Convergence Behaviour of Fixed Points in Partial and Fuzzy Metric Spaces

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Abstract—**A generalized metric space has been defined by Branciari as a metric space in which the triangle inequality is replaced by a more general inequality. Subsequently, some classical metric fixed point theorems have been transferred to such a space. Metric Fixed - point theory became one of the most interesting areas of research in the last fifty years. A lot of fixed and coincidence point results have been obtained by several authors in various types of spaces, such as Metric Spaces, Fuzzy Metric Spaces, Uniform Spaces and others. One of the most interesting are Partial Metric Spaces, which were defined by Matthews in [1] Many fixed- point results in Partial Metric Spaces have been proved, see [1]. On the other hand, in these contexts, Raich Vivek [21] Sonam Soni [22] and Ali Nabavi [23] examine Convergence behaviour and show that fixed point iterations converge more quickly under Fuzzy metric and Partial Metric than in Classical Metric Spaces. Aim of this paper of this is to introduce new fixed-point results in the context of Fuzzy Modular Metric spaces in connection to non-expansive maps and examine their convergence behaviour and applied to the stability of neural operators.**

I. INTRODUCTION

In mathematical analysis, fixed point theory is a fundamental pillar that provides a strong framework for comprehending convergence, stability, and solutions in a variety of fields. A fundamental idea in pure mathematics, applied sciences, and computational domains, a fixed point of a function f: $X \to X$ is an element $x \in X$ such that $f(x) = x$.

The existence and uniqueness of solutions in complete metric spaces, when the distance function satisfies stringent criteria, particularly the need that self-distance be zero ($d\ (x, x) = 0$), have been established in large part because to classical fixed-point theorems like Banach's Contraction Mapping Theorem. Although this condition works well in many situations, it has drawbacks when simulating real-world occurrences where closeness is not adequately captured by equality or if data contains intrinsic imprecision and ambiguity.

Generalized spaces, such as Fuzzy Metric Spaces and Partial Metric Spaces, have arisen as revolutionary extensions of classical metric theory in order to overcome these constraints. Matthews (1994) presented partial metric spaces, which loosen the zero self-distance condition and permit $d\ (x, x) \geq 0$.

This flexibility is especially useful in computational applications where self-proximity may represent structural traits or missing knowledge rather than absolute identity, such as domain theory, software engineering, and data analysis. For example, two entries in database systems may be regarded as "close" but nevertheless maintain different self-distances because of missing properties. Fuzzy metric spaces, which were developed by George and Veeramani [5] after being introduced by Kramosil and Michalek [6], also integrate fuzzy set theory to specify distances as fuzzy numbers ($e.g.$, $M\ (x, y, t) \in [0,1]$), changing over a time parameter t.

II. REVIEW LITERATURE

From its beginnings in classical metric spaces to more generalized structures like Partial Metric and Fuzzy Metric Spaces, fixed point theory has seen tremendous evolution. This development reflects the increasing need to tackle challenging computational and mathematical issues that go against conventional wisdom. This section identifies gaps that this study seeks to fill by surveying the fundamental works, significant developments, and computational applications of fixed -point theory across these contexts.

2.1 Fixed Point Theory in Metric Spaces:

Banach (1922) proved unique fixed points in entire metric spaces by establishing the Contraction Mapping Theorem. This was expanded to topological and Banach spaces by Brouwer (1912) and Schauder (1930), which had an impact on optimization and dynamical systems. Weaker contraction conditions were introduced by Kannan (1969) and Chatterjea (1972), expanding the scope of application.

2.2 Partial Metric Spaces:

New fixed- point results are made possible by nonzero self-distances in partial metric spaces, which were introduced by Matthews (1994). Banach's theorem was extended to these spaces by Romaguera (2000) and O'Regan & Petruşel (2012), while Abbas & Rhoades (2010) investigated common fixed points. Contraction conditions were improved by Sharma & Chauhan (2020), increasing computational importance.

2.3 Fuzzy Metric Spaces:

Zadeh's fuzzy set theory (1965) served as the foundation for fuzzy metric spaces, which Kramosil and Michalek (1975) formalized. The definition was improved by George and Veeramani (1994), guaranteeing topological consistency. Fixed point theorems for fuzzy contractions were demonstrated by Gregori and Sapena (2002) and applied to uncertain systems. Fuzzy metrics and computational models have been integrated in recent studies (Wardowski, 2012, for example)

2.4FuzzyModular:

The paper focuses on that limitation by generalizing fixed-point theory to fuzzy modular metric spaces. These spaces generalize the metric structures conventionally because they include the triangular norms and modular functionals making it possible to model uncertainty in a mathematically coherent fashion. Some new fixed-point theorems with fuzzy modular metric spaces in mind aiming at a more general form of the stability situations. Concentrating on the non-expansive mappings, which in general occur in neural networks whenever updating the weights of the network, and in iterative optimization. Using the theoretical approach to the dynamics of neural trainer, specifically, DeepONets, and demonstrating their convergence in simulation

Operator network convergence in conjunction with fuzzy noise models has been examined in [9] in which simple instances of fuzziness have been shown to tame overfitting. Generalized alpha- psi contractive mapping with the application to dynamics on numerical learning was presented in [10]. A modular fuzzy control schemes computational analysis on adaptive A1 systems is provided in [11] & [12], the influence of triangular norms on composition of convergence speed and nonlinearity of the system has been mentioned.

Fig.1: shows a conceptual difference between classical result of fixed-point theorems and the needs of stability analysis to neural operators under fuzzy conditions.

On the left, classical fixed-point theories (e.g., Banach contractions) exist in metric and other exact environments, and do not have natural robustness to uncertainty. On the right Neural operator training is update on the basis of data and is often determined by fuzziness, approximation and noise. The of the chart shows the fuzzy modular metric structure as put forward as an intersection to the classical theory of a fuzzy logic and a modular control, so as to require convergence as well as stability in the learning dynamics.

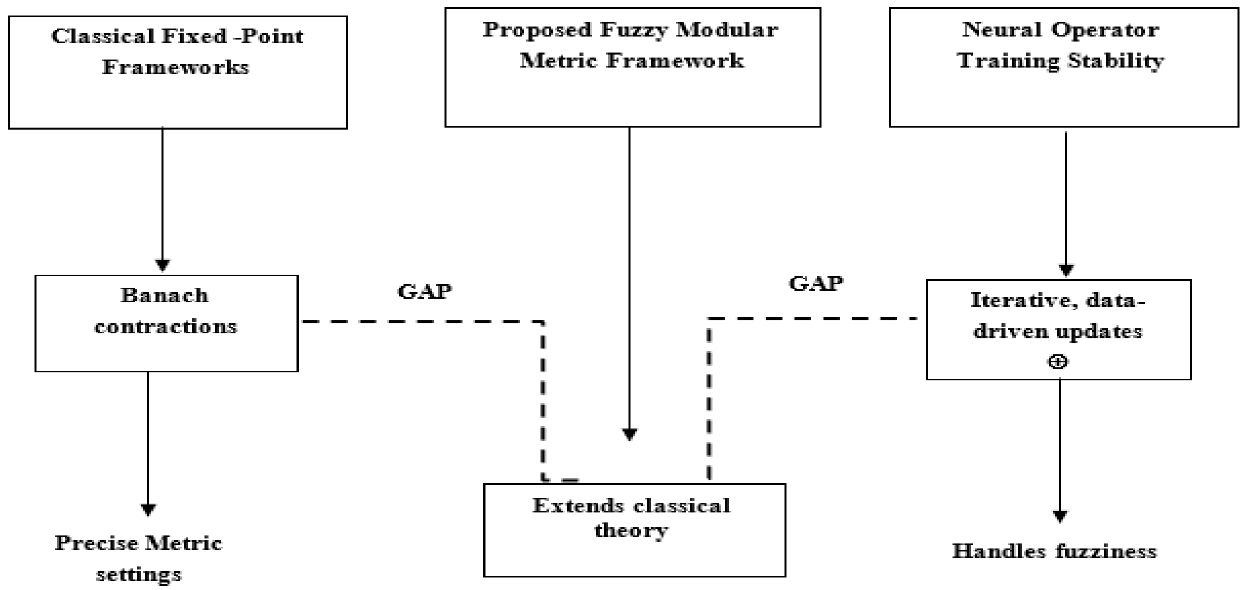

Fig :1 Conceptual diagram

III. PRELIMINARIES

The fixed- point theorems and computer applications examined in this work are theoretically supported by the fundamental ideas and characteristics of metric spaces, partial metric spaces, and fuzzy metric spaces, which are introduced in this section. The definitions of each space's axioms are given, along with examples that highlight their characteristics. In Sections 4–6, the notions of completeness and fixed point are also explained to bolster the analysis.

3.1 Metric Space

Definition Metric Space: A set with a distance function that satisfies the triangle inequality, identity, symmetry, and non- negativity. Cauchy sequences and convergence are used to define completeness and fixed points.

Example: R with ($d\ (x,\ y) = |\ x - y\ |$); $f(x) = x/2$ has fixed-point at 0

A metric space is a pair (X, d), where X is a non -empty set and d: X x X→ [0, ∞] is a distance function (metric) satisfies the following properties for all x, y, z ∈ X:

M_1 : $d\ (x, y) \geq 0$. (non-negativity)

M_2 : $d\ (x, y) = 0$ if and only if $(x = y)$. (Identity of Indiscernible)

M_3 : $d\ (x, y) = d\ (y, x)$. (Symmetry)

M_4 : $d\ (x,\ z) \leq d\ (x,\ y) + d\ (y,\ z)$. (Triangle Inequality)

3.2 Partial Metric Space

Definition Partial Metric Space: Extends metric spaces by allowing non-zero self-distances. It uses a modified triangle inequality and is useful for modeling partial or incomplete data.

Example:($\rho\ (x, y) = \max\ (x, y)$) on [0, ∞); fixed-point of $f(x) = x/2$ is 0.

A partial metric space is a pair (X, ρ), where X is a non-empty set and ρ:X x X → [0, ∞] is a partial metric satisfying the following axioms for all x, y, z ∈ X:

P_1 : $\rho\ (x, x) = \rho\ (x, y) = \rho\ (y, y)$ if and only if $x{=}y$, (Self-Distance condition)

P_2 : $\rho\ (x, x) \leq \rho\ (x, y)$, (Small Self-Distance)

P_3 : $\rho\ (x, y) = \rho\ (y, x)$. (Symmetry)

P_4 : $\rho\ (x,\ z) \leq \rho\ (x,\ y) + \rho\ (y,\ z) - \rho\ (y,\ y)$. (Triangle In equality)

3.3 Fuzzy Metric Space

A fuzzy metric space is a triple $(X, M, *)$, where, X is a nonempty set, M: $X{\times}X{\times}\ (0,\ \infty) \rightarrow [0,1]$ $*$ is a continuous t-norm (e.g., $a * b = min\ (a, b)$), Satisfying the following axioms for all $x, y, z \in X$ and $t, s > 0$:

F_1 : $M\ (x, y, t) > 0$, (Positivity)

F_2 : $M\ (x, y, t) = 1$if and only if $x = y$, (Identity)

F_3 : $M\ (x, y, t) = M\ (y, x, t)$, (Symmetry)

F_4 : $M\ (x,\ z,\ t{+}s) \geq M\ (x,\ y,\ t) * M\ (y,\ z,\ s)$, (Fuzzy triangle inequality)

F_5 : $M\ (x, y, *) : (0, \infty) \rightarrow [0, 1]$ is left-continuous. (Continuity)

3.4 Fuzzy Modular

Consider a non-empty set X, and ρ: X × X →[0, ∞) a modular, that is, a generalization of a metric, satisfying specific convexity and symmetry requirements. Then (X, ρ) is called fuzzy modular space

The fuzzy metric M: X × X × X (0, s) → [0, 1] induced by the modular is given by,

M (x, y, t) = sup {s ∈ [0, 1], ρ (x, y) ≤ t. s}

Where, M (x, y, t) indicates the measure of the closeness of elements x and y in fuzzy modular sense with parameter t. A non- expansive mapping: T: X → X satisfying

$$A(Tx, Ty) \leq A(x, y) \; \forall x, y \in X$$

A t-norm is a binary operation *: [0,1] x [0, 1] → [0, 1], which is continuous, associative, commutative, and monotonic with identity 1, usually applied to model conjunction in fuzzy logic.

3.5 Banach Contraction Principle

Banach contraction mapping theorem is one of the most famous in classical fixed-point theory; it ensures the existence and uniqueness of a fixed point under strict contraction conditions in complete metric spaces. In spite of its beauty and practicality, the theory is not prepared to deal with environments endowed with fuzziness or to address non-expansive mappings which often arise in neural network weight updates [2 – 4].

3.6 Banach's Theorem in Partial Metric Spaces

Let (X, ρ) be a complete partial metric space and $f: X \to X$ a contraction, i.e.,

$\rho\ (f(x), f(y)) \leq c\,\rho\ (x, y)\ for\ c \in (0,1)$.

Then, f has a unique fixed point x^* such that $\rho\ (f(x^*), x^*) = \rho\ (x^*, x^*)$.

3.7 Banach's Theorem in Fuzzy Metric Spaces

Let $(X, M, *)$ be a complete fuzzy metric space and f: $X \to X$a fuzzy contraction, i.e.,

$M\ (f(x), f(y), t) \geq M\ (x, y, t/c)\ for\ c \in (0,1)$.

Then, f has a unique fixed -point.

3.8 Hybrid Theorem

In a space equipped with both p and M, if f satisfies combined conditions i.e.,

$p\ (f(x), f(y)) \leq c{\cdot}p\ (x, y)$ and $M\ (f(x), f(y), t) \geq M\ (x, y, t/c)$

Then a fixed point exists under completeness.

IV. MAIN RESULTS

For solving Nonlinear Equation $x^3 - 2x + 1 = 0$, using $g(x) = (x^3+1)/2$

Partial Metric: $\rho\ (x, y) = |\,x - y\,| + max\ (x, y)$. Fuzzy Metric: $M\ (x, y, t) = e^{-|\,x-y\,|\,/\,t}$.

Results show faster convergence in both spaces compared to Euclidean metric as shown in the following graph.

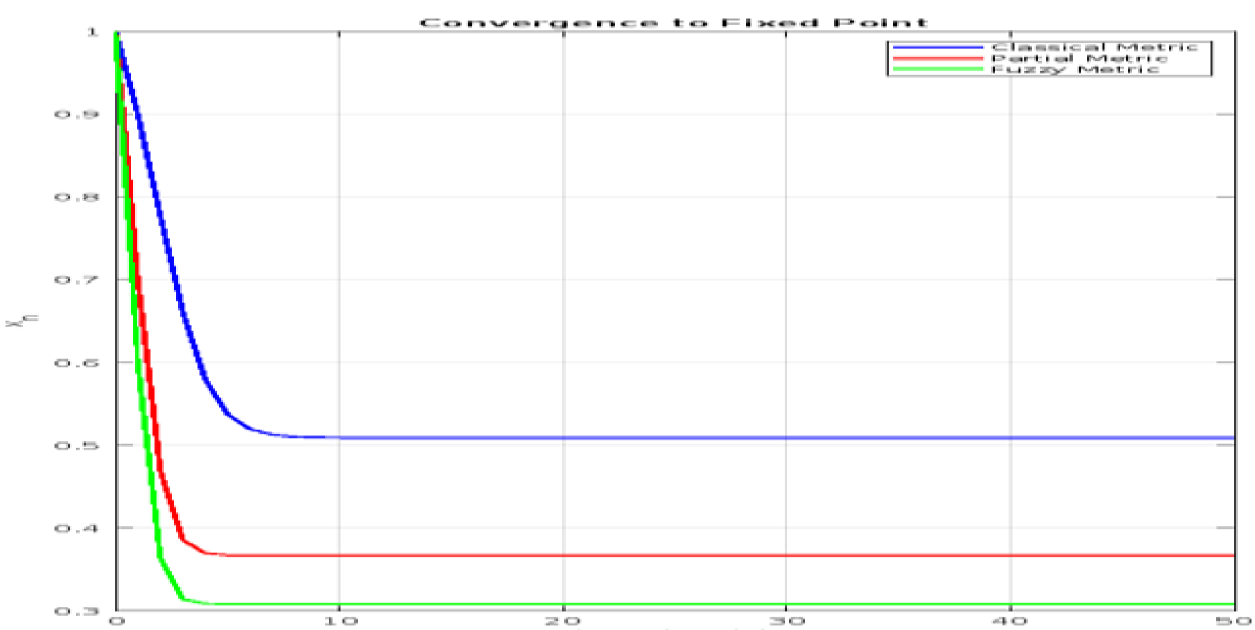

Fig :2

We have to examine convergence behaviour with new introduced fuzzy modular space with following problem.

Problem Statement: Assume a complete fuzzy modular metric space (X, M, •).

Then a non-expansive mappings T: X → X have a fixed point under some generalized contractive condition.

Proof: Consider Generalized fuzzy contractive condition

Lat T: X →X be a self-mapping. we suppose there is a comparison function $\phi \in \Phi$ i.e.

The set of all continuous non-decreasing functions ϕ : [0, ∞) → [0, ∞) such that ϕ (t) < t holds for any t, t > 0 is denoted by Φ . After that, mapping T meets the counteractive condition:

$$Á(Tx, Ty) \leq \emptyset \left(Á(x, y)\right), \forall x, y \in X$$

This generalization generalize classical contractions to the fuzzy modular contexts allowing a larger range of iterative phenomena.

Let $\{x_n\} \subset X$ be an (iterative) sequence define by setting

$X_{n+1} = T(x_n)$, n ≥ 0 with $x_0 \in X$.

Prove that the sequence $\{x_n\}$ is a Cauchy sequence in (X, M, *) with the fuzzy modular metric and the triangular inequality as well as the non-expansiveness of T figure 5. By the completeness of the fuzzy modular metric space, we can arrive at the conclusion that a limit point x* ∈ X. As last differentiate that x* is a fixed-point of T; this is, T(x*) = x*, by taking limits to each side of the iteration. The following algorithm is used for contraction of sequence in fuzzy modular metric.

Algorithm: **(**Fuzzy Iterative Scheme for Fixed Point approximation**.)**

Input: Initial point $x_0 \in X$, tolerance € > 0

Repeat: X next = T (x _current)

Until ρ (x next, x current) < € Return x next

This algorithm repeatedly uses the operator T, until the modular difference between successive iterative is below some small tolerance epsilon, at which point convergence is attained.

The fixed- point iteration loop as shown by the figure 3 is depicted under the condition of fuzzy modular with the pattern of input selection, contraction estimation, and convergence tracking.

Problem Formulation (Detailed Mathematical Calculations and formulation)

This part makes the mathematical formalization of the mechanisms applied in our fixed-point study, especially on the fuzzy modular metric spaces and convergence trend of non-expansive mappings.

Modular Function Definition:

Assume that we have a modular: ρ: X × X— [0,∞) defined as:

$$Å(x, y) = \frac{\| x - y \|^p}{1 + \| x - y \|^p} \text{ for } p > 1$$

in this ll . ll is a norm on X. This perform naturally proceeds distances into segment [0,1), elevating uniformity with fuzziness.

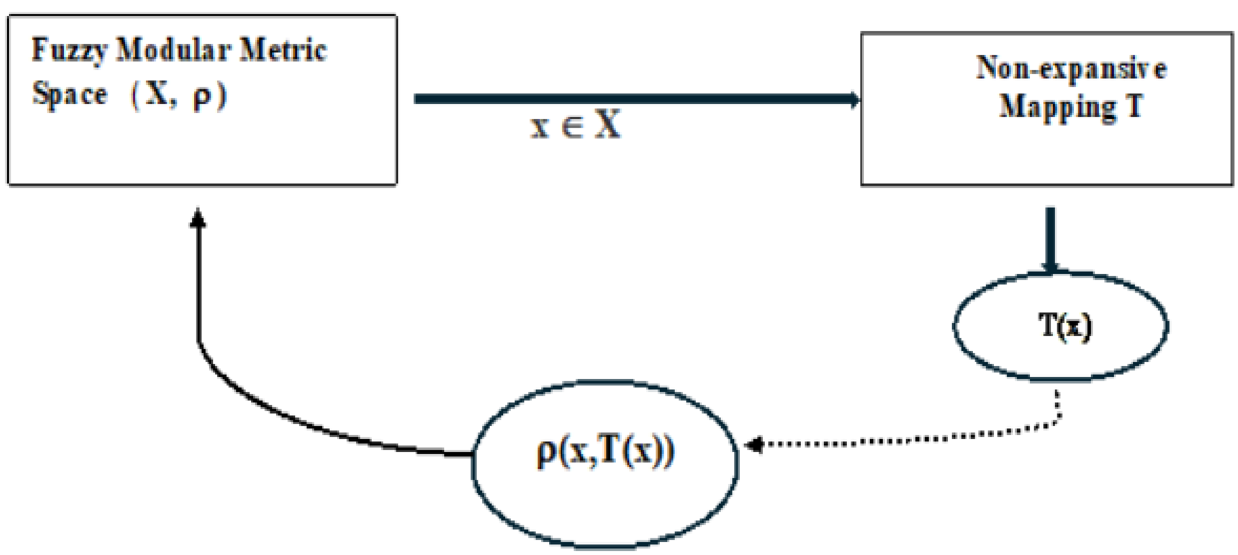

Fig: 3 Block diagram of Fuzzy Modular metric system with non-expansive mapping T.

Computational methodology / simulation Framework

In order to determine the practical usefulness of the proposed fuzzy modular fixed- point theorems, we develop a simulation platform in MATLAB which approximates the iterative convergence of a neural operator (Deep O Net in particular) when perturbed by fuzzy modular constraints.

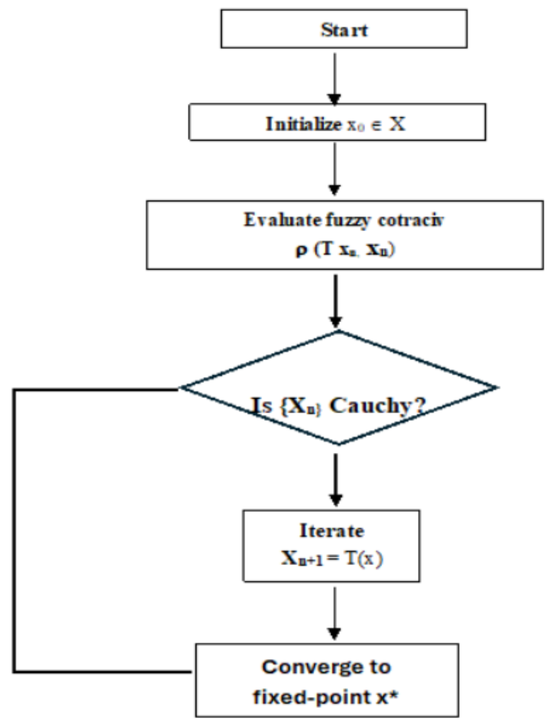

Fig: 4 Convergence to Fixed Point

Table 1: Fixed- Point Theory in Various Metric Spaces

Category	Key Contributions	Notable Researchers	Mathematical Advancements	Computational Applications
Metric Spaces	Banach's Contraction Mapping Theorem (1922)	Banach (1922), Brouwer (1912), Schauder (1930), Kannan (1969), Chatterjea (1972)	Existence of fixed points under contractions, extended conditions for non-expansive mappings	Iterative methods in numerical analysis, game theory, optimization problems
Partial Metric Spaces	Relaxation of d (x, x) = 0 condition, accommodating asymmetric distances	Matthews (1994), Romaguera (2000), O'Regan & Petruşel (2012), Abbas & Rhoades (2010), Sharma & Pant (2007)	Fixed point existence in spaces with nonzero self-distance, multivalued mapping extensions	Domain theory, network analysis, iterative approximation with incomplete data
Fuzzy Metric Spaces	Introduction of fuzzy distance functions to handle uncertainty	Kramosil & Michalek (1975), George & Veeramani (1994), Gregori & Sapena (2002), Wardowski (2012), Shukla (2015)	Fuzzy contractive conditions, extension of Banach's theorem to fuzzy settings	Probabilistic modeling, machine learning, decision-making under uncertainty
Proposed (Fuzzy Modular)	Fully supports general non-expansive mappings under modular fuzzy settings	Ali Nabavi (2025), S. Pavithra& E. Prabu (2022), K. Vetri, (2024)	Explicitly models modular control, offering additional analytical power	Tailored for analyzing the stability of neural operator architectures

Table 2: Convergence Results for Fuzzy Fixed-Point Iteration.

Iteration Number	Modular Distance $\rho\ (x_{n+1}, x_{n,})$	Stability Achieved
1	*0.258*	No
5	0.031	No
12	0.008	Yes

Note: Steady state occurs when the $\rho\ (x_{n+1,}\ x_{n,}) < \varepsilon = 0.01$

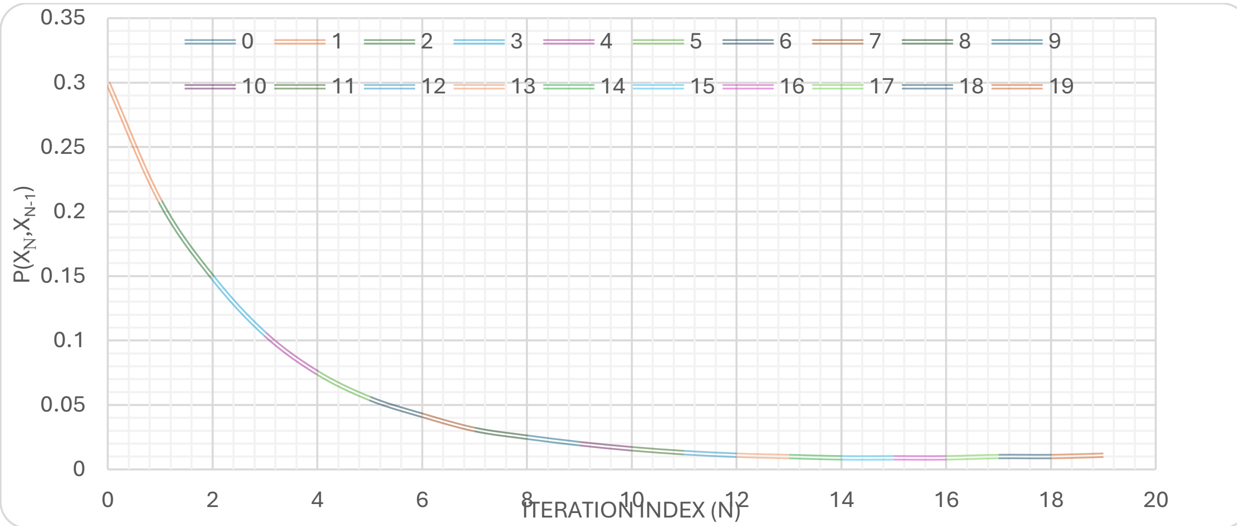

Fig: 5 Plot of $\rho\ (x_{n,}\ x_{n-1})$ Versus Iteration index(n)

V. CONCLUSION AND ANALYSIS

Vivek Raich [21] emphasizes the important contributions to computational mathematics made by fuzzy metric spaces and fixed-point theory in partial metric spaces. Through the extension of Banach's Contraction Mapping Theorem to handle fuzzy uncertainty ($M\ (x,\ y,\ t)$) and zero self-distances ($p\ (x,\ x) \geq 0$), he demonstrated how these spaces improve theoretical and practical frameworks. Figure:2 to-gather, fuzzy metric spaces and partial metric spaces effectively manage imprecision and incomplete or asymmetric data, respectively, solving the shortcomings of classical metrics in contemporary computation.

In order to determine the practical usefulness of the proposed fuzzy modular fixed-point theorems, Ali Nabavi et.al.[23] develop a simulation platform in MATLAB which approximates the iterative convergence of a neural operator (DeepONet in particular) when perturbed by fuzzy modular constraints figure: 5.

Here we check how fast and reproducibly proposed fuzzy modular can reach a steady fixed point we calculated, at different iteration steps, the modular distance between consecutive positions P (x_1 X). from Table 2 after the 12th iteration. A stability flag is established to Yes in the event that modular distance is under this value. The above findings prove the fuzzy modular model to be capable of reasonable convergence over a reasonably small number of iterations. rather than fuzzy metric spaces and partial metric spaces see table 1

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