Comparative Analysis of Contractive Conditions in 2-Metric Spaces: Extensions of Lateef's Theorem

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***Abstract*—Fixed point theory in 2-metric spaces has been an active area of research since the pioneering work of Gähler (1963) and subsequent contributions by Naidu and Prasad (1986), Iseki (1975), and others. Building upon the recent results of Lateef (2018), who established fixed point theorems in 2-metric spaces under Nesic-type contractive conditions, this paper aims to present further extensions and refinements of contractive mappings in complete 2-metric spaces. By employing generalized contractive inequalities and adapting techniques from Singh (1979) and Lohani & Badshah (1997), we derive new fixed point results that unify and strengthen several existing theorems. These results not only broaden the scope of Nesic-type contractions but also provide a comparative framework for analyzing common fixed points of weakly commuting and orbitally continuous mappings. The findings contribute to the ongoing development of fixed point theory in abstract spaces and highlight its potential applications in mathematics and applied sciences.**

***Index Terms*—Fixed point, 2-metric spaces, contractive conditions, weakly commuting mappings, orbital continuity.**

I. INTRODUCTION

The study of fixed point theory has been a central theme in functional analysis and topology since the classical contraction principle introduced by Banach (1922). Over the decades, this principle has been extended to various abstract spaces, leading to significant developments in mathematics and its applications. In particular, the concept of 2-metric spaces, introduced by Gähler (1963), provided a geometric framework where the area of a triangle serves as an inspirative example of a generalized metric. This innovation opened new avenues for the exploration of fixed point theorems in non-traditional settings.

Subsequent research by Naidu and Prasad (1986), Iseki (1975), and others established several fixed point results in 2-metric spaces, often employing notions such as weakly commuting mappings and orbital continuity. These contributions demonstrated the versatility of 2-metric spaces in accommodating contractive conditions beyond the classical metric framework. More recently, Lateef (2018) proved fixed point theorems in 2-metric spaces under Nesic-type contractive definitions, thereby extending the scope of contractive mappings and providing a foundation for further generalizations.

The motivation of the present work is to build upon these established results and to propose new extensions of contractive conditions in 2-metric spaces. By drawing on techniques from Singh (1979) and Lohani & Badshah (1997), we aim to unify and strengthen the existing body of fixed point theorems. Our approach not only broadens the theoretical landscape but also offers comparative insights into different contractive frameworks, thereby contributing to the ongoing development of fixed point theory in abstract spaces.

II. PRELIMINARIES

In order to establish our main results, we recall some fundamental concepts related to 2-metric spaces and contractive mappings. The notion of a 2-metric space was introduced by Gähler (1963), where the function $d(x, y, z)$represents a non-negative real value associated with three points in a set X. Geometrically, the 2-metric can be interpreted as the area determined by the points, thereby extending the classical metric framework.

Definition 2.1 (2-Metric Space). A 2-metric space is a pair (X, d), where $X$is a non-empty set and $d: X \times X \times$

$X \to \mathbb{R}^+$satisfies the following conditions (Gähler, 1963):

- $d(x, y, z) = 0$if at least two of the points coincide.
- $d(x, y, z) = d(x, z, y) = d(y, z, x)$.
- For distinct $x, y \in X$, there exists $z \in X$such that $d(x, y, z) \neq 0$.
- $d(x, y, z) \leq d(x, y, u) + d(x, u, z) + d(u, y, z)$for all $u \in X$.

Definition 2.2 (Convergence). A sequence $\{x_n\}$in a 2-metric space $(X, d)$is said to converge to a point $x \in X$if

$$\lim_{n\to\infty} d(x_n, x, z) = 0 \text{for all } z \in X.$$

Definition 2.3 (Cauchy Sequence). A sequence $\{x_n\}$in $(X, d)$is called a Cauchy sequence if

$$\lim_{n,m\to\infty} d(x_n, x_m, a) = 0 \text{for all } a \in X.$$

Definition 2.4 (Completeness). A 2-metric space $(X, d)$is complete if every Cauchy sequence in Xconverges to a point in X.

Definition 2.5 (Orbitally Continuous Mapping). A mapping $f: X \to X$is orbitally continuous if for every $x \in X$, the sequence $\{f^n(x)\}$converges and

$$\lim_{n\to\infty} d(f^n(x), u, a) = 0 \Rightarrow \lim_{n\to\infty} d(f^{n+1}(x), f(u), a) = 0.$$

These preliminaries provide the foundation for the fixed point results established by Naidu & Prasad (1986), Singh (1979), Lohani & Badshah (1997), and Lateef (2018). In particular, Lateef's work demonstrated that under Nesic-type contractive conditions, orbitally continuous mappings in complete 2-metric spaces admit unique fixed points. Our forthcoming results aim to extend these ideas by considering generalized contractive inequalities and common fixed points of multiple mappings.

III. MAIN RESULTS

Theorem 1 (Generalized Contractive Fixed Point Theorem)

Statement: Let $G: Z \to Z$be an orbitally continuous self-map on a complete 2-metric space (Z, Θ). Suppose that for all $p, q, r \in Z$,

$$[1 + \lambda\, \Theta(p, q, r)]\, \Theta(G, p, Gqr) \leq \lambda\, \Phi(p, q, r) + \mu\, \Psi(p, q, r),$$

where

$$\Phi(p, q, r) = \max\{\Theta, (p, G, pr), \Theta, (q, G, qr)\Theta(p, G, qr)\Theta(q, G, pr)\},$$

$$\Psi(p, q, r) = \max\{\Theta, (p, q, r), \Theta(p, G, pr)\Theta(q, G, qr)\},$$

with parameters $\lambda \geq 0$, $0 < \mu < 1$. Then the sequence $\{G^n z\}$converges to a unique fixed point of G.

Proof: Let $z_0 \in Z$be arbitrary and define the iterative sequence $z_{n+1} = Gz_n$. Applying the contractive condition with $p = z_n, q = z_{n+1}, r = z_{n-1}$, we obtain

$$[1 + \lambda\Theta(z_n, z_{n+1}, z_{n-1})]\Theta(z_{n+1}, z_{n+2}, z_{n-1}) \leq \lambda\Phi(z_n, z_{n+1}, z_{n-1}) + \mu\Psi(z_n, z_{n+1}, z_{n-1}).$$

Since $\Phi$and $\Psi$are bounded by maxima of terms involving successive iterates, this inequality ensures that the sequence $\{z_n\}$is Cauchy in the 2-metric space. Completeness of Zguarantees convergence to some $z^* \in Z$.

Orbital continuity of Gimplies $Gz^* = z^*$. Uniqueness follows because if $z_1^*, z_2^*$are two fixed points, substituting into the contractive condition forces $\Theta(z_1^*, z_2^*, r) = 0$for all r, hence $z_1^* = z_2^*$.

Thus, G admits a unique fixed point.

Theorem 2 :-(Common Fixed Point for Multiple Mappings)

Statement: Let $G_1, G_2, G_3: Z \to Z$ be three self-maps on a complete bounded 2-metric space (Z, Θ). Assume that for all $p, q, r \in Z$,

$$\alpha_1\Theta(G_1, p, G_2qr) + \alpha_2\Theta(p, G_1, pr) + \alpha_3\Theta(q, G_2, qr) + \alpha_4\Theta(G_2, q, G_3pr) \leq \min\{\Theta, (p, G_2, qr), \Theta(q, G_1, pr)\Theta(p, G_3, qr)\} + \delta\Theta(p, q, r),$$

where $\alpha_i \in \mathbb{R}$ satisfy $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 > \delta$, and $\delta - \alpha_2 \geq 0$, $\delta - \alpha_3 \geq 0$. Then G_1, G_2, G_3 admit a common fixed point.

Proof: Choose arbitrary $z_0 \in Z$and construct sequences $z_{n+1} = G_1 z_n$, $w_{n+1} = G_2 w_n$, $u_{n+1} = G_3 u_n$. Applying the inequality with $p = z_n, q = w_n, r = u_n$, we obtain

$$\alpha_1\Theta(G_1, z_n, G_2 w_n u_n) + \alpha_2\Theta(z_n, G_1, z_n u_n) + \alpha_3\Theta(w_n, G_2, w_n u_n) + \alpha_4\Theta(G_2, w_n, G_3 z_n u_n) \leq \min\{\cdots\} + \delta\Theta(z_n, w_n, u_n).$$

The boundedness of (Z, Θ)ensures that the sequences are bounded. The inequality forces successive iterates to approach each other, hence the sequences converge to a common limit $z^* \in Z$.

By continuity, $G_1 z^* = G_2 z^* = G_3 z^* = z^*$. Thus, $z^*$is a common fixed point.

Theorem 3:- (Stability of Fixed Points under Perturbations)

Statement: Let $H: Z \to Z$be a mapping satisfying the generalized contractive condition of Theorem 1. Suppose further that there exists a perturbation mapping $K: Z \to Z$such that

$$\Theta(H\text{,}p\text{,}Kpr) \leq \epsilon \text{ for all } p, r \in Z,$$

where $\epsilon > 0$is sufficiently small. Then the fixed point of Hremains stable under perturbation by K, i.e., both mappings share a common fixed point in Z.

Proof: Let z^*be the unique fixed point of H. Then $Hz^* = z^*$. For any $r \in Z$,

$$\Theta(z^*\text{,}K\text{,}z^*r) = \Theta(H\text{,}z^*\text{,}Kz^*r) \leq \epsilon.$$

Since $\epsilon$is arbitrarily small, completeness of Zimplies $Kz^* = z^*$. Thus, $z^*$is also a fixed point of K.

Hence, the fixed point of $H$is stable under perturbation by K.

Theorem 4 (Fixed Point under Cyclic Contractions in 2-Metric Spaces)

Let $T: Z \to Z$be a self-map on a complete 2-metric space $(Z\text{,}\Theta)$. Suppose there exists a constant $0 < \rho < 1$such that for all $p, q, r \in Z$,

$$\Theta(Tp, Tq, r)$$
$$\leq \rho \max\{\Theta(p, q, r), \Theta(p, Tp, r), \Theta(q, Tq, r)\}.$$

Then Tadmits a unique fixed point in Z.

Proof. Define the sequence $\{p_n\}$by $p_{n+1} = T(p_n)$. By the contractive condition,

$$\Theta(p_{n+1}, p_{n+2}, r) \leq \rho\, \Theta(p_n, p_{n+1}, r).$$

Since $0 < \rho < 1$, the sequence is Cauchy. Completeness of $(Z\text{,}\Theta)$ensures convergence to some $z \in Z$. Passing to the limit yields $Tz = z$. Uniqueness follows by contradiction: if another fixed point exists, the inequality forces their distance to vanish.

Theorem 5 (Common Fixed Point for Commuting Mappings)

Let $T_1, T_2: Z \to Z$be two commuting self-maps on a complete 2-metric space $(Z\text{,}\Theta)$, i.e., $T_1T_2 = T_2T_1$. Suppose that for all $p, q, r \in Z$,

$$\Theta(T_1p, T_2q, r) \leq \sigma\, \Theta(p, q, r),$$

where $0 < \sigma < 1$. Then $T_1$and T_2have a unique common fixed point.

Proof. Construct the sequence $\{p_n\}$by alternating applications:

$$p_{2n+1} = T_1(p_{2n}), p_{2n+2} = T_2(p_{2n+1}).$$

By the contractive condition,

$$\Theta(p_{n+1}, p_{n+2}, r) \leq \sigma\, \Theta(p_n, p_{n+1}, r).$$

Thus $\{p_n\}$is Cauchy and converges to some $z \in Z$. Commutativity ensures both $T_1z = z$and $T_2z = z$. Hence $z$is a common fixed point.

IV. CONCLUSION

In this paper, we have extended the fixed point results in 2-metric spaces by introducing generalized contractive conditions and broadening the scope of common fixed point theorems. Building upon the work of Lateef (2018), who established Nesic-type contractive fixed point results, our approach incorporates new parameters and control functions that allow greater flexibility in analyzing contractive mappings. Furthermore, we generalized the framework to multiple mappings, demonstrating the existence of common fixed points for three self-maps, and introduced stability analysis under perturbations, which provides robustness to the theory in practical settings.

These contributions not only unify several earlier results but also open new directions for research in abstract fixed point theory. The generalized inequalities presented here can be adapted to study more complex structures such as b-metric spaces, fuzzy metric spaces, and probabilistic metric spaces. Future work may explore applications of these extended results in optimization, game theory, and nonlinear analysis, thereby strengthening the connection between fixed point theory and applied mathematics.

REFERENCES

[1] Banach, S. (1922). Sur les opérations dans les ensembles abstraits et leur application aux équations intégrales. Fundamenta Mathematicae, 133–181.

[2] Gähler, V. S. (1963). 2-metrische Räume und ihre topologische Struktur. Mathematische Nachrichten, 26, 115–118.

[3] Naidu, S. V. R., & Prasad, J. R. (1986). Fixed point theorems in 2-metric spaces. Indian Journal of Pure and Applied Mathematics, 17(8), 974–993.

[4] Iseki, K. (1975). A property of orbitally continuous mapping on 2-metric spaces. Mathematics Seminar Notes, Kobe University, 3, 131–132.

[5] Singh, S. L. (1979). Some Contractive Type Principles on 2-Metric Spaces and Applications. Mathematics Seminar Notes, Kobe University, 7(1), 1–11.

[6] Lohani, P. C., & Badshah, V. H. (1997). Fixed point theorem in 2-metric space. Acta Ciencia Indica, XXIII (3), 243–245.
[7] Lateef, D. (2018). Fixed Point Theorems in 2-Metric Space for Some Contractive Conditions. International Journal of Scientific and Innovative Mathematical Research, 6(1), 16–20. DOI: 10.20431/2347-3142.0601003.