The Application of Black Hole String Solutions in Three Dimensions

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***Abstract*—In this paper, we have presented that a simple gauged WZW model gives changed black hole string solutions in three dimensions. For 0< | q | < M, the solutions are qualitatively similar to the Reissner – Nordstrom solution. The solution describes an event horizon, an inner horizon and a time like singularity. For |q| = M the space-time shows an event horizon but no singularity. For |q| > M both the horizon and the curvature singularity vanish. The concept of black hole solution to three-dimensional with general relativity is analysed. We demonstrate that an exact solution to string theory can be obtained by slightly altering this answer. This black hole solution is equivalent to the three-dimensional black string solution.**

***Index Terms*—Black Hole String Solution, WZW model, Conformal field, Space-time, Inner horizon. Axion Charge (q), Larentz metric, Dilation, Curvature singularity. Lorentz signature.**

I. INTRODUCTION

The string theory was presented by Strominger and Horowitz [2] has a rich variety of solutions representing extended objects surrounded by event horizon. Three black hole string solutions in ten dimensions, especially the mass, axion charging per unit length, and asymptotic value of the dilation are distinguished by three parameters. Although it is sufficient to confirm the existence of exact black hole string solutions with these qualitative properties, it was not obvious how to obtain directly the conformal field theory which these features. Witten [3] obtained that a simple gauged WZW model gives a two-dimensional black hole. This presented the possibility of using similar construction to obtain exact conformal field theory corresponding to higher dimensional black holes or extended black holes. Giddings and Stromige [1] obtained the conformal field theory associated with an external limit of the charged black hole fivebranes. Danny Birmingham et.al [9] has deliberate three-dimensional black holes and string theory and Hu, B.L. and Verdagur, E [5] has analysed about the Stochastic Gravity. Stochastic semiclassical gravity is based on the Einstein-Langevin equation, which has additional sources, as opposed to semi classical gravity, which is based on the semiclassical Einstein equation with sources determined by the expectation value of the stress-energy tensor of quantum fields.

Dawood, A.K. and Ghosh, S.G [4] has studied about the generating dynamical black hole solutions using Einstein equations. M. Banados, C. Teitelboim, and J. Zanelli [10] has observed about the black hole in three-dimensional space time and Salgado, M [6] proved a theorem characterizing a three-parameter family of static and spherically symmetric black hole solutions to Einstein equations by imposing certain conditions on the energy-momentum tensor [EMT]. Stephane Detournay, P. Marios Petropoulos and Celine Zwikel [7] has discussed about asymptotic symmetries of three-dimensional black strings. Byon N. Jayawiguna [8] has studied about three-dimensional black hole in the low energy heterotic string theory.

The development of string theory has been greatly aided by supersymmetric supergravity solutions. The present state of black hole solutions in higher-dimensional supergravity theories is reviewed in this article. We focus on supersymmetric black holes' and their relatives' gravitational properties in several dimensions. Prime examples are used to discuss supersymmetric solutions and their systematic derivation. Additionally, using dimensional reduction and duality transformations, we look at the stationary or dynamically crossing branes in 10 and eleven dimensions, which give rise to a number of intriguing black objects.

II. MATHEMATICAL FORMULATION

In this paper, we are interested to show that a simple extension of Witten's construction gives three-dimensional charged black hole string solutions. These solutions are characterized by three parameters, namely, the mass 'M', axion charge 'q' per unit length and a constant 'k' related to the asymptotic value of the derivative of the dilation. The low energy metric, antisymmetric tensor and dilation assume the form

$$ds^2 = -\left(1-\frac{M}{r}\right)dt^2 + \left(1-\frac{q^2}{Mr}\right)dx^2 + \left(1-\frac{M}{r}\right)^{-1}\left(1-\frac{q^2}{Mr}\right)^{-1}\frac{k\,dr^2}{8r^2}, \quad (1)$$

$$Q_{rts} = \frac{q}{r^2}, \quad (2)$$

$$\phi = \ell n r + \frac{1}{2}\ell n\frac{k}{2} = \ell n\left(r\left(\frac{k}{2}\right)^{1/2}\right) \quad (3)$$

Black Hole String Solutions:

Let us describe the conformal field theory that gives the black hole string solutions. One may use a Larentz metric

$ds^2 = 2d\sigma_+ + d\,\sigma_-$ (4)

On the world sheet Σ because our target space has Lorentz signature. Let g be the element of group G, then ungauged WZW (Wess –Zumino -Witten) action assumes the form

$$L(g) = \frac{k}{4\pi}\int_\Sigma d^2\sigma T_r\left(g^{-1}\partial_+ g g^{-1}\,\partial_-\, g\right) - \frac{k}{12\pi}\int_B T_r\left(g^{-1}\,d\,g \wedge g^{-1}\;\,d\,g \wedge g^{-1}dg\right) \quad (5)$$

where B represents three-manifold with boundary Σ. The one-dimensional subgroup H of the symmetry group of equation (5) may be gauged with action g→hgh, as the global symmetry. This global symmetry may be obtained locally by introducing a gauge field A_i which takes values of H in the lie algebra. Let us consider ∈ as the infinitesimal gauge parameter, and then the local axial symmetry is generated by

$\delta\, g = \in g + g \quad \in$ (6)

$A_i = -\partial_i \in$ (7)

This local axial symmetry is symmetry of the gauged WZW action

$$L(g, A) = L(g) + \frac{k}{2\pi}\int_\Sigma d^2\sigma \times T_r\left(A_+ + \partial_- g\, g^{-1} + A_-\, g^{-1}\partial_+\, g + A_+ A_- + A_+\, g\, A_- g^{-1}\right) \quad (8)$$

It has been demonstrated by Witten that if G is SL (2, R) and H is the subgroup generated by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (9)$$

then the gauged WZW action gives a black hole in two-dimensions.

In view of equations (6) and (7), one obtains

$\delta\,(g+A_i) = \in g + g\in - \partial_i \in$. (10)

Let us now generalize this construction by adding one free boson × to the action which is equivalent to let

G = SL (2, R) × R (11)

then by gauging the one-dimensional subgroup generated by

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (12)$$

together with a translation of x.

Hence, we get

$$g_{SL}\,(2, R) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (13)$$

with

a b + u v = 1 (14)

$$L(g) = -\frac{k}{4\pi}\int_\Sigma d^2\sigma\left(\partial_+ u\partial\_v + \partial\_u\partial_+ v + \partial_+ a\partial\_b + \partial\_a\partial_+ b\right) + \frac{k}{2\pi}\int_\Sigma d^2\sigma\, \ell gu\left(\partial_+ a\partial\_b - \partial\_a\partial_+ b\right) + \frac{1}{\pi}\int_\Sigma d^2\sigma\, \partial_+ x\partial\_x \quad (15)$$

Let us gauge

$\delta\, a = 2 \in a,$ (16)

$\delta\, b = -2 \in b,$ (17)

$\delta\, u = \delta\, v = 0$ (18)

$\delta\, x = 2 \in c$ (19)

$\delta\, A_i = -\partial_i \in$ (20)

where c be an arbitrary constant.

Now one may obtain the full action

$$L(g, A) = L(g) + \frac{k}{2\pi}\int_\Sigma d^2\sigma\, A_+\left(b\partial_- - a\partial\_b - u\partial\_v + v\partial\_u + \frac{4c}{k}\partial\_x\right) + A_-\left(b\partial_+ a^- a\partial_+ b + u\partial_+ v - v\partial_+ u + \frac{4c}{k}\partial_+ x\right) + 4A_+A_-\left(1 + \frac{2C^2}{k} - uv\right) \quad (21)$$

In view of Witte [6], one may fix gauge by setting a = ±b, depending on the sing 1–uv. Along this gauge choice and eliminating A, the action reads

$$L = -\frac{k}{8\pi}\int_{\Sigma} d^2\sigma \times \lambda v^2 \partial_+ u\, \partial\_u + \lambda u^2 \partial_+ v \partial\_v$$

$$+\left(2-2uv+2\lambda-\lambda uv\right)\frac{\left(\partial_+ u\partial\_v+\partial\_u\partial_+ v\right)}{(1-uv)(1+\lambda-uv)}$$

$$+\frac{1}{\pi}-\int_{\Sigma} d^2\sigma\,\frac{(1-uv)}{(1+\lambda-uv)}\partial_+ x\,\partial\_x+\frac{1}{2\pi}\int_{\Sigma} d^2\sigma$$

$$\frac{c\left(v\partial_+ u\partial_- - v\partial\_u\partial_+ x - u\partial_+ v\partial\_x + u\partial\_v\partial_+ x\right)}{(1+\lambda-uv)} \quad (22)$$

Where

$$\lambda = \frac{2C^2}{k} \quad (23)$$

Let us now redefine the field

$$u = \exp\left[\frac{\sqrt{2}t}{\sqrt{k(1+\lambda)}}\right]\sqrt{\bar{r}-(1+\lambda)} \quad (24)$$

$$v = -\exp\left[\frac{\sqrt{2}t}{\sqrt{k(1+\lambda)}}\right]\sqrt{\bar{r}-(1+\lambda)} \quad (25)$$

In the view of equations (24) and (25) the action assumes the form

$$L = \frac{1}{\pi}\int_{\Sigma} d^2\sigma\,\frac{k\,\partial_+\bar{r}\,\partial\_\bar{r}}{8\bar{r}^2\left(\left(1-\frac{\lambda}{\bar{r}}\right)\left(1-\frac{1+\lambda}{r}\right)\right)}-\left(1-\frac{1+\lambda}{r}\right)\partial_+ t\,\partial\_t$$

$$+\left(1-\frac{\lambda}{r}\right)\partial_+ x\,\partial\_x+\sqrt{\frac{\lambda}{1+\lambda}}\left(1-\frac{1+\lambda}{\bar{r}}\right)\left(\partial_+ x\,\partial\_t-\partial\_x\,\partial_+ t\right) \quad (26)$$

The above action describes a string propagating in space time with the line element

$$ds^2 = -\left(1-\frac{1+\lambda}{\bar{r}}\right)dt^2+\left(1-\frac{\lambda}{\bar{r}}\right)dx^2+\left(1+\frac{1+\lambda}{\bar{r}}\right)^{-1}\left(1-\frac{\lambda}{\bar{r}}\right)^{-}\frac{k}{8r^2}d\bar{r}^2 \quad (27)$$

with an antisymmetric field

$$D_{tx} = \sqrt{\frac{\lambda}{1+\lambda}}\left(1-\frac{1+\lambda}{\bar{r}}\right) \quad (28)$$

It is obvious that the exact central charge of this gauged WZW model reads

$$\frac{3}{k-2} \quad (29)$$

which is larger than the two-dimensional black hole, since we have added an extra boson. Equations (27) and (28) provide the lowest order relations for the metric and antisymmetric field, but quantum corrections will give corrections. There is also dilation which comes from effects. It may be obtained from the requirement that fields must be an extreme of the low-energy string action

$$S = \int e^{\phi}\left[R-(\nabla\phi)^2-\frac{1}{12}H^2+\frac{8}{k}\right] \quad (30)$$

The antisymmetric tensor field be extremum if

$$\phi = \ell n\bar{r} + a \quad (31)$$

where a be an arbitrary constant.

For

$$\bar{\mathrm{r}} = 0,\ \lambda,\ 1+\lambda \quad (32)$$

The metric components are well behaved.

Let us evaluate scalar curvature

$$R = \frac{4\left(2\bar{r}+4\lambda\bar{r}-7\lambda-7\lambda^2\right)}{k\bar{r}^2} \quad (33)$$

Hence, $\bar{r} = 0$ gives curvature singularity. It is obvious from equation (33), the difficulties at $\bar{r} = \lambda$ and $\bar{r} = 1+\lambda$ may be removed by an appropriate change of coordinates. In deed the original coordinates (u, v, x) are well behaved at uv = 0 which corresponds to $\bar{r} = 1+\lambda$. But for uv = 1, $\bar{r} = \lambda$, then u, v, x coordinates are not well behaved. In means our gauge fixing breaks down there.

Let us call $\bar{r} = 1+\lambda$ as an event horizon. For large $\bar{r}$ the metric becomes asymptotically flat.

Hence, the solution describes static, black hole string, because the solution is invariant under the translations of t and x.

As $\bar{r}$ goes to infinity, g_{ij} and g^{ij} the metric components approach to unity. Similarly, it is not possible to fix the overall scaling of the coordinate $\bar{r}$ since the metric asymptotically approaches $\frac{kdr^{-2}}{8\bar{r}^2}$.Hence, the dilation φ must be of the form

$$\phi = \ell nr + \frac{1}{2}\ell n\left(\frac{k}{2}\right) \quad (34)$$

Hence, we may put

$$\bar{r} = r\,e^{-a}\sqrt{\frac{k}{2}} \quad (35)$$

In equations (27), (28) and (31).
Let us no evaluate the axion charge for n dimensions the action given by equation (30) that the n–3 form

$$K = \frac{1}{2} e^{\phi} + H \tag{36}$$

Be curl free and + denotes the Hodge dual. K must be constant for three dimensions. Hence, the axion charge per unit length be the value of this constant. For black hole string solutions, one obtains

$$q = e^{a} \sqrt{\frac{2\lambda(1+\lambda)}{k}} \tag{37}$$

For $\lambda = 1$, we get
$K = 2C^2$, (38)

$$q = e^{a}\sqrt{\frac{4}{k}} = e^{a}\sqrt{\frac{4}{C^2}} \tag{39}$$

Let us compute the mass per unit length of the string by using ADM standard method. For large r the black hole string solutions approach the asymptotic solution
$ds^2 = -dt^2 + dx^2 + dp^2$ (40)
where

$$P = \sqrt{\frac{k}{8}} \square n\left(r\sqrt{\frac{k}{2}} \right) \tag{41}$$

and H = 0, (42)
with

$$\phi = P\sqrt{\frac{8}{k}} \tag{43}$$

Let us extremis the action given by equation (30) to obtain field equations and also line arise this expression about the asymptotic solutions given by equation (40) and (43). Now integrating the time-time component of this equation over a constant time surface. It comes out as a surface integral at infinity because the integrand is a total time derivative.

The antisymmetric field must vanish in the background. Hence, their contributions to the field equations are

$$e^{\phi}\left[R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu} - \nabla_{\mu}\nabla_{\nu}\phi + g_{\mu\nu}\left(\nabla^2\phi + \frac{1}{2}(\nabla\phi)^2 - \frac{4}{k} \right) \right] \tag{44}$$

Let us line arise this expression in view that and k may not be perturbed. So, we perturb the metric

$$g_{\mu\nu} = \eta_{\mu\nu} + \gamma_{\mu\nu} \tag{45}$$

The line arised form of the equation (44) may be integrated over space like surface to obtain total mass formula

$$M_{total} = \frac{1}{2}\oint e^{\phi}\left[\partial^{i}\gamma_{ij} - \partial_{j}\gamma + \gamma_{ij}\partial^{i}\phi \right] ds \tag{46}$$

Where be the trace of spatial components of $\gamma_{\mu\nu}$.

In view of specific form of and the fact that x measures proper distance at infinity, one obtains the mass per unit length

$$M = \sqrt{\frac{2}{k}}(1+\lambda)e^{a} \tag{47}$$

Hence, in view of the above results, we get the black hole string solutions

$$ds^2 = -\left(1 - \frac{M}{r}\right)dt^2 + \left(1 - \frac{q^2}{Mr}\right)dx^2 + \left(1 - \frac{M}{r}\right)\left(1 - \frac{q^2}{Mr}\right)^{-1}\frac{k}{8r^2}dr^2 \tag{48}$$

$$Q_{rts} = \frac{q}{r^2} \tag{49}$$

$$\phi = \square nr + \frac{1}{2}\square n\left(\frac{k}{2}\right) \tag{50}$$

For q = 0 implies Q = 0 and we get

$$ds^2 = -\left(1 - \frac{M}{r}\right)dt^2 + \left(1 - \frac{M}{r}\right)^{-1}\frac{k}{8r^2}dr^2, \tag{51}$$

which is exactly Witten's black hole solution in two-dimensions.

Let us again introduce coordinate transformation
$r = M\cosh^2 p$ (52)

$$t = \sqrt{\frac{k}{2}}t, \tag{53}$$

III. RESULTS AND DISCUSSION

To obtain the exact form of the Witten's metric. It is observed from the metric given by equation (51) that the region beyond the singularity i.e., r < 0 has negative mass.

REFERENCES:

[1] S. B. Giddings and A. Strominger, "UCSB Preprint," UCSBTH-91-35, 1991.

[2] A. Strominger and G. T. Horowitz, “Title unavailable,” Nuclear Physics B, vol. 360, p. 197, 1991.
[3] E. Witten, “Title unavailable,” Physical Review D, vol. 44, p. 314, 1991.
[4] A. K. Dawood and S. G. Ghosh, “Title unavailable,” IUCAA Preprint IUCAA-55/P/2004, 2004.
[5] B. L. Hu and E. Verdaguer, “Stochastic gravity: A primer with applications,” Living Reviews in Relativity, vol. 7, no. 3, 2004.
[6] M. Salgado, “The Cauchy problem of scalar-tensor theories of gravity,” Classical and Quantum Gravity, vol. 20, pp. 4551–4566, 2003.
[7] S. Detournay et al., “Title unavailable,” Journal of High Energy Physics (JHEP), vol. 6, Art. no. 131, 2019.
[8] B. N. Jayawiguna, “Title unavailable,” Journal of High Energy Physics (JHEP), arXiv:2006.12663 [gr-qc], 2020.
[9] D. Birmingham, I. Sachs, and S. Sen, “Three-dimensional black holes and string theory,” Physics Letters B, vol. 424, pp. 275–280, 1998.
[10] M. Bañados, C. Teitelboim, and J. Zanelli, “The Black Hole in Three-Dimensional Space-Time,” Physical Review Letters, vol. 69, no. 13, pp. 1849–1851, 1992.