j \otimes b_j \in \mathcal{A}^0$ with a_j, b_j bounded and $\sum_j \| a_j \|_\infty \| b_j \|_\infty < \infty$. Then h_p^θ extends from $\mathcal{S}$ to a bounded operator on $\mathcal{H}_\mu$ with

$$\| h_p^\theta \|_{\mathcal{H}_\mu \to \mathcal{H}_\mu} \leq \sum_j \| a_j \|_\infty \| b_j \|_\infty.$$

More generally, for $p \in \mathcal{A}^l$ (so that each $a_j \in \mathcal{O}$) the operator $h_p^\theta : \mathcal{H}_\theta^s \to \mathcal{H}_\theta^{s-l}$ is bounded for every $s \in \mathbb{R}$.

Proof. For a single product symbol $a \otimes b$, Definition 8.2 gives, for $\varphi \in \mathcal{S}$,

$$(h_{a\otimes b}^\theta \varphi)(x) = a(x)\left[(h^\theta)^{-1}(b\, h^\theta \varphi)\right](x), \qquad \text{i.e.} \qquad h_{a\otimes b}^\theta = M_a\, (h^\theta)^{-1} M_b\, h^\theta,$$

because $a(x)$ does not participate in the inverse transform. This is a composition of the unitary h^θ, the multiplication M_b of norm $\| b \|_\infty$, the unitary $(h^\theta)^{-1}$ and the multiplication M_a of norm $\| a \|_\infty$; hence $\| h_{a\otimes b}^\theta \| \leq \| a \|_\infty \| b \|_\infty$. Summing over j and using the triangle inequality together with the assumed absolute convergence gives the first statement, the sum converging in operator norm.

For the second statement, fix s and write $h_{a\otimes b}^\theta = M_a \circ \left[(h^\theta)^{-1} M_b h^\theta\right]$. By Definition 7.4, h^θ is an isometry of $\mathcal{H}_\theta^t$ onto $L^2(I, (1 + y\csc^2\theta)^t y^{-\mu}\, dy)$ for every t, and multiplication by $b \in \mathcal{M}^l$ carries the weight-s space into the weight-$(s-l)$ space with norm at most $\sup_y (1 + y\csc^2\theta)^{-l}|b(y)| < \infty$; hence the bracketed operator is bounded $\mathcal{H}_\theta^s \to \mathcal{H}_\theta^{s-l}$.

It remains to show that M_a with $a \in \mathcal{O}$ is bounded on $\mathcal{H}_\theta^t$ for every $t \in \mathbb{R}$. Put $P := \mathrm{Id} - \Delta_{\mu,\nu,\theta}^*$; by (29), h^θ conjugates P into multiplication by $1 + y\csc^2\theta \geq 1$, so P is self-adjoint, positive and boundedly invertible, and $\mathcal{H}_\theta^t = \mathrm{Dom}(P^{t/2})$ with $\| f \|_{t,\theta} = \| P^{t/2} f \|_{\mathcal{H}_\mu}$ (Proposition 7.5). For $t = 0$ boundedness of M_a is clear, with norm $\| a \|_\infty \leq \pi_0(a)$. For $t = 2$ we compute the commutator from (26):

$$[\Delta_{\mu,\nu,\theta}^*, M_a] = 2xa'\, D + (xa'' + [(1-\mu) + 2ix\cot\theta]a'),$$

a first-order operator whose coefficients satisfy $|xa'| \leq \pi_1(a)x(1+x)^{-1} \leq \pi_1(a)$, $|xa''| \leq \pi_2(a)x(1+x)^{-2} \leq \pi_2(a)$ and $|xa'| \leq \pi_1(a)$ again for the chirp term. Moreover, integrating by parts in $\langle -\Delta_{\mu,\nu} u, u\rangle_{\mathcal{H}_\mu}$ and using (28) gives, for $\mu = 0$, $\| \sqrt{x}\, Du \|_{\mathcal{H}_\mu}^2 \leq \langle -\Delta_{0,\nu} u, u\rangle \leq \| u \|_{1,\theta}^2$, and the general case follows by conjugation with M_μ; hence $\| 2xa' Du \|_{\mathcal{H}_\mu} \leq 2 \| \sqrt{x} a' \|_\infty \| \sqrt{x} Du \| \leq C\pi_1(a) \| u \|_{1,\theta}$, using $|\sqrt{x} a'| \leq \pi_1(a)$. Therefore $\| [P, M_a]u \|_{\mathcal{H}_\mu} \leq C\pi_2(a) \| u \|_{1,\theta} \leq C\pi_2(a) \| u \|_{2,\theta}$, and

$$\| PM_a u \| \leq \| M_a P u \| + \| [P, M_a]u \| \leq C\pi_2(a) \| u \|_{2,\theta},$$

so M_a is bounded on $\mathcal{H}_\theta^2$. Iterating gives boundedness on $\mathcal{H}_\theta^{2k}$ for every $k \in \mathbb{N}$, duality gives it on $\mathcal{H}_\theta^{-2k}$, and complex interpolation of the domain scale of the positive self-adjoint operator P [14, §IX.4] gives it for all real t. Composing the two maps gives the claim. ▫

E. The symbolic calculus

Theorem 8.10 (Exact calculus for multipliers). Let $p \in \mathcal{M}^l$, $p' \in \mathcal{M}^{l'}$ and $s \in \mathbb{R}$. Then:

1. *(Composition)* $h_p^\theta h_{p'}^\theta = h_{pp'}^\theta$, with $pp' \in \mathcal{M}^{l+l'}$; in particular these operators commute and $p \mapsto h_p^\theta$ is an injective algebra homomorphism $\mathcal{M}^\bullet \to \mathcal{L}(\mathcal{S})$.
2. *(Adjoint)* $(h_p^\theta)^* = h_{\bar{p}}^\theta$ in $\mathcal{H}_\mu$ (for $l \leq 0$, as bounded operators; in general as an identity of operators on $\mathcal{S}$ and $\mathcal{S}'$).
3. *(Mapping)* $h_p^\theta : \mathcal{H}_\theta^s \to \mathcal{H}_\theta^{s-l}$ is bounded with norm at most $\sup_{y\geq 0}(1 + y\csc^2\theta)^{-l}|p(y)|$; and $h_p^\theta : \mathcal{S} \to \mathcal{S}$, $\mathcal{S}' \to \mathcal{S}'$ continuously.
4. *(Smoothing)* $p \in \mathcal{M}^{-\infty}$ if and only if h_p^θ maps $\mathcal{H}_\theta^s$ boundedly into $\mathcal{H}_\theta^t$ for all $s, t \in \mathbb{R}$.
5. *(Ellipticity and parametrix)* Suppose p is elliptic: there are $c, R > 0$ with $|p(y)| \geq c(1 + y)^l$ for $y \geq R$. Then there is $\varrho \in \mathcal{M}^{-l}$ with

$$h^\theta_\varrho h^\theta_p = h^\theta_p h^\theta_\varrho = \mathrm{Id} - h^\theta_r, \qquad r \in C_c^\infty([0,\infty)) \subset \mathcal{M}^{-\infty}.$$

6. *(Hypoellipticity)* With p as in (e): if $u \in \mathcal{H}^{s_0}_\theta$ for some s_0 and $h^\theta_p u \in \mathcal{H}^s_\theta$, then $u \in \mathcal{H}^{s+l}_\theta$; if $h^\theta_p u \in \mathcal{S}$ and $u \in \mathcal{S}'$ has coefficients of polynomial growth, then $u \in \mathcal{S}$.
7. *(Isomorphism)* If in addition $p(y) \neq 0$ for every $y \geq 0$, then $h^\theta_p: \mathcal{H}^{s+l}_\theta \to \mathcal{H}^s_\theta$ is a Hilbert space isomorphism with inverse $h^\theta_{1/p}$.

Proof. Throughout, Definition 8.2 gives for x-independent symbols the clean formula

$$h^\theta_p = (h^\theta)^{-1} M_p\, h^\theta,$$

since $p(x,y) = p(y)$ does not involve x and the diagonal evaluation is trivial.

(a) That $pp' \in \mathcal{M}^{l+l'}$ follows from Leibniz: $|D^s(pp')| \leq \sum_i \binom{s}{i} C_i(p) C_{s-i}(p')(1+y)^{l-i}(1+y)^{l'-s+i} = C(1+y)^{l+l'-s}$. Then by (40), $h^\theta_p h^\theta_{p'} = (h^\theta)^{-1} M_p h^\theta (h^\theta)^{-1} M_{p'} h^\theta = (h^\theta)^{-1} M_{pp'} h^\theta = h^\theta_{pp'}$. Injectivity: if $h^\theta_p = 0$ then $M_p = 0$, so $p = 0$ a.e., hence everywhere by continuity.

(b) $M_p^* = M_{\overline{p}}$ in $\mathcal{H}_\mu$, and h^θ is unitary, so $((h^\theta)^{-1} M_p h^\theta)^* = (h^\theta)^{-1} M_{\overline{p}} h^\theta$.

(c) h^θ is by definition an isometry from $\mathcal{H}^s_\theta$ onto $L^2((1 + y\csc^2\theta)^s y^{-\mu} dy)$; multiplication by p maps this space into $L^2((1 + y\csc^2\theta)^{s-l} y^{-\mu} dy)$ with the stated norm, because $|p(y)|(1 + y\csc^2\theta)^{(s-l)/2} \leq [\sup(1 + y\csc^2\theta)^{-l}|p|](1 + y\csc^2\theta)^{s/2}$; and $(h^\theta)^{-1}$ is an isometry back onto $\mathcal{H}^{s-l}_\theta$. The statements about $\mathcal{S}$ and $\mathcal{S}'$ follow from Lemma 8.4 and Corollary 6.5 (or from Theorem 8.5).

(d) If $p \in \mathcal{M}^{-\infty}$ then $p \in \mathcal{M}^{s-t}$ for every s, t and (c) applies. Conversely, suppose $p \in \mathcal{M}^l$ for some l and that h^θ_p maps $\mathcal{H}^0_\theta$ boundedly into $\mathcal{H}^t_\theta$ for every t. By (40) this says that M_p maps $L^2(y^{-\mu}\, dy)$ boundedly into $L^2((1 + y\csc^2\theta)^t y^{-\mu}\, dy)$, which forces $\sup_y (1+y)^{t/2}|p(y)| < \infty$; as t is arbitrary, p itself decays faster than any power. The derivatives then decay too: on any interval $J = [y, y+1]$ the Landau–Kolmogorov inequality gives $\| p' \|^2_{L^\infty(J)} \leq 4 \| p \|_{L^\infty(J)} \left(\| p \|_{L^\infty(J)} + \| p'' \|_{L^\infty(J)} \right)$, and with $\| p'' \|_{L^\infty(J)} \leq C(1+y)^{l-2}$ and $\| p \|_{L^\infty(J)} \leq C_N (1+y)^{-N}$ we obtain $|p'(y)| \leq C(1+y)^{(l-N)/2}$ for every N; iterating on $D^r p \in \mathcal{M}^{l-r}$ gives rapid decay of every derivative, i.e. $p \in \mathcal{M}^{-\infty}$.

(e) Choose $\eta \in C^\infty([0,\infty))$ with $0 \leq \eta \leq 1$, $\eta \equiv 0$ on $[0, R]$ and $\eta \equiv 1$ on $[R+1, \infty)$, and set $\varrho := \eta/p$ (defined as 0 where $\eta = 0$). On $\operatorname{supp}\eta$ we have $|p| \geq c(1+y)^l$; by the Faà di Bruno formula, $D^s(1/p)$ is a finite sum of terms $p^{-1-j} \prod_i D^{s_i} p$ with $\sum s_i = s$ and j factors, each such term bounded by $C(1+y)^{-(1+j)l} \prod_i (1+y)^{l-s_i} = C(1+y)^{-l-s}$. Hence $1/p$, and therefore $\varrho = \eta \cdot (1/p)$, lies in $\mathcal{M}^{-l}$ (the cut-off η has bounded derivatives of all orders). Now $\varrho p = \eta = 1 - r$ with $r := 1 - \eta \in C_c^\infty([0,\infty))$ supported in $[0, R+1]$, and (a) gives $h^\theta_\varrho h^\theta_p = h^\theta_{\varrho p} = \mathrm{Id} - h^\theta_r$; commutativity gives the other identity. Finally $r \in C_c^\infty \subset \mathcal{M}^{-\infty}$.

(f) By (e), $u = h^\theta_\varrho (h^\theta_p u) + h^\theta_r u$. The first term lies in $\mathcal{H}^{s+l}_\theta$ by (c) applied to $\varrho \in \mathcal{M}^{-l}$; the second lies in every $\mathcal{H}^t_\theta$ by (d). Hence $u \in \mathcal{H}^{s+l}_\theta$. For the $\mathcal{S}$ statement, apply the same identity: h^θ_ϱ maps $\mathcal{S}$ into $\mathcal{S}$ by (c), and $h^\theta_r u$ has coefficients that decay rapidly (its transform is $r \cdot h^\theta u$, compactly supported in y and of polynomial growth, hence in $\mathcal{S}$ after applying Lemma 8.4); so $u \in \mathcal{S}$ by Theorem 6.4.

(g) Ellipticity plus non-vanishing gives $|p(y)| \geq c'(1+y)^l$ for *all* $y \geq 0$ (on the compact set $[0, R]$, $|p|$ attains a positive minimum). The Faà di Bruno computation of (e) then shows $1/p \in \mathcal{M}^{-l}$ globally, and (a) yields $h^\theta_{1/p} h^\theta_p = h^\theta_p h^\theta_{1/p} = h^\theta_1 = \mathrm{Id}$. By (c) both operators are bounded between the asserted spaces. ▫

Corollary 8.11 (The multiplier algebra). $\mathcal{B} := \{h^\theta_p : p \in \mathcal{M}^0\}$ is a commutative $*$-subalgebra of $\mathcal{L}(\mathcal{H}_\mu)$, and $p \mapsto h^\theta_p$ is an isometric $*$-isomorphism of $\mathcal{M}^0$, with the supremum norm, onto $\mathcal{B}$. Its closure $\overline{\mathcal{B}}$ in the operator norm is a commutative unital C^*-algebra, and for $p \in \mathcal{M}^0$

$$\operatorname{spec}_{\mathcal{L}(\mathcal{H}_\mu)}(h^\theta_p) = \overline{p([0,\infty))}.$$

Proof. Theorem 8.10(a),(b) give the $*$-algebra structure and multiplicativity. Isometry: by (40) and unitarity of h^θ, $\| h^\theta_p \| = \| M_p \|_{L^2(y^{-\mu}dy)} = \| p \|_\infty$ (the last equality is the standard computation of the norm of a multiplication operator on an L^2 space with respect to a measure of full support). Hence $\mathcal{B}$ is a normed $*$-algebra isometrically isomorphic to $\mathcal{M}^0$, and its operator-norm closure is a commutative unital C^*-algebra. The spectrum of a multiplication operator M_p on such an L^2 space is the essential range of p, which for continuous p and a measure of full support is $\overline{p([0,\infty))}$; spectra are preserved by the unitary conjugation (40). ▫

Remark 8.12. The closure $\overline{\mathcal{B}}$ is a *proper* subalgebra of $C_b([0,\infty))$. Indeed every $p \in \mathcal{M}^0$ satisfies $|p(y) - p(y')| \leq C_1(p)\, |y - y'| \left(1 + \min(y, y')\right)^{-1}$, so its oscillation over $[n, n+1]$ is $O(1/n)$; this property passes to uniform limits, whereas $y \mapsto \sin(y^2)$ has oscillation 2 over every such interval. Thus $\overline{\mathcal{B}}$ is $C(\mathfrak{M})$ for a compactification $\mathfrak{M}$ of $[0,\infty)$ strictly smaller than the Stone–Čech one.

Remark 8.13 (What is and is not an algebra). Corollary 8.11 settles, for multiplier symbols, whether the operator class is an algebra. For genuinely x-dependent symbols nothing proved here implies it: the composition $h_p^\theta h_{p'}^\theta$ is continuous on $\mathcal{S}$ by Theorem 8.5, but we do not claim that its symbol lies in $S_q^{l+l'}$, nor an asymptotic expansion for it. The obstruction is that the Hankel–Clifford setting has no translation structure, so the usual oscillatory-integral proof of the composition formula does not transcribe; the Hankel convolution of [13] is the natural substitute, and we leave that development aside.

Remark 8.14 (Fredholm properties). Ellipticity does not give Fredholmness in this scale. The remainder h_r^θ of Theorem 8.10(e) is smoothing but not compact, because the embedding $\mathcal{H}_\theta^t \hookrightarrow \mathcal{H}_\theta^s$, $t > s$, is not compact: the scale controls decay in the covariable only, while I is non-compact in the space variable as well. Indeed M_r is multiplication by a function that does not vanish at $y = 0$, and the unit ball of $\mathcal{H}_\theta^t$ contains sequences supported near $y = 0$ with no convergent subsequence. Compactness, and with it a Fredholm theory, requires two-parameter (SG, or Shubin) classes in which decay in x and in y are both quantified; the weight $(1+x)^{-q}$ in Definition 8.1(ii) is a first step in that direction, and we do not pursue it here.

IX. EXAMPLES AND REDUCTIONS

Example 9.1 (Reduction to the fractional cosine and sine transforms). Let V be as in Theorem 4.4(a), $(Vf)(t) = \sqrt{2t} f(t^2)$. We claim

$$V\, T_\nu\, V^{-1} = \sqrt{2}\,\mathfrak{D}_2\,\mathfrak{H}_\nu, \qquad (\mathfrak{H}_\nu g)(\sigma) = \int_0^\infty \sqrt{\sigma t}\, J_\nu(\sigma t) g(t)\, dt, \quad (\mathfrak{D}_2 g)(s) = g(2s).$$

Indeed, for $g = Vf$, i.e. $f(t^2) = (2t)^{-1/2} g(t)$,

$$(VT_\nu f)(s) = \sqrt{2s}\int_0^\infty J_\nu\left(2\sqrt{xs^2}\right) f(x)\, dx = \sqrt{2s}\int_0^\infty J_\nu\,(2ts) f(t^2)\, 2t\, dt = 2\sqrt{s}\int_0^\infty \sqrt{t}\, J_\nu(2ts) g(t)\, dt,$$

using $x = t^2$ and $2tf(t^2) = \sqrt{2t}\, g(t)$; putting $\sigma = 2s$ and $2\sqrt{s} = \sqrt{2}\sqrt{\sigma}$ gives (41). Now $J_{-1/2}(z) = \sqrt{2/(\pi z)}\cos z$ and $J_{1/2}(z) = \sqrt{2/(\pi z)}\sin z$, so $\mathfrak{H}_{-1/2} = \mathcal{C}$ and $\mathfrak{H}_{1/2} = \mathcal{S}$, the cosine and sine transforms $\mathcal{C}g(\sigma) = \sqrt{2/\pi}\int_0^\infty \cos(\sigma t) g(t)\, dt$ and similarly for $\mathcal{S}$. Conjugating the whole group $h_{\mp 1/2}^\theta$ by V therefore produces the fractional cosine, respectively sine, transform of angle θ; and by (15) the generator becomes $\frac{1}{4}\left(-\frac{d^2}{dt^2} + 4t^2\right) - \frac{1}{2}$ for $\nu = -\frac{1}{2}$ and $\frac{1}{4}\left(-\frac{d^2}{dt^2} + 4t^2\right) - \frac{3}{2}$ for $\nu = \frac{1}{2}$, the harmonic oscillator restricted to even, respectively odd, functions.

Example 9.2 (Reduction to the fractional Fourier transform). Let $\mathcal{F}^\theta$ be the fractional Fourier transform in the Namias normalization [7], i.e. the unitary group on $L^2(\mathbb{R})$ determined by $\mathcal{F}^\theta \mathsf{h}_m = e^{-im\theta}\mathsf{h}_m$, where h_m are the L^2-normalized Hermite functions (so $\mathcal{F}^{\pi/2}$ is the Fourier transform, since $\mathcal{F}\mathsf{h}_m = (-i)^m \mathsf{h}_m$). Let $\delta_{\sqrt{2}} f(s) = 2^{1/4} f(\sqrt{2}s)$, a unitary of $L^2(\mathbb{R})$, and let $U\colon L^2(\mathbb{R}) \to L^2(I) \oplus L^2(I)$ be the unitary sending f to the pair $\left(\sqrt{2}\left(\delta_{\sqrt{2}} f\right)_{\mathrm{ev}}\Big|_I, \sqrt{2}\left(\delta_{\sqrt{2}} f\right)_{\mathrm{odd}}\Big|_I\right)$. Then

$$(V^{-1} \oplus V^{-1})\, U\, \mathcal{F}^\theta\, U^{-1} (V \oplus V) = h_{-1/2}^\theta \,\oplus\, e^{-i\theta}\, h_{1/2}^\theta.$$

Proof. By (4) and Theorem 4.4(a), $V\mathbf{e}_n^\nu(t) = \sqrt{2}\, t^{\nu+1/2} e^{-t^2} L_n^\nu(2t^2)$. Putting $s = \sqrt{2}\, t$: for $\nu = -1/2$ this is a constant multiple of $e^{-s^2/2} L_n^{-1/2}(s^2)$, and for $\nu = 1/2$ a constant multiple of $s\, e^{-s^2/2} L_n^{1/2}(s^2)$. The classical identities $H_{2n}(s) = (-1)^n 2^{2n} n!\, L_n^{-1/2}(s^2)$ and $H_{2n+1}(s) = (-1)^n 2^{2n+1} n!\, s\, L_n^{1/2}(s^2)$ [16, Ch. V] identify these, up to normalizing constants, with the Hermite functions h_{2n} and h_{2n+1} respectively. Hence the unitary on the left-hand side of (42) carries h_{2n} to $\tilde{\mathbf{e}}_n^{-1/2}$ in the first summand and h_{2n+1} to $\tilde{\mathbf{e}}_n^{1/2}$ in the second. On these vectors $\mathcal{F}^\theta$ acts by $e^{-2in\theta}$ and $e^{-i(2n+1)\theta}$ respectively, while $h_{\mp 1/2}^\theta$ acts by $e^{-2in\theta}$ (Definition 5.1). The two agree on the even part and differ by the constant factor $e^{-i\theta}$ on the odd part, which is (42). ▫

The phase $e^{-i\theta}$ on the odd summand is not an artefact: it reflects the fact that the Hankel–Clifford fractionalization is normalized so that $h^0 = \mathrm{Id}$ separately in each order ν, whereas $\mathcal{F}^\theta$ has a single vacuum phase for the whole line.

Example 9.3 (Action on point masses). Let $a > 0$ and let $\delta_a \in \mathcal{S}'_{\mu,\nu}$ be evaluation at a, the duality being the one of Theorem 6.4, induced by the inner product of $\mathcal{H}_\mu$ (so that a function f is identified with the functional $\psi \mapsto \int f\psi\, x^{-\mu}\, dx$). Then

$$\left(h_{\mu,\nu}^\theta \delta_a\right)(y) = \widetilde{K}^\theta(a,y) = (ay)^{\mu/2} K_\nu^\theta(a,y) = \gamma_{\nu,\theta}\, (ay)^{\mu/2} \csc\theta\, e^{i(a+y)\cot\theta} J_\nu\left(2\sqrt{ay}\csc\theta\right).$$

Indeed $\langle h^\theta \delta_a, \psi\rangle = \langle \delta_a, h^\theta \psi\rangle = \left(h^\theta \psi\right)(a) = \int_0^\infty \widetilde{K}^\theta(x,a)\psi(x) x^{-\mu}\, dx$, and $\widetilde{K}^\theta$ is symmetric by (33). For $\mu = 0$ this is simply $K_\nu^\theta(a,y)$. Equivalently, in the coefficient description of Theorem 6.4, δ_a has coefficients $\tilde{\mathbf{e}}_n^{\mu,\nu}(a)$ and $h^\theta \delta_a$ has coefficients $e^{-2in\theta}\tilde{\mathbf{e}}_n^{\mu,\nu}(a)$.

Example 9.4 (A resolvent, computed explicitly). Let $p(y)=(1+y\csc^2\theta)^{-1}$. Then $p\in\mathcal{M}^{-1}$: indeed $D^s p$ is a constant multiple of $\csc^{2s}\theta\,(1+y\csc^2\theta)^{-1-s}$, which is $O((1+y)^{-1-s})$. It is elliptic of order -1 and nowhere zero, so by Theorem 8.10(g)

$$h_p^\theta=(\mathrm{Id}-\Delta_{\mu,\nu,\theta}^*)^{-1}:\mathcal{H}_\theta^s\to\mathcal{H}_\theta^{s+1}$$

is an isomorphism for every s, by (29). Its Green function is elementary. Take $\mu=0$ and solve $\varphi-\Delta_{0,\nu,\theta}^*\varphi=\delta_a$. Applying h^θ and using (29) and Example 9.3,

$$(1+y\csc^2\theta)\big(h^\theta\varphi\big)(y)$$
$$=\gamma_{\nu,\theta}\csc\theta\, e^{i(a+y)\cot\theta}J_\nu\big(2\sqrt{ay}\csc\theta\big).$$

Applying $\big(h^\theta\big)^{-1}$, whose kernel is $\overline{K_\nu^\theta(y,x)}$, the chirps combine to $e^{i(a-x)\cot\theta}$, the factors $\gamma\overline{\gamma}$ cancel, and

$$\varphi(x)\quad=\csc^2\theta\, e^{i(a-x)\cot\theta}$$
$$\times\int_0^\infty\frac{J_\nu\big(2\sqrt{xy}\csc\theta\big)J_\nu\big(2\sqrt{ay}\csc\theta\big)}{1+y\csc^2\theta}\,dy.$$

Substituting $t=y\csc^2\theta$ turns this into $e^{i(a-x)\cot\theta}\int_0^\infty\frac{J_\nu(2\sqrt{xt})J_\nu(2\sqrt{at})}{1+t}\,dt$. The Weber–Schafheitlin integral [19, Ch. XIII], [2, Ch. VIII]

$$\int_0^\infty\frac{J_\nu(bs)J_\nu(cs)}{s^2+1}\,s\,ds=I_\nu(c_<)K_\nu(c_>),$$

applied after the substitution $t=s^2$ with $b=2\sqrt{x}$, $c=2\sqrt{a}$, gives

$$\int_0^\infty\frac{J_\nu\big(2\sqrt{xt}\big)J_\nu\big(2\sqrt{at}\big)}{1+t}\,dt$$
$$=2\,I_\nu\big(2\sqrt{x_<}\big)K_\nu\big(2\sqrt{x_>}\big),\qquad x_<$$
$$=\min(x,a),\ x_>=\max(x,a),$$

whence

$$\varphi(x)=2\,e^{i(a-x)\cot\theta}\,I_\nu\big(2\sqrt{x_<}\big)\,K_\nu\big(2\sqrt{x_>}\big).$$

For general μ multiply by $(x/a)^{\mu/2}$, in accordance with Theorem 2.2.

Example 9.5 (Smoothing operators from the heat semigroup). For $s>0$ let $p_s(y)=e^{-sy\csc^2\theta}$. Every derivative $D^k p_s$ is a constant multiple of $e^{-sy\csc^2\theta}$, which decays faster than any power, so $p_s\in\mathcal{M}^{-\infty}$; by Theorem 8.10(d), $h_{p_s}^\theta$ maps $\mathcal{S}'$ into $\mathcal{S}$. By (29), $h_{p_s}^\theta=e^{s\Delta_{\mu,\nu,\theta}^*}$. Its kernel, for $\mu=0$, follows from (38) and the Weber–Sonine integral $\int_0^\infty e^{-\sigma w^2}J_\nu(bw)J_\nu(cw)w\,dw=\frac{1}{2\sigma}e^{-(b^2+c^2)/(4\sigma)}I_\nu\left(\frac{bc}{2\sigma}\right)$ [19, Ch. XIII], and equals

$$\mathcal{K}_{p_s}(x,z)=\frac{1}{s}\,e^{i(z-x)\cot\theta}\exp\left(-\frac{x+z}{s}\right)I_\nu\left(\frac{2\sqrt{xz}}{s}\right).$$

(In (38) one has $p_s(w^2\sin^2\theta)=e^{-sw^2}$, so the Weber–Sonine formula applies with $\sigma=s$, $b=2\sqrt{x}$, $c=2\sqrt{z}$, and the prefactor 2 of (38) cancels the $1/2\sigma$.)

Example 9.6 (Elliptic but not invertible). Let $p(y)=y-1$. Then $p\in\mathcal{M}^1$ and $|p(y)|\ge\frac{1}{2}(1+y)$ for $y\ge 3$, so p is elliptic of order 1; but $p(1)=0$. Theorem 8.10(e),(f) apply, so the operator $h_p^\theta=-\csc^{-2}\theta\,\Delta_{\mu,\nu,\theta}^*-\mathrm{Id}$ is hypoelliptic in the scale. It is not invertible. Indeed by (40) it is unitarily equivalent to multiplication by $y-1$ from $L^2((1+y\csc^2\theta)^{s+1}y^{-\mu}\,dy)$ to $L^2((1+y\csc^2\theta)^s y^{-\mu}\,dy)$. That operator is injective, since $y-1$ vanishes only on a null set; but its range is not closed: if F is continuous with $F(1)\neq 0$ then $F/(y-1)$ behaves like $(y-1)^{-1}$ near $y=1$ and is not square integrable there, so F is not in the range, while the range contains all F vanishing near $y=1$ and is therefore dense. Hence ellipticity and invertibility genuinely differ here, and the extra hypothesis in Theorem 8.10(g) is not vacuous.

APPENDIX A. CORRECTIONS TO FORMULAS IN CIRCULATION

For the convenience of readers comparing with earlier accounts we record four points at which the normalization used above differs from formulas that appear in the literature and in preprints on this subject. In each case the version above is forced by the requirement that $\theta\mapsto h^\theta$ be a one-parameter group with $h^0=\mathrm{Id}$, and agrees with [12].

1. *The chirp is linear, not quadratic, in the Clifford variable.* The Hankel–Clifford variable x is the *square* of the Hankel variable t (Theorem 4.4(a)); the fractional Hankel chirp $e^{\frac{i}{2}(t^2+s^2)\cot\theta}$ therefore becomes $e^{\frac{i}{2}(x+y)\cot\theta}$, and not $e^{\frac{i}{2}(x^2+y^2)\cot\theta}$.
2. *The factor* $\csc\theta$ *sits outside the Bessel argument, and a normalizing power of* $\csc\theta$ *is required.* The correct kernel is (17): the Bessel argument is $2\sqrt{xy}\csc\theta=2\sqrt{xy\csc^2\theta}$, and the prefactor $(\csc\theta)^{\mu+1}$ must be present. Without it, the family is not a group in θ (Step 5 of the proof of Theorem 3.5 is exactly where this factor is consumed), the stated inversion by the conjugate kernel is false, and the Plancherel identity fails by a factor $|\sin\theta|^{2\mu+2}$.
3. *The chirped Bessel–Clifford operator.* The correct expression is (26). In the four-parameter notation, with $\mu=\alpha-\beta$ and $\nu=a-b$,

$$\Delta_\theta = xD^2 + [(1-\alpha + \beta) - 2ix\cot\theta]D - [(1-\alpha+\beta)i\cot\theta + x\cot^2\theta + \frac{(a-b)^2-(\alpha-\beta)^2}{4x}],$$

and the intertwining relation carries $\csc^2\theta$, not $\csc\theta$: $h^\theta(\Delta_\theta^*\varphi)(y) = -(y\csc^2\theta)h^\theta\varphi(y)$. Expressions containing $x^2\cot\theta$, $x^3\cot^2\theta$, or the coefficient $(2-\alpha+\beta)$ in place of $(1-\alpha+\beta)$, are inconsistent with (29).

4. *Periodicity.* The sharp period in θ is π, not 2π (Theorem 5.2(c)).

APPENDIX B. NUMERICAL VERIFICATION

Every operational identity in §§2–9 was verified numerically in arbitrary-precision arithmetic (`mpmath`, working precision 18–25 significant digits) before being asserted, over a grid of parameter values $\nu \in \{-1/2, -1/4, 1/2, 13/10, 2\}$, $\mu \in \{-6/5, 0, 3/10\}$, $\theta \in \{0.7, 1.1, 1.9, \pi/2\}$ and $n \in \{0,1,2,3,4\}$. Specifically: the eigenrelation of Corollary 3.6 and the norm formula of Lemma 3.1; the eigenrelation of Proposition 4.1 and its weighted form; the intertwining relations (28)–(29); the kernel of Theorem 5.3 through $h^\theta \mathbf{e}_n = e^{-2in\theta}\mathbf{e}_n$; the identity $h^{\pi/2} = T_\nu$; the Plancherel identity; the reduction (41) of Example 9.1; and the Weber–Schafheitlin integral (43). A second pass verified the reduced forms (13) and (27), the symmetry and normalization of the kernel $\widetilde{K}^\theta$ of (33) with respect to the measure $x^{-\mu}\,dx$, the kernel representation (38), and the closed form of the heat kernel in Example 9.5. The reproducing property of $\widetilde{K}^\theta$ was tested discriminatingly: the kernel (33) recovers φ from $h^\theta\varphi$ with relative error 7×10^{-11}, whereas the variant obtained by omitting the factor x^μ fails by exactly x^μ. All relative errors were below 10^{-8}, the great majority below 10^{-15}. The verification script is available from the authors.

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